Hamiltonian Control Systems

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Lecture 3

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Outline

- Passivity-based control (PBC)
- Standard PBC
- Passivity of pH systems
- Stabilization by damping injection
- Energy-balancing PBC
- Application to mechanical systems
- Electrical example
- Dissipation obstacle

Passivity-Based Control (PBC)

- Physical systems satisfy

  stored energy $=$ **supplied energy** $+$ **dissipated energy**

- This suggest the natural control objective

  desired stored energy $=$ **new supplied energy**
  $+$ **desired dissipated energy**

- Essence of PBC (Ortega/Spong, 1989)

  PBC $=$ **energy shaping** $+$ **damping assignment**

- Main objective: rendering the (closed-loop) system passive w.r.t. some desired storage function.

Classification of PBC

- Standard PBC
  - fix *a priori* a desired storage function
  - add damping in the errors
  - partial system inversion
- Energy-Balancing Control
  - total energy = (energy of the open-loop) - (energy extracted from the environment)
- IDA-PBC
  - interconnection and damping structure free to choose
  - solving a PDE

Classification of PBC

- Power-Shaping Control
  - Instead of energy shaping
  - Brayton-Moser models

Definition: Passivity

- General (f, g, h) system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned}$$

with state $x \in \mathbb{R}^n$, and port-variables $u, y \in \mathbb{R}^m$, is passive if

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t u^T(\tau) y(\tau) \mathrm{d}\tau,}_{\text{supplied energy}}$$

for some $H : \mathbb{R}^n \to \mathbb{R}^+$.

Passivation Objective

Select a control action $u = \beta(x) + v$ such that

$$E_d[x(t)] - E_d[x(0)] = \int_0^t v^T(\tau) \hat{y}(\tau) \mathrm{d}\tau - d_d(\cdot),$$

where

- E_d: desired closed-loop energy function,
- $d_d \geq 0$: desired dissipation (damping), and
- $\hat{y}$: (possibly new) passive output

$\Leftrightarrow$ Energy-shaping plus damping injection.

Standard PBC

Procedure:

- Identify the passivity properties of the system.
- Fix the desired energy function in terms of the error variables (actual state minus desired state)

$$E_d(x, x_d) = \frac{1}{2}(x - x_d)^T Q_d (x - x_d).$$

- Make a 'copy' of the system to create error system with desired passivity properties.

Standard PBC (Cont'd)

Example: Consider a mass-spring system driven by a force F

$$m\dot{v} = F - f, \quad k^{-1}\dot{f} = v.$$

- System is passive wrt supply rate Fv (supplied power) and storage (co-energy) function

$$E(v, f) = \frac{1}{2}mv^2 + \frac{1}{2k}f^2$$

since (assume $v(0) = f(0) = 0$)

$$\underbrace{\frac{1}{2}m[v(t)]^2 + \frac{1}{2k}[f(t)]^2}_{\text{stored energy}} = \underbrace{\int_0^t F(\tau)v(\tau)\mathrm{d}\tau}_{\text{supplied energy}}.$$

Standard PBC (Cont'd)

Example (cont'd):

- Let v_d and f_d resp. be the desired velocity and force,

$$E_d(v, f, v_d, f_d) = \frac{1}{2}m(v - v_d)^2 + \frac{1}{2k}(f - f_d)^2.$$

- Copy of the system + add damping in errors:

$$\begin{aligned} m\dot{v}_d &= F - f_d + R(v - v_d) \\ k^{-1}\dot{f}_d &= v_d, \end{aligned}$$

for some $R \geq 0$.

Control objective: $v_d = 0 \Rightarrow \dot{v}_d = 0 \Rightarrow F = -Rv$.

