Hamiltonian Control Systems

Lecture 4, Elgersburg School 2012

Arjan van der Schaft and Dimitri Jeltsema

March 12–16, 2012

Outline

Energy-Balancing PBC

Proposition: Consider the port-Hamiltonian system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u,$$

$$y = g^T(x)\frac{\partial H}{\partial x}(x).$$

If we can find a function $\beta(x)$ and a vector function $K(x)$ satisfying

$$[J(x) - R(x)]K(x) = g(x)\beta(x)$$

such that

i) $\dfrac{\partial K}{\partial x}(x) = \dfrac{\partial^T K}{\partial x}(x)$ (integrability);

Energy-Balancing PBC

Proposition (cont'd):

ii) $K(x^*) = -\dfrac{\partial H}{\partial x}(x^*)$ (equilibrium assignment);

iii) $\dfrac{\partial K}{\partial x}(x^*) \succ -\dfrac{\partial^2 H}{\partial x^2}(x^*)$ (Lyapunov stability).

Then the closed-loop system is a port-Hamiltonian system of the form

$$\dot{x} = [J(x) - R(x)]\frac{\partial H_d}{\partial x}(x),$$

with $H_d(x) = H(x) + H_a(x)$, $K(x) = \frac{\partial H_a}{\partial x}(x)$, and x^* (locally) stable.

Energy-Balancing PBC

- Note that ($R(x) = R^T(x) \succeq 0$)

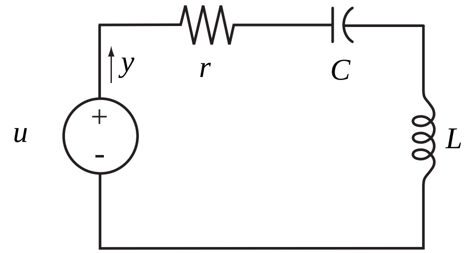
$$\dot{H}_d(x) = -\frac{\partial^T H_d}{\partial x}(x) R(x) \frac{\partial H_d}{\partial x}(x) \leq 0.$$

- Also note that x^* is (locally) asymptotically stable if, in addition, the largest invariant set is contained in

$$\left\{ x \in \mathbb{D} \;\middle|\; \frac{\partial^T H_d}{\partial x}(x) R(x) \frac{\partial H_d}{\partial x}(x) = 0 \right\},$$

where $\mathbb{D} \subset \mathbb{R}^n$.

Example: Linear RLC circuit



- Write the dynamics in PH form.
- Determine the equilibrium point.
- Find the passive input and output.
- Design an Energy-Balancing PBC that stabilizes an admissible equilibrium.

Casimir Functions

In general, for any Hamiltonian dynamics

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x),$$

one may search for conserved quantities C, called **Casimirs**, as being solutions of

$$\frac{\partial^T C}{\partial x}(x) J(x) = 0.$$

Then $\frac{d}{dt} C = 0$ for **every** H, and thus also $H + C$ is a **candidate Lyapunov function**. [Note that the minimum of $H + C$ may now be **different** from the minimum of H].

Example: Euler Body

Recall port-Hamiltonian model:

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} u,$$

A Casimir function $C(p)$ for the Euler body can be found by solving

$$\begin{bmatrix} \frac{\partial C}{\partial p_1} & \frac{\partial C}{\partial p_2} & \frac{\partial C}{\partial p_3} \end{bmatrix} \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} = [0 \quad 0 \quad 0],$$

which yields $C(p) = p_1^2 + p_2^2 + p_3^2$.

Control by interconnection: set-point stabilization

Consider first a lossless Hamiltonian plant system

$$P: \begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x}(x) + g(x)u \\ y = g^T(x)\dfrac{\partial H}{\partial x}(x) \end{cases}$$

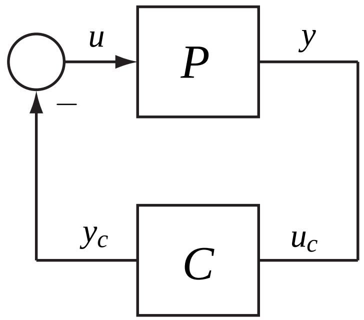
where the desired set-point x^* is **not** a minimum of the Hamiltonian H, while the Hamiltonian dynamics $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ does not possess useful Casimirs.

How to (asymptotically) stabilize x^* ?

Control by interconnection

Consider a **controller** port-Hamiltonian system

$$C: \begin{cases} \dot{\xi} = J_c(\xi)\dfrac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c, \\ y_c = g_c^T(\xi)\dfrac{\partial H_c}{\partial \xi}(\xi) \end{cases}$$



via the standard feedback interconnection

$$u = -y_c, \quad u_c = y.$$

Control by interconnection

Then the closed-loop system is the port-Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} J(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}$$

with total Hamiltonian $H(x) + H_c(\xi)$.

