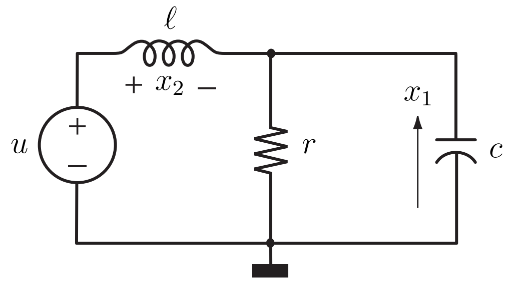
Exercises "Hamiltonian Control Systems"

Thursday

1. Consider the electrical RLC circuit shown in the figure below.



   a) Show that the port-Hamiltonian equations for this circuit are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/r & 1 \\ -1 & 0 \end{bmatrix}}_{J-R} \begin{bmatrix} \dfrac{\partial H}{\partial x_1} \\ \dfrac{\partial H}{\partial x_2} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial x_1} \\ \dfrac{\partial H}{\partial x_2}, \end{bmatrix}, \quad H(x_1, x_2) = \frac{x_1^2}{2c} + \frac{x_2^2}{2\ell},$$

   where x_1 and x_2 represent the charge associated with the capacitor and the flux linkage associated with the inductor, respectively.

   b) Show that a feedback of the form $u = \alpha(x)$, such that

$$g\alpha(x) = [J - R]\frac{\partial H_a}{\partial x}(x),$$

   yields the closed-loop dynamics

$$\dot{x} = [J - R]\frac{\partial H_d}{\partial x}(x),$$

   with $H_d(x) = H(x) + H_a(x)$, where $H_a(x)$ denotes the controller energy.

c) Apply Proposition 6.9 of the lecture notes (part 2), see also slide 3 and 4 of lecture 4, to design a controller $u = \alpha(x)$ such that the closed-loop energy $H_d(x)$ has a minimum at the desired equilibrium point $x^* = (cu^*, \ell u^*/r)^T$.

d) Show that x^* is asymptotically stable.

e) Is the equilibrium point $x^* = (cu^*, \ell u^*/r)^T$ also stabilizable using control by interconnection using a passive controller? Motivate your answer!

2. Consider an n-dimensional stable linear system

$$\dot{x} = Ax + Bu, \qquad (*)$$

with state x, input u, and constant matrices A and B of appropriate dimensions.

a) Prove that $(*)$ can be written as a linear port-Hamiltonian system of the form

$$\dot{x} = [J - R]\frac{\partial H}{\partial x}(x) + Bu,$$

where $J = -J^T$, $R = R^T \succeq 0$, and $H(x) = \frac{1}{2}x^T Q x$, with $Q = Q^T \succ 0$.

b) What does $R \succ 0$ imply for the system $(*)$? Motivate your answer!

3. Consider a fully actuated mass m with Cartesian coordinates q_1, q_2 moving over a horizontal plane:

$$m\ddot{q}_1 = \bar{u}_1,$$
$$m\ddot{q}_2 = \bar{u}_2,$$

and apply the feedback $\bar{u}_2 = -k(q_2 - q_2^*) + u_2, \bar{u}_1 = u_1$.

a) Show that the closed-loop system is given by the port-Hamiltonian equations

$$\begin{aligned} \dot{q}_1 &= \frac{\partial H}{\partial p_1}, & \dot{q}_2 &= \frac{\partial H}{\partial p_2}, \\ \dot{p}_1 &= -\frac{\partial H}{\partial q_1} + u_1, & \dot{p}_2 &= -\frac{\partial H}{\partial q_2} + u_2, \\ y_1 &= \frac{\partial H}{\partial p_1}, & y_2 &= \frac{\partial H}{\partial p_2}, \end{aligned}$$

with $H(q_1, q_2, p_1, p_2) = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}k(q_2 - q_2^*)^2$.

b) Apply now the feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2 \\ y_1 y_2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Show that the closed-loop system satisfies $\frac{d}{dt}H = 0$, while

$$p_2(t) \to 0, \quad q_2(t) \to q_2^*,$$

for $t \to \infty$. What does this imply for the trajectory of the mass?