Standard PBC (Cont'd)

Example (cont'd):

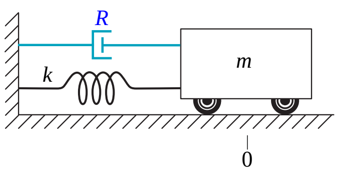
- Closed-loop energy-balance:

$$\underbrace{\frac{1}{2}m[v(t) - v_d(t)]^2 + \frac{1}{2k}[f(t) - f_d(t)]^2}_{\text{stored energy}}$$

$$= -\underbrace{\int_0^t R[v(\tau) - v_d(\tau)]^2\mathrm{d}\tau}_{\text{dissipated energy}}.$$

- Stabilized system:

$$\begin{aligned} m\dot{v} &= -f - Rv \\ k^{-1}\dot{f} &= v. \end{aligned}$$



Standard PBC (Cont'd)

Example (cont'd):

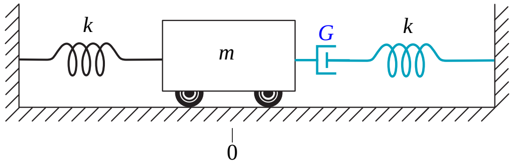
- Alternative choice of damping:

$$\begin{aligned} m\dot{v}_d &= F - f_d \\ k^{-1}\dot{f}_d &= v_d + G(f - f_d), \end{aligned}$$

for some $G \geq 0$.

- Stabilized system:

$$\begin{aligned} m\dot{v} &= -f + f_d \\ k^{-1}\dot{f} &= v \\ k^{-1}\dot{f}_d &= G(f - f_d). \end{aligned}$$



Add some more structure . . .

- Port-Hamiltonian input-state-output systems

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u, \quad y = g^T(x)\frac{\partial H}{\partial x}(x).$$

Here:

- H — Stored energy
- J — Internal interconnection structure ($J = -J^T$)
- R — Dissipation structure ($R = R^T$)

- Power-balance:

$$\frac{d}{dt}H = \left(\frac{\partial H}{\partial x}(x)\right)^T \dot{x} = \left(\frac{\partial H}{\partial x}(x)\right)^T \left([J - R]\frac{\partial H}{\partial x}(x) + g(x)u\right)$$

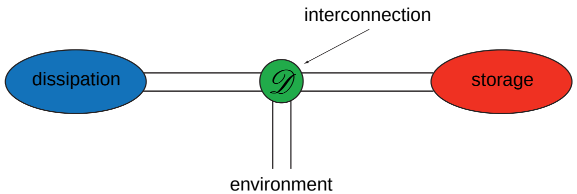
$$= y^T u - \left(\frac{\partial H}{\partial x}(x)\right)^T R \frac{\partial H}{\partial x}(x)$$

Energy-Balance

- Integrating the power-balance from 0 to t yields the **energy-balance**:

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} = \underbrace{\int_0^t y^T(s)u(s)ds}_{\text{supplied energy}}$$

$$\underbrace{- \int_0^t \left(\frac{\partial H}{\partial x}(x(s))\right)^T R(x(s))\frac{\partial H}{\partial x}(x(s))ds}_{\text{dissipated energy}}$$



Passivity of pH Systems

The port-Hamiltonian system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u,$$

$$y = g^T(x)\frac{\partial H}{\partial x}(x),$$

with state $x \in \mathbb{R}^n$ and port-variables $u, y \in \mathbb{R}^m$, is **passive** if

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau}_{\text{supplied energy}}, \qquad (*)$$

for some Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}^+$.

Stabilization by Damping Injection

- Use storage function (read: Hamiltonian) as Lyapunov function for the uncontrolled system.
- Passive systems can be asymptotically stabilized by adding damping via the control. In fact, for a passive port-Hamiltonian system we have

$$\dot{H}(x) \leq u^T y.$$

Hence letting $u = -K_d y$, with $K_d = K_d^T \succ 0$, we obtain

$$\dot{H}(x) \leq -y^T K_d y,$$

$\Rightarrow$ **asymptotic stability**, provided an observability condition is met (i.e., zero-state detectability of the output).