Main idea: design the controller system in such a manner that the closed-loop system has useful Casimirs $C(x,\xi)$!

This may lead to a candidate Lyapunov function

$$V(x,\xi) := H(x) + H_c(\xi) + C(x,\xi),$$

with H_c to be determined.

Control by interconnection

Thus we look for functions $C(x,\xi)$ satisfying

$$\begin{bmatrix} \frac{\partial^T C}{\partial x}(x,\xi) & \frac{\partial^T C}{\partial \xi}(x,\xi) \end{bmatrix} \begin{bmatrix} J(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) \end{bmatrix} = 0$$

such that the candidate Lyapunov function

$$V(x,\xi) := H(x) + H_c(\xi) + C(x,\xi)$$

has a minimum at (x^*,ξ^*) for some (or a set of) $\xi^* \Rightarrow$ **stability**.

Subsequently, one may add extra damping (directly or in the dynamics of the controller) to achieve **asymptotic stability**.

Example: the ubiquitous pendulum

Consider an actuated pendulum (1DOF robot) with Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + 1 - \cos q$$

actuated by a torque u, with output $y = p$ (angular velocity). Suppose we wish to stabilize the pendulum at $q^* \neq 0$ and $p^* = 0$.

Apply the nonlinear integral control

$$\begin{aligned} \dot{\xi} &= u_c \\ y_c &= \frac{\partial H_c}{\partial \xi}(\xi), \end{aligned}$$

which is a port-Hamiltonian controller system with $J_c = 0$.

Example: the ubiquitous pendulum

Casimirs $C(q,p,\xi)$ are found by solving

$$\begin{bmatrix} \frac{\partial C}{\partial q} & \frac{\partial C}{\partial p} & \frac{\partial C}{\partial \xi} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = [0 \quad 0 \quad 0]$$

leading to Casimirs $C(q,p,\xi) = K(q-\xi)$, and candidate Lyapunov functions

$$V(q,p,\xi) = \frac{1}{2}p^2 + 1 - \cos q + H_c(\xi) + K(q-\xi),$$

with the functions H_c and K to be determined.

Example: the ubiquitous pendulum

For a local minimum, determine K and H_c such that

- **Equilibrium assignment**

$$\begin{aligned} \tfrac{\partial V}{\partial q}(q^*,0,\xi^*) &= \sin q^* + \tfrac{\partial K}{\partial z}(q^* - \xi^*) = 0, \quad \tfrac{\partial V}{\partial p}(q^*,0,\xi^*) = 0, \\ \tfrac{\partial V}{\partial \xi}(q^*,0,\xi^*) &= \tfrac{\partial H_c}{\partial \xi}(\xi^*) - \tfrac{\partial K}{\partial \xi}(q^* - \xi^*) = 0. \end{aligned}$$

- **Minimum condition**

$$\begin{bmatrix} \cos q^* + \frac{\partial^2 K}{\partial q^2}(q^* - \xi^*) & 0 & -\frac{\partial^2 K}{\partial q \partial \xi}(q^* - \xi^*) \\ 0 & 1 & 0 \\ -\frac{\partial^2 K}{\partial \xi \partial q}(q^* - \xi^*) & 0 & \frac{\partial^2 K}{\partial \xi^2}(q^* - \xi^*) + \frac{\partial^2 H_c}{\partial \xi^2}(\xi^*) \end{bmatrix} \succ 0$$

$\Rightarrow$ Many possible solutions.

A state feedback perspective: shaping the Hamiltonian

Restrict (without much loss of generality) to Casimirs of the form

$$C(x,\xi) = \xi - G(x).$$

It follows that for all time instants

$$\xi = G(x) + \kappa, \quad \kappa \in \mathbb{R}.$$

Suppose that in this way **all** control states ξ can be expressed as a function $\xi = G(x)$ of the plant state x.

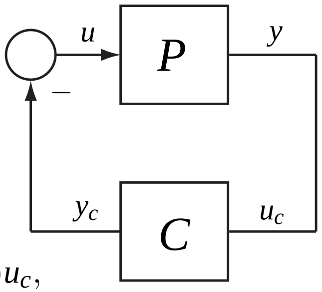
Then, the dynamic feedback reduces to a **state feedback**, and the Lyapunov function $H(x) + H_c(\xi) + C(x,\xi)$ reduces to the **shaped** Hamiltonian $H(x) + H_c(G(x))$.

The dissipation obstacle

Surprisingly, the presence of dissipation $R \neq 0$ may pose a problem!

