Exercises "Hamiltonian Control Systems", Tuesday

1. Consider a (constant) Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with $\dim \mathcal{F} = n$.

   (a) Show that the Dirac structure can be represented in kernel representation
   $$\mathcal{D} = \{(f, e) \mid Ff + Ee = 0\}$$
   for certain $n \times n$ matrices F, E satisfying
   $$\begin{aligned} FE^T + EF^T &= 0 \\ \operatorname{rank} \begin{bmatrix} F & E \end{bmatrix} &= n \end{aligned}$$
   Hint: Any n-dimensional subspace of $\mathcal{F} \times \mathcal{F}^*$ can be represented as $\operatorname{im} \begin{bmatrix} E^T \\ F^T \end{bmatrix}$ for certain square matrices F, E satisfying $\operatorname{rank} \begin{bmatrix} F & E \end{bmatrix} = n$.

   (b) Show that the Dirac structure can be represented in constrained input-output representation as
   $$\mathcal{D} = \left\{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je + G\lambda,\ G^T e = 0\right\},$$
   for a skew-symmetric mapping $J : \mathcal{F}^* \to \mathcal{F}$ and a linear mapping G such that $\operatorname{im} G = \left\{f \mid (f, 0) \in \mathcal{D}\right\}$. Furthermore, $\ker J = \left\{e \mid (0, e) \in \mathcal{D}\right\}$.
   bf Hint: The proof that this defines a Dirac structure is straightforward. Conversely, let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ be a Dirac structure. Define the subspace
   $$\mathcal{F}^*_{\mathcal{D}} = \{e \in \mathcal{F}^* \mid \exists f \text{ s.t. } (f, e) \in \mathcal{D}\}$$
   Define now the matrix G such that $\mathcal{F}^*_{\mathcal{D}} = \ker G^T$.

   (c) A special type of kernel representation of a Dirac structure occurs if not only $EF^* + FE^* = 0$ but in fact $FE^* = 0$ (while still $\operatorname{rank}(F + E) = \dim \mathcal{F}$). Show that this implies that $\operatorname{im} E^* = \ker F$, and that the Dirac structure is the product of the subspace $\ker F \subset \mathcal{F}$ and the subspace $\ker F^\perp = \ker E \subset \mathcal{F}^*$.

2. Consider the motion of a vertical wheel (unicycle), which can arbitrarily (with slipping) move over a horizontal plane. The position of the wheel is described by the two Cartesian coordinates q_1, q_2 of the contact point of the wheel with the plane, and the orientation angle q_3 of the wheel, say with the q_1-axis. Setting all constants equal to one it follows that the

dynamical equations of the wheel are given by the standard Hamiltonian system

$$
\begin{aligned}
\dot{q}_1 &= \frac{\partial H}{\partial p_1}(q,p) \\
\dot{q}_2 &= \frac{\partial H}{\partial p_2}(q,p) \\
\dot{q}_3 &= \frac{\partial H}{\partial p_3}(q,p) \\
\dot{p}_1 &= -\frac{\partial H}{\partial q_1}(q,p) \\
\dot{p}_2 &= -\frac{\partial H}{\partial q_2}(q,p) \\
\dot{p}_3 &= -\frac{\partial H}{\partial q_3}(q,p)
\end{aligned}
$$

with Hamiltonian $\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2$ (kinetic energy).

Now consider the *non-slipping* kinematic constraint

$$\dot{q}_1 \sin q_3 = \dot{q}_2 \cos q_3$$

expressing that the wheel can not move sideways. The constrained Hamiltonian system equals

$$
\begin{aligned}
\dot{q}_1 &= \frac{\partial H}{\partial p_1}(q,p) \\
\dot{q}_2 &= \frac{\partial H}{\partial p_2}(q,p) \\
\dot{q}_3 &= \frac{\partial H}{\partial p_3}(q,p) \\
\dot{p}_1 &= -\frac{\partial H}{\partial q_1}(q,p) + \lambda \sin q_3 \\
\dot{p}_2 &= -\frac{\partial H}{\partial q_2}(q,p) - \lambda \cos q_3 \\
\dot{p}_3 &= -\frac{\partial H}{\partial q_3}(q,p) \\
0 &= \frac{\partial H}{\partial p_1}(q,p) \sin q_3 - \frac{\partial H}{\partial p_2}(q,p) \cos q_3
\end{aligned}
$$

