

Exercises "Hamiltonian Control Systems", Monday

1. Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with $\dim \mathcal{F} = n$.
Show that $\mathcal{D} = \mathcal{D}^\perp$ if and only if $e^T f = 0$ for all $(f, e) \in \mathcal{D}$ and $\text{rank } \mathcal{D} = n$.

Hint: 'Only if' is easiest. For the 'if' direction let $(f^a, e^a), (f^b, e^b)$ be any vectors contained in \mathcal{D} . Then by linearity also $(f^a + f^b, e^a + e^b) \in \mathcal{D}$.

2. Consider a (constant) Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with $\dim \mathcal{F} = n$.

- (a) Show that the Dirac structure can be represented in kernel representation

$$\mathcal{D} = \{(f, e) \mid Ff + Ee = 0\}$$

for certain $n \times n$ matrices F, E satisfying

$$\begin{aligned} FE^T + EF^T &= 0 \\ \text{rank} \begin{bmatrix} F & E \end{bmatrix} &= n \end{aligned}$$

Hint: Any n -dimensional subspace of $\mathcal{F} \times \mathcal{F}^*$ can be represented as $\text{im} \begin{bmatrix} E^T \\ F^T \end{bmatrix}$ for certain square matrices F, E satisfying $\text{rank} \begin{bmatrix} F & E \end{bmatrix} = n$.

- (b) Show that the Dirac structure can be represented in constrained input-output representation as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je + G\lambda, G^T e = 0\},$$

for a skew-symmetric mapping $J : \mathcal{F} \rightarrow \mathcal{F}^*$ and a linear mapping G such that $\text{im } G = \{f \mid (f, 0) \in \mathcal{D}\}$. Furthermore, $\ker J = \{e \mid (0, e) \in \mathcal{D}\}$.

bf Hint: The proof that this defines a Dirac structure is straightforward. Conversely, let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ be a Dirac structure. Define the subspace

$$\mathcal{F}_{\mathcal{D}}^* = \{e \in \mathcal{F}^* \mid \exists f \text{ s.t. } (f, e) \in \mathcal{D}\}$$

Define now the matrix G such that $\mathcal{F}_{\mathcal{D}}^* = \ker G^T$.

- (c) A special type of kernel representation of a Dirac structure occurs if not only $EF^* + FE^* = 0$ but in fact $FE^* = 0$ (while still $\text{rank}(F + E) = \dim \mathcal{F}$). Show that this implies that $\text{im } E^* = \ker F$, and that the Dirac structure is the product of the subspace $\ker F \subset \mathcal{F}$ and the subspace $\ker F^\perp = \ker E \subset \mathcal{F}^*$.

3. Consider an electrical circuit with n edges with current-voltage pairs $(I_i, V_i), i = 1, \dots, n$. Kirchhoff's current laws can be expressed as

$$BI = 0$$

with I the n -dimensional vector of currents and B the incidence matrix of the circuit graph. (A matrix with only $0, -1, 1$ elements.) Dually, Kirchhoff's voltage laws can be expressed as

$$V = B^T \varphi$$

with V the n -dimensional vector of voltages and φ the vector of vertex (node) potentials. Equivalently, $I \in \ker B$ and $V \in \text{im } B^T$.

- (a) Show that this defines a Dirac structure in the space \mathbb{R}^{2n} of currents and voltages. Show that this is of similar type as obtained in the last part of the previous exercise. ('Similar' because the matrix B is in general not square.)
- (b) Write a general LC-circuit as a port-Hamiltonian system. Do the same for a general (linear) RLC-circuit.
4. A switching electrical circuit can be defined by associating a number of the circuit edges with switches. The switch i is *open* if $I_i = 0$, while it is *closed* if $V_i = 0$. Write a general *RLC - circuit* with switches as a switching port-Hamiltonian system.
5. Show that a transformer $I_1 = \alpha I_2, V_2 = -\alpha V_1$ defines a Dirac structure in the 4-dimensional space of currents and voltages.
6. Consider the motion of a vertical wheel (unicycle), which can arbitrarily (with slipping) move over a horizontal plane. The position of the wheel is described by the two Cartesian coordinates q_1, q_2 of the contact point of the wheel with the plane, and the orientation angle q_3 of the wheel, say with the q_1 -axis. Setting all constants equal to one it follows that the dynamical equations of the wheel are given by the standard Hamiltonian system