Stabilization by Damping Injection

- If $H(x)$ non-negative, total amount of energy that can be extracted from a passive system is bounded, i.e.,

$$-\int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau \leq H[x(0)] < \infty.$$

Example: Actuated Euler Body

The Euler equations for the rotational dynamics of a rigid body about its center of mass with a single control input u are given by

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + g_1 u \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + g_2 u \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + g_3 u, \end{aligned}$$

where ω_k, for $k = 1,2,3$, are the angular velocities relative to the axis of a frame fixed to the body, and $I_1 > I_2 > I_3$ are the principle moments of inertia.

For $u = 0$ the system has its equilibrium at $\omega^* = (0,0,0)^T$.

Example: Actuated Euler Body

From linear theory we know that if a pair (A,B) of a linear system is controllable, then there exists a feedback matrix F s.t. $A+BF$ has all its eigenvalues in the left-half of the complex plane.

$\Rightarrow$ This implies that if the linearization of a nonlinear system is controllable, the nonlinear system is (locally) **stabilizable**.

However, for the actuated Euler body, we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I_1^{-1}g_1 \\ I_2^{-1}g_2 \\ I_3^{-1}g_3 \end{bmatrix} \quad \Rightarrow \quad \text{inconclusive.}$$

Example: Actuated Euler Body

Defining the angular momenta $p_k = I_k\omega_k$, $k = 1,2,3$, we can rewrite the system in port-Hamiltonian form as

$$\begin{bmatrix}\dot p_1\\ \dot p_2\\ \dot p_3\end{bmatrix} = \begin{bmatrix}0 & -p_3 & p_2\\ p_3 & 0 & -p_1\\ -p_2 & p_1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial p_1}\\ \frac{\partial H}{\partial p_2}\\ \frac{\partial H}{\partial p_3}\end{bmatrix} + \begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix}u,$$

with natural output

$$y = g_1\frac{\partial H}{\partial p_1} + g_2\frac{\partial H}{\partial p_2} + g_3\frac{\partial H}{\partial p_3},$$

and Hamiltonian the total kinetic energy $H(p) = \frac{1}{2}\left(\frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + \frac{p_3^2}{I_3}\right)$.

Example: Actuated Euler Body

Since for $u = 0$, $\dot H(p) = 0$ and $H(p)$ has its min. at $p = 0 \Rightarrow$ the origin $p_k^* = I_k\omega_k^* = 0$, for $k = 1,2,3$, is stable.

Now, to render the closed-loop as. stable we apply the feedback

$$u = -y = -\sum_{k=1}^{3} g_k\frac{p_k}{I_k} = -\sum_{k=1}^{3} g_k\omega_k,$$

yielding convergence to the largest invariant set contained in

$$\mathbb{O} := \{p \in \mathbb{R}^3 \mid \dot H(p) = 0\} = \left\{p \in \mathbb{R}^3 \,\middle|\, \sum_{k=1}^{3} g_k\frac{p_k}{I_k} = 0\right\},$$

which actually is $p = 0$ iff $g_k \neq 0$ for all $k = 1,2,3$.

Energy-Balancing PBC

- Usually, the point where the open-loop energy is minimal is not of interest. Instead some nonzero eq. point, say x^*, is desired.
- Standard formulation of PBC: Find $u = \beta(x) + v$ s.t.

$$\underbrace{H_d[x(t)] - H_d[x(0)]}_{\text{stored energy}} = \underbrace{\int_0^t v^T(\tau)z(\tau)\mathrm{d}\tau}_{\text{supplied energy}} - \underbrace{d_d(t)}_{\text{diss. energy}},$$

where the desired energy $H_d(x)$ has a minimum at x^*, and z is the new output (which may be equal to y).

- Hence control problem consist in finding $u = \beta(x) + v$ s.t. energy supplied by the controller is a function of the state x.

Energy-Balancing PBC

Proposition: Consider the port-Hamiltonian system

$$\dot x = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u,$$
$$y = g^T(x)\frac{\partial H}{\partial x}(x).$$

If we can find a function $\beta(x)$ and a vector function $K(x)$ satisfying

$$[J(x) - R(x)]K(x) = g(x)\beta(x)$$

such that

i) $\frac{\partial K}{\partial x}(x) = \frac{\partial^T K}{\partial x}(x)$ (integrability);

Energy-Balancing PBC

Proposition (cont'd):

ii) $K(x^*) = -\frac{\partial H}{\partial x}(x^*)$ (equilibrium assignment);

iii) $\frac{\partial K}{\partial x}(x^*) \succ -\frac{\partial^2 H}{\partial x^2}(x^*)$ (Lyapunov stability).

