# ELGERSBURG SCHOOL

## Pseudospectra and Nonnormal Dynamical Systems

## Exercise Sheet 2

*Each of these problems can solved independently, so please explore those you find most interesting.*

The numerical range is the set

$$W(\mathbf{A}) = \{\mathbf{x}^*\mathbf{A}\mathbf{x} : \|\mathbf{x}\| = 1\}.$$

The spectral abscissa and spectral radius are

$$\alpha(\mathbf{A}) = \sup_{\lambda\in\sigma(\mathbf{A})} \operatorname{Re}\lambda, \qquad \rho(\mathbf{A}) = \sup_{\lambda\in\sigma(\mathbf{A})} |\lambda|.$$

The $\varepsilon$-pseudospectral abscissa and $\varepsilon$-pseudospectral radius are

$$\alpha_\varepsilon(\mathbf{A}) = \sup_{z\in\sigma_\varepsilon(\mathbf{A})} \operatorname{Re} z, \qquad \rho_\varepsilon(\mathbf{A}) = \sup_{z\in\sigma_\varepsilon(\mathbf{A})} |z|.$$

1. Solve the following theoretical problems.

   (a) For any $\mathbf{A} \in \mathbb{C}^{n\times n}$, prove that $\frac{1}{2}\|\mathbf{A}\| \le \mu(\mathbf{A}) \le \|\mathbf{A}\|$.
   *Hint*: Consider the Cartesian decomposition $\mathbf{A} = \mathbf{H} + i\mathbf{S}$, where $\mathbf{H} = (\mathbf{A} + \mathbf{A}^*)/2$ and $\mathbf{S} = (\mathbf{A} - \mathbf{A}^*)/(2i)$.

   (b) Professor Illmann asked about lower bounds on $\|e^{t\mathbf{A}}\|$ when $t$ is restricted to a *finite interval*. Here you will prove such a bound (see *Spectra and Pseudospectra*, Section 15). Suppose $\mathbf{A}$ is a stable matrix ($\alpha(\mathbf{A}) < 0$). Let $T > 0$ and define

   $$M_T := \sup_{t\in[0,T]} \|e^{t\mathbf{A}}\|.$$

   Abbreviate $\alpha_\varepsilon := \alpha_\varepsilon(\mathbf{A})$. Prove that, for all $\varepsilon > 0$ for which $M_T < e^{T\alpha_\varepsilon}$ and $\varepsilon \le \alpha_\varepsilon$,

   $$M_T \ge \frac{\alpha_\varepsilon}{\varepsilon + e^{-T\alpha_\varepsilon}(\alpha_\varepsilon - \varepsilon)}.$$

   (This formula reduces to the bound in the lectures when $T \to \infty$.)
   *Hint:* See the proof of the analogous result over $t \in [0, \infty]$ from today's lecture. Split the integral into the sum of integrals, each of length $T$.

   (c) For diagonalizable

   $$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \widehat{\mathbf{v}}_j^*,$$

   prove these versions of the Bauer–Fike theorem:

   $$\sigma_\varepsilon(\mathbf{A}) \subseteq \bigcup_{j=1}^{n} \lambda_j + \Delta_{\varepsilon\|\mathbf{V}\|\|\mathbf{V}^{-1}\|}$$

and

$$\sigma_\varepsilon(\mathbf{A}) \subseteq \bigcup_{j=1}^{n} \lambda_j + \Delta_{\varepsilon n \kappa(\lambda_j)},$$

where $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$ is the open disk of radius $r > 0$ and

$$\kappa(\lambda_j) = \frac{\|\widehat{\mathbf{v}}\|\|\mathbf{v}\|}{|\widehat{\mathbf{v}}^*\mathbf{v}|}.$$

(d) Recall Gerschgorin's theorem: The eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$ are contained in

$$\bigcup_{j=1}^{n} a_{j,j} + \Delta_{r_j}$$

where where $a_{j,k}$ denotes the $(j,k)$ entry of $\mathbf{A}$,

$$r_j := \sum_{k=1,k\neq j}^{n} |a_{j,k}|,$$

and $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$ is the open disk of radius $r > 0$. Show that

$$\sigma_\varepsilon(\mathbf{A}) \subseteq \bigcup_{j=1}^{n} a_{j,j} + \Delta_{r_j+\varepsilon\sqrt{n}}.$$

2. The MATLAB code `jordcomp.m` creates the matrix

$$\mathbf{A} = \begin{bmatrix} 601 & 300 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3000 & 1201 & 0 & 0 & 0 & 0 & 0 & 300 \\ 475098 & 110888 & 301 & 100 & -15266 & -202 & 4418 & 27336 \\ 4594766 & 1185626 & -900 & -299 & -130972 & -3404 & 34846 & 272952 \\ -22800 & -6000 & 0 & 0 & 601 & 0 & 0 & -1800 \\ -3776916 & -968379 & 0 & 0 & 108597 & 2402 & -28800 & -222896 \\ -292663 & -71665 & 0 & 0 & 8996 & 200 & -2398 & -16892 \\ -37200 & -14400 & 0 & 0 & 300 & 0 & 0 & -2399 \end{bmatrix}.$$

