ELGERSBURG SCHOOL

Pseudospectra and Nonnormal Dynamical Systems

Exercise Sheet 3

Each of these problems can solved independently, so please explore those you find most interesting.

1. Explorations of Toeplitz matrices.

   (a) Recall from Lecture 3 that the *symbol* of a Toeplitz matrix is

   $$a(z) = \sum_{j=-n}^{n} a_j z^j.$$

   Conduct MATLAB experiments: plot $a(\mathbb{T})$ for various choices of the Toeplitz coefficients until you find a few choices you like. Here are a few departure points: the "Grcar matrix" has symbol

   $$a(z) = -z^{-1} + 1 + z + z^2 + z^3;$$

   one of the examples from Lecture 3 was

   $$a(z) = iz^{-4} + z^{-2} + 2z^{-1} + 5z^2 + iz^5.$$

   To generate the unit circle in MATLAB, use `T = exp(linspace(0,2i*pi,500));`

   (b) For your favorite symbol, construct corresponding Toeplitz matrices of dimension $N = 50, 100, 200, 400$. In each case, plot the eigenvalues of these matrices, as computed by the `eig` command. Do the computed values tend to an asymptotic limiting set, as theoretically expected?

   You can use MATLAB's built-in `toeplitz` command to build these matrices. See `help toeplitz` for details.

   (c) Use EigTool to compute the pseudospectra for your matrices in (b). Do these plots reflect the expected exponential growth of the norm of the resolvent?

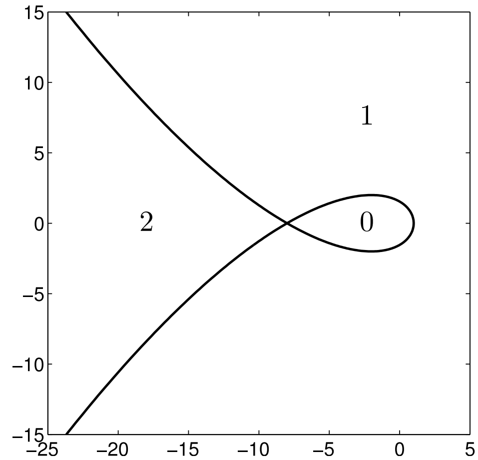
2. Consider the third-order ($d = 3$) constant coefficient differential operator $\mathbf{A}_\ell$ on $L^2(0, \ell)$ with

   $$\mathbf{A}_\ell u = u + 3u' + 3u'' + u'''$$

   with β homogeneous boundary conditions at $x = 0$ and $\gamma = d - \beta$ homogeneous boundary conditions at $x = \ell$. The symbol for this operator is

   $$a(k) = 1 + 3(-ik) + 3(-ik)^2 + (-ik)^3$$

   with symbol curve and winding numbers shown below.



(a) Based on the theory presented in today's lecture, in what regions of the complex plane will $\|(z - \mathbf{A}_\ell)^{-1}\|$ grow exponentially with ℓ when (i) $\beta = 1$ and $\gamma = 2$; (ii) $\beta = 2$ and $\gamma = 1$.

(b) To get a *crude* estimate of the pseudospectra – to confirm your answer to part (a) – use the code `thirdord.m` to generate low-accuracy Toeplitz matrix approximations for these two sets of boundary conditions, and use EigTool to compute the pseudospectra for the same complex values shown in the plot above. Start with `N = 100` and `ell = 10`, and increase `ell`.

(These discretizations use finite difference approximations to arrive at Toeplitz matrices that approximate $\mathbf{A}_\ell$. Note that the code simply uses the Euclidean norm: we will talk about using the proper $L^2(0, \ell)$ norm on Friday.)

3. Computing pseudospectra of a damped, vibrating string.

This exercise asks you to perform some pseudospectral computations for several damped vibrating strings. Codes to generate the excellent discretizations of this problem are provided.

We model a string via the differential equation

$$u_{tt}(x, t) = u_{xx}(x, t) - 2a(x)u_t(x, t)$$

on $x \in [0, 1]$, $t \geq 0$ with homogeneous boundary conditions $u(0, t) = u(1, t) = 0$. Write this equation in first-order form as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ \mathrm{d}^2/\mathrm{d}x^2 & -2a \end{bmatrix} \begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix},$$

where the operator

$$A = \begin{bmatrix} 0 & I \\ \mathrm{d}^2/\mathrm{d}x^2 & -2a \end{bmatrix}$$

has domain $\mathrm{Dom}(A) = (H_0^1(0,1) \cap H^2(0,1)) \times H_0^1(0,1)$. The energy norm on this space is defined by

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_E = \int_0^1 \left(\overline{f'(x)}u'(x) + \overline{g(x)}v(x)\right) \mathrm{d}x.$$

We wish to understand the nonnormality of this system, in the physically relevant norm, given several choice of the damping parameter a.

(a) [for those who prefer paperwork]
Let a be a constant. Determine, by hand, the eigenvalues λ_k and eigenfunctions V_k of this operator. At what values of a is there a double eigenvalue? How many linearly independent eigenfunctions are associated with such a double eigenvalue? The adjoint of this operator is

$$A^* = \begin{bmatrix} 0 & -I \\ -\mathrm{d}^2/\mathrm{d}x^2 & -2a \end{bmatrix}.$$

Compute the eigenfunctions $\widehat{V}_k$ of A^* (which are "left eigenfunctions" of A). Estimate the condition number of each eigenvalue,

$$\kappa(\lambda_k) = \frac{\|V_k\|\|\widehat{V}_k\|}{\langle V_k, \widehat{V}_k\rangle}.$$

(b) Use the code `make_Aconst` to generate the discretization matrix for the operator with constant $a \geq 0$, then apply EigTool to compute the pseudospectra of this matrix over $\mathrm{Re}\, z, \mathrm{Im}\, z \in [-100, 100]$. (EigTool will use the Euclidean norm on $\mathbb{C}^{\mathtt{N}}$ here, by default.) Pick the discretization size `N` sufficiently large that you are confident that the pseudospectra have convergence in this region. This is a spectral discretization, so the matrices do not need to be particularly large.