Indeed, if we consider a plant system with dissipation ($R \succeq 0$)

$$P: \begin{cases} \dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u, \\ y = g^T(x)\frac{\partial H}{\partial x}(x), \end{cases}$$



and a controller with dissipation ($R_c \succeq 0$)

$$C: \begin{cases} \dot{\xi} = [J_c(\xi) - R_c(\xi)]\frac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c, \\ y_c = g_c^T(\xi)\frac{\partial H_c}{\partial \xi}(\xi), \end{cases}$$

The dissipation obstacle

...we now obtain closed-loop port-Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} J(x) - R(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) - R_c(\xi) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}$$

with total Hamiltonian $H(x) + H_c(\xi)$.

Now, look for Casimir's $C(x,\xi)$ satisfying

$$\begin{bmatrix} \frac{\partial^T C}{\partial x}(x,\xi) & \frac{\partial^T C}{\partial \xi}(x,\xi) \end{bmatrix} \begin{bmatrix} J(x) - R(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) - R_c(\xi) \end{bmatrix} = 0,$$

The dissipation obstacle

...which yields

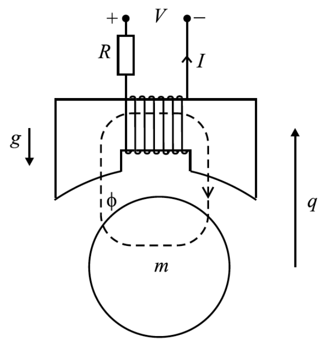
$$-\frac{\partial^T C}{\partial x}(x,\xi)R(x)\frac{\partial C}{\partial x}(x,\xi) = \frac{\partial^T C}{\partial \xi}(x,\xi)R_c(\xi)\frac{\partial C}{\partial \xi}(x,\xi).$$

However, since we assumed semi-positivity of $R(x)$ and $R_c(x)$:

$$\frac{\partial^T C}{\partial x}(x,\xi)R(x) = 0, \quad \frac{\partial^T C}{\partial \xi}(x,\xi)R_c(\xi) = 0.$$

This is the **dissipation obstacle**, which implies that one cannot shape the Lyapunov function in the coordinates that are directly affected by energy dissipation.

Example: Levitated Ball



Port-Hamiltonian equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} V$$

$$y = \frac{\partial H}{\partial x_1} = I, \quad H(x_1,x_2,x_3) = \frac{1}{2k}(1 - x_2)x_1^2 + \frac{x_3^2}{2m} + mgx_2,$$

with flux $x_1 = \phi$, displacement $x_2 = q$, and momentum $x_3 = p$.

Desired equilibrium: $x^* = [\sqrt{2kmg}, x_2^*, 0]^T$.

Example: Levitated Ball

Control by interconnection and energy-balancing (EB) do not work:

- Coordinates that need to be shaped are x_1 and x_2. However,

$$\begin{bmatrix} \frac{\partial C}{\partial x_1} & \frac{\partial C}{\partial x_2} & \frac{\partial C}{\partial x_3} \end{bmatrix} \begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [0 \quad 0 \quad 0].$$

- Solving the EB PDE: $[J-R]K(x) = g\beta(x)$ yields

$$\begin{aligned} -rK_1(x) &= \beta(x), \\ K_2(x) &= 0, \\ K_3(x) &= 0. \end{aligned}$$

Example: Levitated Ball

...this means the function H_a (recall $K(x) = \frac{\partial H_a}{\partial x}(x)$) can only depend on x_1. Thus, the resulting closed-loop energy (read: Lyapunov function) would be of the form

$$H_d(x) = \frac{1}{2k}(1-x_2)x_1^2 + \frac{x_3^2}{2m} + mgx_2 + H_a(x_1).$$

Even though, with a suitable selection of $H_a(x_1)$, we can satisfy the equilibrium assignment condition, the Hessian will be

$$\frac{\partial^2 H_d}{\partial x^2}(x) = \begin{bmatrix} \frac{1-x_2}{k} + \frac{\partial^2 H_a}{\partial x_1^2}(x_1) & -\frac{x_1}{k} & 0 \\ -\frac{x_1}{k} & 0 & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \not> 0.$$

This calls for a more radical approach

Generalization: IDA-PBC

- IDA-PBC aims at forcing the closed-loop dynamics to be:

$$\dot{x} = [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x)$$

Now:

- H_d — Closed-loop stored energy
- J_d — Desired interconnection structure
- R_d — Desired dissipation

Closed-loop energy-balance

$$H_d(x(t)) - H_d(x(0)) = -\int_0^t \left(\frac{\partial H_d}{\partial x}(x(s))\right)^T R_d(x(s))\frac{\partial H_d}{\partial x}(x(s))ds.$$

IDA-PBC

To accomplish this, we just match the original (open-loop) pH system to the desired one, i.e.,

$$[J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\beta(x) \equiv [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x),$$

via the control $u = \beta(x)$.