Using MATLAB's `eig` command, do your best to approximate the true eigenvalues of $\mathbf{A}$ and the dimensions of the associated Jordan blocks. (You might consider how quickly the eigenvalues split under a parameterized perturbation $\widehat{\mathbf{A}}(t) := \mathbf{A} + t\mathbf{E}$ for some fixed $\mathbf{E}$; cf. Braconnier, Chatelin and Dunyach, 1995.) Justify your answer as best as you can. The matrices $\mathbf{A}$ and $\mathbf{A}^*$ have identical eigenvalues. Does MATLAB agree? (Do *not* use the `jordan` command in MATLAB's Symbolic Toolbox for this problem – it can compute the Jordan form exactly due to the structure of this matrix.)

3. Testing Crouzeix's Conjecture

Today we stated *Crouzeix's Theorem*: For $f$ analytic on the numerical range $W(\mathbf{A})$,

$$\|f(\mathbf{A})\| \leq 11.08 \max_{z\in W(\mathbf{A})} |f(z)|.$$

Crouzeix has conjectured that the constant 11.08 can be lowered. Conduct random experiments (vary $\mathbf{A}$ and $f$) to conjecture the smallest $C \geq 1$ such that

$$\|f(\mathbf{A})\| \leq C \max_{z \in W(\mathbf{A})} |f(z)|.$$

You may use Nick Higham's routine `fv.m` to compute $W(\mathbf{A})$ to facilitate your experiments. (`W = fv(A)` will return a list of points on the boundary of $W(\mathbf{A})$.)

4. Consider Demmel's matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -100 & -10000 \\ 0 & -1 & -100 \\ 0 & 0 & -1 \end{bmatrix}.$$

Estimate the distance to instability

$$\begin{aligned} d(\mathbf{A}) &:= \min\{\|\mathbf{E}\| : iy \in \sigma(\mathbf{A}+\mathbf{E}) \text{ for some } y \in \mathbb{R}\} \\ &= \sup\{\varepsilon > 0 : \sigma_\varepsilon(\mathbf{A}) \text{ is contained in the left half plane}\} \end{aligned}$$

in three different ways.

(a) Make various random perturbations $\mathbf{E}$ to $\mathbf{A}$ to estimate $d(\mathbf{A})$.

(b) Compute the largest value of $\|(z - \mathbf{A})^{-1}\|$ for $z$ on the imaginary axis, and explain why this leads to $d(\mathbf{A})$.

(c) Use the Byers algorithm discussed today's lecture to compute $d(\mathbf{A})$.
Recall: The matrix $(x + iy) - \mathbf{A}$ has a singular value $\varepsilon$ if and only if $iy$ is an eigenvalue of the Hamiltonian matrix

$$\begin{bmatrix} x - \mathbf{A}^* & \varepsilon\mathbf{I} \\ -\varepsilon\mathbf{I} & \mathbf{A} - x \end{bmatrix}.$$

Explain the relative merits of each approach. Which gives you the greatest confidence?

5. Use EigTool to compute the pseudospectra for the *stable* Boeing 767 matrix (available from the "Demos/Dense matrices" menu). Estimate the maximum growth of the matrix exponential via the lower bound

$$\sup_{t \leq 0} \|e^{t\mathbf{A}}\| \geq \frac{\alpha_\varepsilon(\mathbf{A})}{\varepsilon},$$

then plot $\|e^{t\mathbf{A}}\|$ using the "Transients" menu.

You may compute values of $\alpha_\varepsilon(\mathbf{A})$ for various $\varepsilon > 0$ by looking at the plotted pseudospectra, or by using the "Numbers/Pseudospectra Abscissa" menu option (which runs the criss-cross algorithm).

6. Consider the Jordan block

$$\mathbf{A} = \begin{bmatrix} 0.1 & 1.5 & & \\ & 0.1 & \ddots & \\ & & \ddots & 1.5 \\ & & & 0.1 \end{bmatrix} \in \mathbb{C}^{50\times 50}.$$

Use `jordresrad.m` to precisely compute the $\varepsilon$-pseudospectral radius

$$\rho_\varepsilon(\mathbf{A}) = \sup_{z\in\sigma_\varepsilon(\mathbf{A})} |z|$$

for a variety of $\varepsilon$ values (experiment: say, $\varepsilon = 10^{-20}, \ldots, 10^0$).

Produce a `semilogy` plot showing $\|\mathbf{A}^k\|$ versus $k$ for $k = 0, 1, \ldots, 100$.

Superimpose on your plot the upper bound proved in today's lecture

$$\|\mathbf{A}^k\| \le \frac{\rho_\varepsilon(\mathbf{A})^{k+1}}{\varepsilon}$$

for the $\varepsilon$ values for which you computed $\rho_\varepsilon(\mathbf{A})$.