Then the closed-loop system is a port-Hamiltonian system of the form

$$\dot{x} = [J(x) - R(x)]\frac{\partial H_d}{\partial x}(x),$$

with $H_d(x) = H(x) + H_a(x)$, $K(x) = \frac{\partial H_a}{\partial x}(x)$, and x^* (locally) stable.

Energy-Balancing PBC

- Note that $(R(x) = R^T(x) \succeq 0)$

$$\dot{H}_d(x) = -\frac{\partial^T H_d}{\partial x}(x)R(x)\frac{\partial H_d}{\partial x}(x) \leq 0.$$

- Also note that x^* is (locally) asymptotically stable if, in addition, the largest invariant set is contained in

$$\left\{ x \in \mathbb{D} \;\middle|\; \frac{\partial^T H_d}{\partial x}(x)R(x)\frac{\partial H_d}{\partial x}(x) = 0 \right\},$$

where $\mathbb{D} \subset \mathbb{R}^n$.

Mechanical Systems

Consider a (fully actuated) mechanical systems with total energy

$$H(q,p) = \tfrac{1}{2}p^T M^{-1}(q)p + P(q),$$

with generalized mass matrix $M(q) = M^T(q) \succ 0$. Assume that the potential energy $P(q)$ is bounded from below. PH structure:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}}_{J=-J^T} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B(q) \end{bmatrix}}_{g(q)} u,$$

$$y = \begin{bmatrix} 0 & B^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}.$$

Mechanical Systems

Clearly, the system has as passive outputs the generalized velocities:

$$\dot{H}(q,p) = u^T y = u^T B^T(q)\frac{\partial H}{\partial p}(q,p) = u^T M^{-1}(q)p = u^T \dot{q}.$$

Now, the Energy-Balancing PBC design boils down to

$$JK(q,p) = g(q)\beta(q,p), \quad K(x) = \frac{\partial H_a}{\partial x}(x)$$

which in the fully actuated ($B(q) = I_k$) case simplifies to

$$K_2(q,p) = 0$$
$$-K_1(q,p) = \beta(q,p).$$

Mechanical Systems

The simplest way to ensure that the closed-loop energy has a minimum at $(q,p)=(q^*,0)$ is to select

$$\beta(q)=\frac{\partial P}{\partial q}(q)-K_p(q-q^*),\quad K_p=K_p^T\succ 0.$$

This gives the controller energy

$$H_a(q)=-P(q)+\frac{1}{2}(q-q^*)^T K_p(q-q^*)+\kappa,$$

so that the closed-loop energy takes the form

$$H_d(q,p)=\frac{1}{2}p^T M^{-1}(q)p+\frac{1}{2}(q-q^*)^T K_p(q-q^*).$$

Mechanical Systems

To ensure that the trajectories actually converge to $(q^*,0)$ we need to render the closed-loop asymptotically stable by adding some damping

$$v=-K_d\frac{\partial H}{\partial p}(q,p)=-K_d\dot{q},$$

as shown before. Note that the energy-balance of the system is now

$$\underbrace{H_d[q(t),p(t)]-H_d[q(0),p(0)]}_{\text{stored energy}}=-\underbrace{\int_0^t \dot{q}^T(\tau)K_p\dot{q}(\tau)\mathrm{d}\tau}_{\text{diss. energy}}.$$

Mechanical Systems

- Observe that the controller obtained via energy-balancing is just the classical PD + gravity compensation controller.
- However, the design via Energy-Balancing PBC provides a new interpretation of the controller, namely, that the closed-loop energy is (up to a constant) equal to

$$H_d(q,p)=H(q,p)-\int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau,$$

i.e., the difference between the open-loop and controller energy.