$$\begin{aligned} \dot{q}_1 &= \frac{\partial H}{\partial p_1}(q, p) \\ \dot{q}_2 &= \frac{\partial H}{\partial p_2}(q, p) \\ \dot{q}_3 &= \frac{\partial H}{\partial p_3}(q, p) \\ \dot{p}_1 &= \frac{\partial H}{\partial q_1}(q, p) \\ \dot{p}_2 &= \frac{\partial H}{\partial q_2}(q, p) \\ \dot{p}_3 &= \frac{\partial H}{\partial q_3}(q, p) \end{aligned}$$

with Hamiltonian $\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2$ (only kinetic energy).

Now consider the *non-slipping* kinematic constraint

$$\dot{q}_1 \sin q_3 = \dot{q}_2 \cos q_3$$

expressing that the wheel can not move sideways. The constrained Hamiltonian system equals

$$\begin{aligned}\dot{q}_1 &= \frac{\partial H}{\partial p_1}(q, p) \\ \dot{q}_2 &= \frac{\partial H}{\partial p_2}(q, p) \\ \dot{q}_3 &= \frac{\partial H}{\partial p_3}(q, p) \\ \dot{p}_1 &= \frac{\partial H}{\partial q_1}(q, p) + \lambda \sin q_3 \\ \dot{p}_2 &= \frac{\partial H}{\partial q_2}(q, p) - \lambda \cos q_3 \\ \dot{p}_3 &= \frac{\partial H}{\partial q_3}(q, p) \\ 0 &= \dot{q}_1 \sin q_3 - \dot{q}_2 \cos q_3\end{aligned}$$

- (a) In the *kinematic model* of the unicycle one parametrizes the allowed velocities as

$$\begin{aligned}\dot{q}_1 &= v_1 \cos q_3 \\ \dot{q}_2 &= v_1 \sin q_3 \\ \dot{q}_3 &= v_2\end{aligned}$$

where v_1, v_2 may take arbitrary values. One may regard v_1, v_2 as *inputs*. Motivated by this one may define the reduced vector of pseudomomenta as

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos q_3 & \sin q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Show that the constrained system is now equivalently given as

$$\begin{aligned}\dot{q}_1 &= \frac{\partial \tilde{H}}{\partial z_1}(q, p) \cos q_3 \\ \dot{q}_2 &= \frac{\partial \tilde{H}}{\partial z_1}(q, p) \sin q_3 \\ \dot{q}_3 &= \frac{\partial \tilde{H}}{\partial z_2}(q, p) \\ \dot{z}_1 &= \frac{\partial \tilde{H}}{\partial q_1}(q, p) \\ \dot{z}_2 &= \frac{\partial \tilde{H}}{\partial q_2}(q, p)\end{aligned}$$

where

$$\tilde{H}(q_1, q, q_3, z_1, z_2) =$$

- (b) How does this generalize to a unicycle on an inclined plane ?
(c) Extend the previous part to the bicycle model with kinematic model

$$\begin{aligned}\dot{q}_1 &= v_1 \cos(q_3 + q_4) \\ \dot{q}_2 &= v_1 \sin(q_3 + q_4) \\ \dot{q}_3 &= v_1 \sin q_4 \\ \dot{q}_4 &= v_2\end{aligned}$$

- (d) Show that for arbitrary kinematic constraints $A^T(q)\dot{q} = 0$ with $\text{rank} A(q) = k$ this corresponds to constructing an $n \times (n - k)$ matrix $S(q)$ of full rank ($= n - k$) such that

$$A^T(q)S(q) = 0$$

leading to the kinematic model

$$\dot{q} = S(q)v$$

where $v \in \mathbb{R}^{n-k}$ is the kinematic input vector. The constraints of the Hamiltonian system can be eliminated by defining the reduced vector of pseudo-momenta

$$z = S^T(q)p$$

Derive the general equations of motion in this general case.

- (e) How does this generalize to a Hamiltonian which also contains a potential energy term (e.g., corresponding to the motion on an inclined plane) ?
- (f) Show that the resulting equations of motion, even with full actuation on all the momenta, do not satisfy Brockett's necessary conditions for stabilizability. Hence a system with kinematic constraints cannot be stabilized by a continuous state feedback. What is the interpretation of this in the case of holonomic kinematic constraints ?