Exercises "Hamiltonian Control Systems"

Friday

1. Consider the shallow water equations

$$\partial_t \begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \partial_z \begin{bmatrix} h \\ v \end{bmatrix} = 0$$

   with Hamiltonian $H(h,v) = \frac{1}{2}\int_a^b [hv^2 + gh^2]dz$ and boundary variables $hv_{|a,b}$ and $(\frac{1}{2}v^2 + gh)_{|a,b}$. Apply the boundary control (no mass flow at a; mass flow at b negatively proportional to the Bernoulli function at b)

$$h(b)v(b) = 0, \quad h(a)v(a) = -\frac{1}{2}v^2(a) - gh(a)$$

   Argue that this stabilizes the system around the zero-state $h(z) = 0, v(z) = 0, z \in [a,b]$.

2. Consider the series interconnection of a linear resistor (with resistance R), linear inductor (with inductance L), a linear capacitor (with capacitance C), and a voltage source with voltage V. The total stored energy of the circuit is thus $H(Q,\phi) = \frac{1}{2C}Q^2 + \frac{1}{2L}\phi^2$, where Q is the charge of the capacitor and ϕ the flux of the inductor.

   - As before, tthe port-Hamiltonian description of the circuit is given as

$$\Sigma: \quad \begin{aligned} \begin{bmatrix} \dot{Q} \\ \dot{\phi} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V \\ I &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{bmatrix} \end{aligned}$$

     where I is the current through the voltage source.

   - Consider now a constant voltage $\bar{V}$, and the corresponding equilibrium $(\bar{Q},\bar{\phi}) = (C\bar{V}, 0)$. This can be modeled by interconnecting Σ to the 'source system', written in port-Hamiltonian form as

$$S: \quad \begin{aligned} \dot{\xi} &= I_S \\ V_S &= \frac{\partial H_S}{\partial \xi} \end{aligned}, \quad H_S(\xi) = \bar{V}\xi$$

     via the interconnection $I_S = -I, V = V_S$.

- Write the interconnected system as a 3-dimensional port-Hamiltonian system, and show that the function $Q + \xi$ is a Casimir for this interconnected system.
- Determine the constant α such that
$$\frac{1}{2C}Q^2 + \frac{1}{2L}\phi^2 + \bar{V}\xi + \alpha(Q + \xi)$$
has a minimum at the equilibrium $(\bar{Q}, \bar{\phi}) = (C\bar{V}, 0)$. Show that this implies that $Q(t) \to C\bar{V}, \phi(t) \to 0$ for t tending to ∞. What happens with $\xi(t)$?
- Consider now the situation that the circuit is interconnected to the constant voltage source $\bar{V}$ via a (lossless) *transmission line*, given as
$$\begin{aligned} \tfrac{\partial Q}{\partial t}(z,t) &= -\tfrac{\partial}{\partial z}I(z,t) &&= -\tfrac{\partial}{\partial z}\tfrac{\phi(z,t)}{L(z)} \\ \tfrac{\partial \phi}{\partial t}(z,t) &= -\tfrac{\partial}{\partial z}V(z,t) &&= -\tfrac{\partial}{\partial z}\tfrac{Q(z,t)}{C(z)} \\ f_a(t) &= V(a,t), \quad e_a(t) &&= I(a,t) \\ f_b(t) &= V(b,t), \quad e_b(t) &&= I(b,t) \end{aligned}$$
Show that in this case
$$Q + \xi + \int_a^b \tilde{Q}(z)dz$$
is a Casimir for the interconnected system (with $\tilde{Q}(z)$ the charge distribution over the transmission line with spatial domain $[a, b]$). Determine as in (c) a Lyapunov function for the interconnected system showing that for $t \to \infty, Q(t) \to C\bar{V}, \phi(t) \to 0$. What happens with $\tilde{Q}$ for $t \to \infty$?

Consider a port-Hamiltonian system
$$\Sigma : \begin{aligned} \dot{x} &= [J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$
for square matrices $J^T(x) = -J(x), R(x) = R^T(x) \geq 0$. Let $C : X \to \mathbb{R}$ be a function satisfying
$$\frac{\partial^T C}{\partial x}(x)[J(x) - R(x)] = 0$$

(a) Show that
$$\frac{d(H + C)}{dt} \leq u^T\tilde{y}$$
where $\tilde{y} := g^T(x)\left(\frac{\partial H}{\partial x}(x) + \frac{\partial C}{\partial x}(x)\right)$.
Let $H + C$ have a minimum at x^*. Derive conditions under which the system subject to the feedback $u = -\tilde{y}$ is asymptotically stable around x^*.

(b) Show that if $R(x) = R^T(x) \geq 0$ the condition $\frac{\partial^T C}{\partial x}(x)[J(x) - R(x)] = 0$ is equivalent to the two conditions
$$\frac{\partial^T C}{\partial x}(x)J(x) = 0, \quad \frac{\partial^T C}{\partial x}(x)R(x) = 0$$