(c) How do the pseudospectra in (a) behave as $|\mathrm{Im}\, z|$ gets large? Does this suggest that the system can experience transient growth?

(d) Now we shall be more careful, using a discretization of the physically relevant energy norm. The code `getG` will return a matrix `R` such that $\mathbf{G} = \mathbf{R}^*\mathbf{R}$ defines an inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle_{\mathbf{G}} := \mathbf{y}^*\mathbf{G}\mathbf{x} = \mathbf{y}^*\mathbf{R}^*\mathbf{R}\mathbf{x},$$

which gives a norm $\|\mathbf{x}\|_{\mathbf{G}} := \langle \mathbf{x}, \mathbf{x}\rangle_{\mathbf{G}}^{1/2}$ and an induced operator norm

$$\|\mathbf{M}\|_{\mathbf{G}} = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{M}\mathbf{x}\|_{\mathbf{G}}}{\|\mathbf{x}\|_{\mathbf{G}}}.$$

Show that $\|\mathbf{M}\|_{\mathbf{G}} = \|\mathbf{R}\mathbf{M}\mathbf{R}^{-1}\|_2$, where $\|\cdot\|_2$ here denotes the standard Euclidean norm,

(e) Following from (c), compute the energy-norm pseudospectra of $\mathbf{A}$ by calling EigTool with $\mathbf{R}\mathbf{A}\mathbf{R}^{-1}$. Use damping values $a = 0, 1/2, 1, 3/2$. (The operator is skew-adjoint – and hence normal – in this inner product when $a = 0$: can you spot this from the pseudospectra? What interesting phenomenon happens at $a = 1$?)

(f) Now consider the Castro–Cox damping function, $a(x) = 1/(x + 1/k)$, discretized via the `makeAcc` routine. Examine the behavior of the energy-norm pseudospectra for $k = 10, 10^2, 10^3, \ldots$. Assess the potential for transient behavior different from what you would predict based on the spectrum alone.

4. Pseudospectra Approximation: Orr–Sommerfeld operator

The Orr–Sommerfeld operator arises in the stability analysis of fluid flows. In this problem, you will experiment with a discretization of this operator. In order to gain an appreciation for the approximation of spectra and pseudospectra, consider the following experiments.

This exercise uses the MATLAB routines `orr.m` (and the subordinate code `cheb.m`) and `arnoldi_ro.m`.

(a) Compute `A = orrdemo(n)` for values of $n = 16, 32, 64, 128, \ldots$.
To understand the stability of the associated fluid flow, we seek accurate approximations of the rightmost eigenvalues. Plot these eigenvalues in the complex plane.

How do the eigenvalues of `A` evolve as `n` is increased?

Which eigenvalues of `A` are most accurate?

(b) Use EigTool to compute $\sigma_\varepsilon(\mathbf{A})$ for `n = 128`, paying particular attention to the area $\text{Re}(z) \in [-1, 0.2]$ and $\text{Im}(z) \in [-1, 0]$.

(c) Fix `n = 128`. Use `[V,H] = arnoldi_ro(A,k)` to compute an orthonormal basis $\mathbf{V}_k$ for the Krylov subspace for $\texttt{k} = 10, 20, 30, \ldots$. For each k, use EigTool to plot $\sigma_\varepsilon(\mathbf{V}_k^*\mathbf{A}\mathbf{V}_k)$ in the same region as in part (b). How large must k be before this approach gives a decent approximation to $\sigma_\varepsilon(\mathbf{A})$ in the specified region of the complex plane?

(d) Again with `n=128`, use `[V,D] = eig(A)` to compute the eigenvalues and eigenvectors of the Orr–Sommerfeld operator. Let $\mathbf{V}_k$ denote an orthonormal basis for the eigenvectors of `A` corresponding to the *rightmost* eigenvalues. (You can use `orth` to generate this orthonormal basis from the set of (non-orthogonal) eigenvectors.) For $\texttt{k} = 10, 20, 30, \ldots$, compute $\sigma_\varepsilon(\mathbf{V}_k^*\mathbf{A}\mathbf{V}_k)$ as in part (c). When using this invariant subspace, how large must k be before this approach gives a good approximation in the specified region of the complex plane?

5. Consider the Jordan block

$$\mathbf{A} = \begin{bmatrix} 0.1 & 1.5 & & \\ & 0.1 & \ddots & \\ & & \ddots & 1.5 \\ & & & 0.1 \end{bmatrix} \in \mathbb{C}^{50\times 50}.$$

Use `jordresrad.m` to precisely compute the ε-pseudospectral radius

$$\rho_\varepsilon(\mathbf{A}) = \sup_{z\in\sigma_\varepsilon(\mathbf{A})} |z|$$

for a variety of ε values (experiment: say, $\varepsilon = 10^{-20}, \ldots, 10^0$).

Produce a `semilogy` plot showing $\|\mathbf{A}^k\|$ versus k for $k = 0, 1, \ldots, 100$.

Superimpose on your plot the upper bound proved in today's lecture

$$\|\mathbf{A}^k\| \le \frac{\rho_\varepsilon(\mathbf{A})^{k+1}}{\varepsilon}$$

for the ε values for which you computed $\rho_\varepsilon(\mathbf{A})$.