(a) In the *kinematic model* of the unicycle one parametrizes the allowed velocities as

$$
\begin{aligned}
\dot{q}_1 &= v_1 \cos q_3 \\
\dot{q}_2 &= v_1 \sin q_3 \\
\dot{q}_3 &= v_2
\end{aligned}
$$

where v_1, v_2 may take arbitrary values. One may regard v_1, v_2 as *inputs*. Motivated by this one may define the vector of pseudo-momenta as

$$
\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \cos q_3 & \sin q_3 & 0 \\ 0 & 0 & 1 \\ \sin q_3 & -\cos q_3 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}
$$

Show that the constrained system is now equivalently given as

$$
\begin{aligned}
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} &= \begin{bmatrix} \cos q_3 & 0 & \sin q_3 \\ \sin q_3 & 0 & -\cos q_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial z_1}(q,z) \\ \frac{\partial \tilde{H}}{\partial z_2}(q,z) \\ \frac{\partial \tilde{H}}{\partial z_3}(q,z) \end{bmatrix} \\
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} &= -\begin{bmatrix} \cos q_3 & \sin q_3 & 0 \\ 0 & 0 & 1 \\ \sin q_3 & -\cos q_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial q_1}(q,z) \\ \frac{\partial \tilde{H}}{\partial q_2}(q,z) \\ \frac{\partial \tilde{H}}{\partial q_3}(q,z) \end{bmatrix} + (p_2 \cos q_3 + p_2 \sin q_3) p_3 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda \\
0 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial z_1}(q,z) \\ \frac{\partial \tilde{H}}{\partial z_2}(q,z) \\ \frac{\partial \tilde{H}}{\partial z_3}(q,z) \end{bmatrix}
\end{aligned}
$$

where $\tilde{H}(q_1, q_2, q_3, z_1, z_2, z_3)$ is H expressed in the new coordinates. Show that this reduces to the port-Hamiltonian system without constraints

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \cos q_3 & 0 \\ \sin q_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}}{\partial z_1}(q, z_1, z_2) \\ \frac{\partial \bar{H}}{\partial z_2}(q, z_1, z_2) \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} \cos q_3 & \sin q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}}{\partial q_1}(q, z_1, z_2) \\ \frac{\partial \bar{H}}{\partial q_2}(q, z_1, z_2) \\ \frac{\partial \bar{H}}{\partial q_3}(q, z_1, z_2) \end{bmatrix}$$

where $\bar{H}(q, z_1, z_2)$ is obtained by substituting the solution z_3 to the equation $\frac{\partial \tilde{H}}{\partial z_3}(q, z) = 0$ in $\tilde{H}(q, z)$.

(b) How does this generalize to a unicycle on an inclined plane ?

(c) Extend the previous part to the *bicycle* model with kinematic model

$$\begin{aligned} \dot{q}_1 &= v_1 \cos(q_3 + q_4) \\ \dot{q}_2 &= v_1 \sin(q_3 + q_4) \\ \dot{q}_3 &= v_1 \sin q_4 \\ \dot{q}_4 &= v_2 \end{aligned}$$

(d) Show that for arbitrary kinematic constraints $A^T(q)\dot{q} = 0$ with rank $A(q) = k$ this corresponds to constructing an $n \times (n-k)$ matrix $S(q)$ of full rank ($= n - k$) such that

$$A^T(q)S(q) = 0$$

leading to the kinematic model

$$\dot{q} = S(q)v$$

where $v \in \mathbb{R}^{n-k}$ is the kinematic input vector. The constraints of the Hamiltonian system can be eliminated by defining the vector of pseudo-momenta

$$z = \begin{bmatrix} S^T(q) \\ A^T(q) \end{bmatrix} p$$

Derive the general equations of motion (on the constrained state space) in this general case.

(e) How does this generalize to a Hamiltonian which also contains a potential energy term (e.g., corresponding to the motion on an inclined plane) ?

(f) Show that the resulting equations of motion, even with full actuation on all the momenta, do not satisfy Brockett's necessary conditions for stabilizability. Hence a system with kinematic constraints cannot be stabilized by a continuous state feedback. What is the interpretation of this in the case of holonomic kinematic constraints ?