ELGERSBURG SCHOOL

Pseudospectra and Nonnormal Dynamical Systems

Exercise Sheet 4

Each of these problems can solved independently, so please explore those you find most interesting.

1. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding right eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and left eigenvectors $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_n$.

   The derivative of the eigenvalue $\lambda_k(t)$ of $\mathbf{A} + t\mathbf{E}$ with respect to t (for $\|\mathbf{E}\| = 1$) at $t = 0$ is bounded in magnitude by the *eigenvalue condition number*

   $$\kappa(\lambda_k) = \frac{\|\widehat{\mathbf{v}}_k\|\|\mathbf{v}_k\|}{|\widehat{\mathbf{v}}_k^*\mathbf{v}_k|}.$$

   Consider the matrix

   $$\mathbf{A} = \begin{bmatrix} -1 & \alpha \\ 0 & -2 \end{bmatrix},$$

   where $\alpha \geq 1$ is a real constant.

   (a) Compute the eigenvalue condition numbers $\kappa(\lambda_1)$ and $\kappa(\lambda_2)$ for eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

   (b) Compute the matrix exponential $\mathrm{e}^{t\mathbf{A}}$.

   (c) What is $\lim_{t\to\infty} \|\mathrm{e}^{t\mathbf{A}}\|$?

   (d) For fixed $\alpha > 4$, find the value of $t \geq 0$ that maximizes the largest entry in $\mathrm{e}^{t\mathbf{A}}$.

   (e) What does your solution to (d) suggest about the general behavior of $\|\mathrm{e}^{t\mathbf{A}}\|$ for $t \in (0,\infty)$ with $\alpha > 4$? How does $\max_{t\geq 0} \|\mathrm{e}^{t\mathbf{A}}\|$ depend on α?

2. We have seen many linear, time-varying dynamical system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, whose solutions $\mathbf{x}(t)$ decay to zero as $t \to \infty$, provided all eigenvalues of $\mathbf{A}$ have negative real part.

   Is the same true for *variable-coefficient* problems? Suppose $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, and that all eigenvalues of the matrix $\mathbf{A}(t) \in \mathbb{C}^{n\times n}$ have negative real part for all $t \geq 0$. Is this enough to guarantee that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$? This problem asks you to explore this possibility.

   (a) Consider the matrix

   $$\mathbf{U}(t) = \begin{bmatrix} \cos(\gamma t) & \sin(\gamma t) \\ -\sin(\gamma t) & \cos(\gamma t) \end{bmatrix}.$$

   Show that $\mathbf{U}(t)$ is unitary (i.e., $\mathbf{U}(t)^*\mathbf{U}(t) = \mathbf{I}$) for any fixed real values of γ and t.

   (b) Now consider the matrix $\mathbf{A}(t) \in \mathbb{C}^{n\times n}$ defined by

   $$\mathbf{A}(t) = \mathbf{U}(t)\mathbf{A}_0\mathbf{U}(t)^*, \qquad \mathbf{A}_0 = \begin{bmatrix} -1 & \alpha \\ 0 & -2 \end{bmatrix}.$$

   (Notice that $\mathbf{A}_0$ is the matrix that featured in Problem 1.)
   Explain why $\sigma(\mathbf{A}(t)) = \sigma(\mathbf{A}_0)$, $W(\mathbf{A}(t)) = W(\mathbf{A}_0)$, and $\sigma_\varepsilon(\mathbf{A}(t)) = \sigma_\varepsilon(\mathbf{A}_0)$ for all $t \geq 0$. (In other words, show the spectrum, numerical range, and ε-pseudospectra are identical for all t.)

(c) Now we wish to investigate the behavior of the dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t). \qquad (*)$$

Define $\mathbf{y}(t) = \mathbf{U}(t)^*\mathbf{x}(t)$. Explain why equation $(*)$ implies that

$$\dot{\mathbf{y}}(t) = (\mathbf{A}_0 + (\mathbf{U}(t)^*)'\mathbf{U}(t))\mathbf{y}(t). \qquad (**)$$

(Here $(\mathbf{U}(t)^*)' \in \mathbb{C}^{n\times n}$ denotes the t-derivative of the conjugate-transpose of $\mathbf{U}(t)$.)

(d) Compute $(\mathbf{U}(t)^*)'\mathbf{U}(t)$. Does this matrix vary with t?

(e) Define the matrix

$$\widehat{\mathbf{A}} = \mathbf{A}_0 + (\mathbf{U}(t)^*)'\mathbf{U}(t).$$

Fix $\alpha = 7$. Plot the (real) eigenvalues of $\widehat{\mathbf{A}}$ (e.g., in MATLAB) as a function of $\gamma \in [0, 7]$. Do the eigenvalues of $\widehat{\mathbf{A}}$ fall in the left half of the complex plane for all γ?

