ELGERSBURG SCHOOL

Pseudospectra and Nonnormal Dynamical Systems

Exercises Sheet 1

1. Getting to know EigTool.

   (a) Download EigTool (`http://www.cs.ox.ac.uk/pseudospectra/eigtool`)

   (b) Run `eigtool` from a MATLAB prompt.

   (c) Experiment with menu option: `Demos/Dense/...`

   (d) For dense matrices, experiment with menu option: `Transients`

   (e) For dense matrices, experiment with menu option: `Numbers`

   (f) For dense matrices, experiment with `Pmode+epsilon` button

   (g) Experiment with menu option: `Demos/Sparse/...`

   (h) Observe convergence of Restarted Arnoldi method: adjust the ARPACK subspace dimension, and observe how this affects convergence behavior for Arnoldi.

2. Population dynamics: design a transiently growing population.

   The *Leslie matrix* is used for modeling the (female) population of a given species with fixed birth rates and survivability levels (in the absence of immigration). The population is divided into n brackets of y-years each, and an average member of bracket k gives birth to $b_k \geq 0$ females in the next y years, and has probability $s_k \in [0,1]$ of surviving the next y years. Letting $p_k^{(j)}$ denote the population in the kth bracket in the jth generation, we see that the population evolves according to the matrix equation (e.g., for $n = 5$)

$$\begin{bmatrix} p_1^{(j+1)} \\ p_2^{(j+1)} \\ p_3^{(j+1)} \\ p_4^{(j+1)} \\ p_5^{(j+1)} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ s_1 & & & & \\ & s_2 & & & \\ & & s_3 & & \\ & & & s_4 & \end{bmatrix} \begin{bmatrix} p_1^{(j)} \\ p_2^{(j)} \\ p_3^{(j)} \\ p_4^{(j)} \\ p_5^{(j)} \end{bmatrix},$$

   with unspecified entries equal to zero. (We presume the mortality of the last age bracket.)

   Denote the matrix as $\mathbf{A}$, so that $\mathbf{p}^{(j+1)} = \mathbf{A}\mathbf{p}^{(j)}$, and hence $\mathbf{p}^{(j)} = \mathbf{A}^j\mathbf{p}^{(0)}$.

   We shall see that transient growth in matrix powers is linked to the sensitivity of eigenvalues to perturbations in the matrix entries. This problem is designed to reinforce this connection in the context of ‘physically’ meaningful transient behavior.

   (a) Design a set of parameters $b_1, \ldots, b_5 > 0$ and $s_1, \ldots, s_4 > 0$ (for $n = 5$) so that the population will eventually decay in size to zero ($\mathbf{A}^j \to \mathbf{0}$ as $j \to \infty$), but this will be preceded by a period of significant transient growth in the population, where $\mathbf{p}^{(j)} \gg \mathbf{p}^{(0)}$. (Think about what kind of birth and survivability values might suggest this demographic pattern.)

(b) Plot your population for a number of generations to demonstrate the transient growth and eventual decay. (You may modify the `pop.m` code.)

(c) Compute pseudospectra of $\mathbf{A}$ to show that this transient growth coincides with sensitivity of the eigenvalues of your matrix.

3. Pseudospectra Approximation: Orr–Sommerfeld operator

The Orr–Sommerfeld operator arises in the stability analysis of fluid flows. In this problem, you will experiment with a "pseudospectral collocation" discretization of this operator, to gain an appreciation for the approximation of spectra and pseudospectra, consider the following experiments.

This exercise uses the MATLAB routines `orr.m` (and the subordinate code `cheb.m`) and `arnoldi_ro.m`.

(a) Compute `A = orrdemo(n)` for values of $n = 16, 32, 64, 128, \ldots$.
To understand the stability of the associated fluid flow, we seek accurate approximations of the rightmost eigenvalues. Plot these eigenvalues in the complex plane.

How do the eigenvalues of `A` evolve as `n` is increased?

Which eigenvalues of `A` are most accurate?

(b) Use EigTool to compute $\sigma_\varepsilon(\mathbf{A})$ for `n = 128`, paying particular attention to the area $\mathrm{Re}(z) \in [-1, 0.2]$ and $\mathrm{Im}(z) \in [-10]$.

(c) Fix `n = 128`. Use `[V,H] = arnoldi_ro(A,k)` to compute an orthonormal basis $\mathbf{V}_k$ for the Krylov subspace for `k` $= 10, 20, 30, \ldots$. For each k, use EigTool to plot $\sigma_\varepsilon(\mathbf{V}_k^*\mathbf{A}\mathbf{V}_k)$ in the same region as in part (b). How large must k be before this approach gives a decent approximation to $\sigma_\varepsilon(\mathbf{A})$ in the specified region of the complex plane?

(d) Again with `n=128`, use `[V,D] = eig(A)` to compute the eigenvalues and eigenvectors of the Orr–Sommerfeld operator. Let $\mathbf{V}_k$ denote an orthonormal basis for the eigenvectors of `A` corresponding to the *rightmost* eigenvalues. (You can use `orth` to generate this orthonormal basis from the set of (non-orthogonal) eigenvectors.) For `k` $= 10, 20, 30, \ldots$, compute $\sigma_\varepsilon(\mathbf{V}_k^*\mathbf{A}\mathbf{V}_k)$ as in part (c). When using this invariant subspace, how large must k be before this approach gives a good approximation in the specified region of the complex plane?

4. Solve the following theoretical problems.

(a) Consider the block matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}.$$

Show that

$$\sigma_\varepsilon(\mathbf{S}) \subseteq \sigma_\delta(\mathbf{A}) \cup \sigma_\delta(\mathbf{C})$$

for some value of δ (which you should specify).

(b) Prove that $\mathbf{A}$ is normal (i.e., $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$ if and only if there exists a unitary matrix $\mathbf{U}$ such that $\mathbf{U}^*\mathbf{A}\mathbf{U}$ is diagonal.

(c) Recall that the numerical range is given by

$$W(\mathbf{A}) = \{\mathbf{x}^*\mathbf{A}\mathbf{x} : \|\mathbf{x}\| = 1\}.$$

Letting $\Delta_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$, prove that

$$\sigma_\varepsilon(\mathbf{A}) \subset W(\mathbf{A}) + \Delta_\varepsilon.$$

(d) Let $\mathbf{A}$ be diagonalizable, $\mathbf{A} = \mathbf{V}^*\mathbf{A}\mathbf{V}$. Prove this version of the Bauer–Fike Theorem:

$$\sigma_\varepsilon(\mathbf{A}) \subseteq \sigma(\mathbf{A}) + \Delta_{\varepsilon\kappa(\mathbf{V})},$$

where $\kappa(\mathbf{V}) = \|\mathbf{V}\|\|\mathbf{V}^{-1}\|$.