IDA-PBC

Find a vector function $K(x)$, a function $\beta(x)$, a skew-symmetric matrix J_a and a symmetric positive semi-definite matrix R_a such that

$$[J(x) + J_a(x) - R(x) - R_a(x)]K(x) = [J_a(x) - R_a(x)]\frac{\partial H}{\partial x}(x) + g(x)\beta(x), \qquad (*)$$

with $K(x) = \frac{\partial H_a}{\partial x}(x)$.

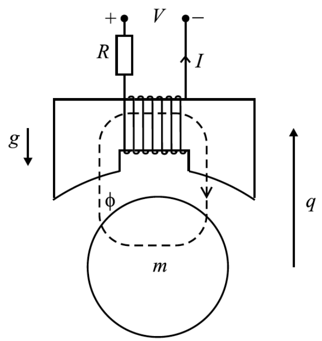
Hence, the closed-loop dynamics with $u = \beta(x)$ is again pH, with

$$H_d(x) = H(x) + H_a(x),\ J_d(x) = J(x) + J_a(x),\ R_d(x) = R(x) + R_a(x).$$

Exploit freedom in selecting J_a and R_a to make solution of $(*)$ easier.

Example: Levitated Ball

Port-Hamiltonian equations:



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-R & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial x_1}\\ \frac{\partial H}{\partial x_2}\\ \frac{\partial H}{\partial x_3}\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}V$$

$$y = \frac{\partial H}{\partial x_1} = I, \quad H(x_1, x_2, x_3) = \frac{1}{2k}(1 - x_2)x_1^2 + \frac{x_3^2}{2m} + mgx_2,$$

with flux $x_1 = \phi$, displacement $x_2 = q$, and momentum $x_3 = p$.

Desired equilibrium: $x^* = [\sqrt{2kmg}, x_2^*, 0]^T$.

Example: Levitated Ball

Control by interconnection and EB did not work.

Apart from the dissipation obstacle, the source of the problem is the lack of an *effective coupling* between the electrical and mechanical subsystems.

For that, modify interconnection structure to

$$J_d = \begin{bmatrix}0 & 0 & -\alpha\\ 0 & 0 & 1\\ \alpha & -1 & 0\end{bmatrix},$$

where α is a constant to be defined.

Example: Levitated Ball

Now, equation $(*)$ becomes

$$-\alpha K_3(x) - rK_1(x) = \tfrac{\alpha}{m}x_3 + \beta(x),$$
$$K_3(x) = 0,$$
$$\alpha K_1(x) - K_2(x) = -\tfrac{\alpha}{k}(1-x_2)x_1.$$

The first equation defines the control, whereas the third one can be solved, with e.g. Maple, as

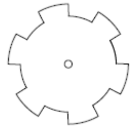
$$H_a(x) = \frac{x_1^3}{6k\alpha} + \frac{1}{2k}(1-x_2)x_1^2 + \Phi\left(x_2 + \frac{x_1}{\alpha}\right).$$

Further info can be found in: R. Ortega et al., **Putting Energy Back in Control**, IEEE Control Systems Magazine, April 2001.

IDA-PBC

Isotropic (smooth rotor) synchronous motors are more efficient that indented rotor motors

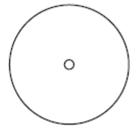
- Open–loop



$$J(x) = \begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & -(x_1+\Phi) \\ -x_2 & x_1+\Phi & 0 \end{bmatrix}$$

Φ is the dq back emf constant.

- Virtual behavior in closed–loop



$$J_d(x) = \begin{bmatrix} 0 & L_0x_3 & 0 \\ -L_0x_3 & 0 & -\Phi \\ 0 & \Phi & 0 \end{bmatrix}$$

L_0 free parameter, represents stator inductance ($L_d = L_q$).

New control paradigms: Energy transfer control

Consider two port-Hamiltonian systems

$$\Sigma_i : \begin{cases} \dot{x}_i = J_i(x_i)\dfrac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)u_i \\ y_i = g_i^T(x_i)\dfrac{\partial H_i}{\partial x_i}(x_i), \quad i = 1,2. \end{cases}$$

Suppose we want to transfer the energy from Σ_1 to Σ_2, while keeping the total energy $H_1 + H_2$ constant. This can be done by using the output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1y_2^T \\ y_2y_1^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

New control paradigms: Energy transfer control

It follows that the closed-loop system is energy-preserving. However, for the individual energies

$$\frac{d}{dt}H_1 = -y_1^Ty_1y_2^Ty_2 = -||y_1||^2||y_2||^2 \leq 0,$$

implying that H_1 is decreasing as long as $||y_1||$ and $||y_2||$ are different from 0. On the other hand,

$$\frac{d}{dt}H_2 = y_2^Ty_2y_1^Ty_1 = ||y_2||^2||y_1||^2 \geq 0.$$

implying that H_2 is increasing at the same rate.