(f) Calculate the eigenvalues of $\widehat{\mathbf{A}}$ for $\gamma = 1$ and $\alpha = 7$.
What can be said of solutions $\mathbf{y}(t)$ to the system $(**)$ as $t \to \infty$ for these α, γ values?
What then can be said of $\|\mathbf{x}(t)\| = \|\mathbf{U}(t)\mathbf{y}(t)\|$, where $\mathbf{x}(t)$ solves $(*)$, as $t \to \infty$?
How does this compare to the similar constant coefficient problem $\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t)$ (where we have seen that $\mathbf{A}_0$ has the same spectrum, numerical range, and pseudospectra as $\mathbf{A}(t)$ for all t)?

(g) Adapt this experiment to the matrix

$$\mathbf{A}_0 = \begin{bmatrix} -1 & M^2 \\ -1 & -1 \end{bmatrix}$$

with $M = 100$, which we saw in the lectures to have a large real distance to instability, but a small complex distance to instability.

[Examples of this sort were perhaps first constructed by Vinograd; see the books by Dekker and Verwer, and Lambert.]

3. Consider the convection-diffusion operator

$$\mathbf{A}u = \nu u'' + u'$$

for $x \in [-1, 1]$ with homogeneous Dirichlet boundary conditions, $u(-1) = u(1) = 0$.

(a) Following the example on Slide 11 of today's lecture, construct a pseudospectral discretization of this operator. The example in the notes discretizes the Laplacian. To discretize the operator here, replace

```
L = D*D;
```

with

```
L = nu*D*D + D;
```

(b) Take $\nu = 0.05$. Compute the rightmost eigenvalues of your matrix `L` for several values of n. (For example, $n = 16, 32, 64, 128$. How do they compare to the exact formula,

$$\lambda_k = -\frac{\nu k^2 \pi^2}{4} - \frac{1}{4\nu}.$$

(c) Use EigTool to plot the L^2-norm ε-pseudospectra of this example for $\nu = 0.05$ and $n = 128$, over the portion of the complex plane with $\operatorname{Re} z \in [-25, 5]$ and $\operatorname{Im} z \in [-15, 15]$, with $\varepsilon = 10^{-6}, \ldots, 10^{1}$.

(d) We now wish to solve time-dependent problems of the form

$$u_t = \nu u'' + u'$$

with $u(-1, t) = u(1, t) = 0$ with $u(x, 0) = u_0(x)$ given.
Plot numerical solutions obtained from the matrix exponential

```
u = expm(t*L)*u0
```

for times $t = 0, 0.1, 0.2, \ldots, 3$ and $\nu = 0.05$, with initial condition

$$u_0(x) = (1 + x)(1 - x)\mathrm{e}^{-(x-1/4)^2}.$$

(Here the vector `u0` is a discretization of the function u_0 on the Chebyshev grid.)
Produce a plot showing the L^2-norm of the solution $u(x, t)$ as a function of $t \in [0, 3]$. (How do you compute the L^2 norm of a vector, using the `R` matrix on Slide 11?)
Do you transient behavior of the solution?

(e) As an alternative to the matrix exponential, one can discretize the problem in time, and use, e.g., the forward Euler method to integrate the equation. Fix at time-step Δt, and let $\mathbf{u}_k$ denote the vector of approximate solutions on the Chebyshev grid at time $k\Delta t$. The forward Euler method steps forward in time via

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{L} \mathbf{u}_k,$$

where $\mathbf{L}$ is the discretization matrix for your operator. It follows that

$$\mathbf{u}_k = (\mathbf{I} + \Delta t \mathbf{L})^k \mathbf{u}_0.$$

Consider the following parameter values: $n = 32$, $\nu = 0.0025$, and $\Delta t = 0.005$.
(These specific values are important to illustrate the desired effects.)

(i) Confirm that the spectral radius of $\mathbf{I} + \Delta t \mathbf{L}$ is less than one, so $(\mathbf{I} + \Delta t \mathbf{L})^k \to \mathbf{0}$ at $k \to \infty$.

(ii) Plot the pseudospectra of $\mathbf{I} + \Delta t \mathbf{L}$ in the vicinity of the largest magnitude eigenvalues.

(iii) Compute the first 500 steps of the Forward Euler method, and plot your solution at each step. The sawtooth oscillations are an example of transient behavior. What happens if you instead use $\Delta t = 0.0025$?

4. In our description of the use of pseudospectra to analyze DAEs, we assumed that $\mathbf{A}$ was invertible. Suppose instead that we only have that $\mathbf{A} - \lambda\mathbf{B}$ is invertible for some *lambda* $\in \mathbb{C}$. (This is true of all so-called *nonsingular* pencils.)

   Premultiply the equation $\mathbf{B}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ by $(\mathbf{A} - \lambda\mathbf{B})^{-1}$. Defining

$$\mathbf{A}_\lambda := (\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{A}, \qquad \mathbf{B}_\lambda := (\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{B}$$

   this gives

$$\mathbf{B}_\lambda\dot{\mathbf{x}}(t) = \mathbf{A}_\lambda\mathbf{x}(t).$$

   (a) Show that $\mathbf{A}_\lambda = \mathbf{I} + \lambda\mathbf{B}_\lambda$.

   (b) Use a Schur factorization for $\mathbf{B}_\lambda$ (like the one used for $\mathbf{A}^{-1}\mathbf{B}$ in the notes) to derive a closed form expression for the solution $\mathbf{x}(t)$ at all times.