Linear Matrix Inequalities in Control

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Class 1

Outline of Part I

Course organization

Course topics

- Facts from convex analysis.
  LMI’s: history, algorithms and software.
- The role of LMI’s in dissipativity
  Stability and nominal performance. Analysis results.
- From analysis to synthesis.
  State-feedback, estimation and output-feedback synthesis.
- From nominal to robust stability, robust performance and robust synthesis.
- IQC’s and multipliers.
  Relations to classical tests and to \( \mu \)-theory.
- Mixed control problems and parameter-varying systems and control design.
Software Issues

Install Yalmip
- Modelling language for solving convex optimization problems
- Consistent with Matlab; free (!)
- Efficient to implement algorithms
- Developed by Johan Löfberg

Install SDP solver
- Small till medium size LP and QP problems: SeDuMi or SPDT3
- High level LP and QP problems: GUROBI
- Generic package for academic use: MOSEK
- Many more commercial and non-commercial solvers . . .

Outline of Part II
- What to expect?
- Convex sets and convex functions
  - Convex sets
  - Convex functions
- Why is convexity important?
  - Examples
  - Ellipsoidal algorithm
  - Duality and convex programs
- Linear Matrix Inequalities
  - Definitions
  - LMI’s and convexity
  - LMI’s in control
- A design example

Outline
- What to expect?
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  - LMI’s in control
- A design example
Merging control and optimization

Classical optimal control paradigm (LQG, $H_2$, $H_\infty$, $L_1$, MPC) is restricted:

- Performance specs in terms of complete closed loop transfer matrix.
- One measure of performance only. Often multiple specs have been imposed for controlled system.
- Cannot incorporate structured time-varying/nonlinear uncertainties.
- Can only design LTI controllers.
- Can only synthesize one controller in one architecture.

Control vs. optimization

View control input and/or feedback controller as decision variable of optimization problem. Desired specifications are imposed as constraints on controlled system.

Outline

1. What to expect?
2. Convex sets and convex functions
   - Convex sets
   - Convex functions
3. Why is convexity important?
   - Examples
   - Ellipsoidal algorithm
   - Duality and convex programs
4. Linear Matrix Inequalities
   - Definitions
   - LMI’s and convexity
   - LMI’s in control
5. A design example

Major goals for control and optimization

- Distinguish easy from difficult problems. (Convexity is key!)
- What are consequences of convexity in optimization?
- What is robust optimization?
- How to check robust stability by convex optimization?
- Which performance measures can be dealt with?
- How can controller synthesis be convexified?
- What are limits for the synthesis of robust controllers?
- How can we perform systematic gain scheduling?

Optimization problems

Casting optimization problems in mathematics requires

- $\mathcal{X}$: decision set
- $\mathcal{S} \subseteq \mathcal{X}$: feasible decisions
- $f : \mathcal{S} \rightarrow \mathbb{R}$: cost function

$f$ assigns to each decision $x \in \mathcal{S}$ a cost $f(x) \in \mathbb{R}$.

Wish to select the decision $x \in \mathcal{S}$ that minimizes the cost $f(x)$. 
Optimization problems

1. What is least possible cost? Compute optimal value
   \[ f_{\text{opt}} := \inf_{x \in S} f(x) = \inf \{ f(x) \mid x \in S \} \geq -\infty \]

   Convention: \( S = \emptyset \) then \( f_{\text{opt}} = +\infty \)

   Convention: If \( f_{\text{opt}} = -\infty \) then problem is said to be unbounded

2. How to determine almost optimal solutions? For arbitrary \( x \in S \) with \( f_{\text{opt}} \leq f(x) \leq f_{\text{opt}} + \varepsilon \).

3. Is there an optimal solution (or minimizer)? Does there exist \( x_{\text{opt}} \in S \) with \( f_{\text{opt}} = f(x_{\text{opt}}) \).

   We write: \( f(x_{\text{opt}}) = \min_{x \in S} f(x) \)

4. Can we calculate all optimal solutions? (Non)-uniqueness
   \[ \arg \min f(x) := \{ x \in S \mid f_{\text{opt}} = f(x) \} \]

Recap: infimum and minimum of functions

**Infimum of a function**

Any \( f : S \to \mathbb{R} \) has infimum \( L \in \mathbb{R} \cup \{-\infty\} \) denoted as \( \inf_{x \in S} f(x) \). It is defined by the properties

- \( L \leq f(x) \) for all \( x \in S \)
- \( L \) finite: for all \( \varepsilon > 0 \) exists \( x \in S \) with \( f(x) < L + \varepsilon \)
- \( L \) infinite: for all \( \varepsilon > 0 \) there exist \( x \in S \) with \( f(x) < -1/\varepsilon \)

**Minimum of a function**

If exists \( x_0 \in S \) with \( f(x_0) = \inf_{x \in S} f(x) \) we say that \( f \) attains its minimum on \( S \) and write \( L = \min_{x \in S} f(x) \).

If exists, the minimum is uniquely defined by the properties

- \( L \leq f(x) \) for all \( x \in S \)
- There exists some \( x_0 \in S \) with \( f(x_0) = L \)

A classical result

**Theorem (Weierstrass)**

If \( f : S \to \mathbb{R} \) is continuous and \( S \) is a compact subset of the normed linear space \( \mathcal{X} \), then there exists \( x_{\text{min}}, x_{\text{max}} \in S \) such that for all \( x \in S \)

\[ \inf_{x \in S} f(x) = f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}) = \sup_{x \in S} f(x) \]

Comments:
- Answers problem 3 for “special” \( S \) and \( f \)
- No clue on how to find \( x_{\text{min}}, x_{\text{max}} \)
- No answer to uniqueness issue
- \( S \) compact if for every sequence \( x_n \in S \) a subsequence \( x_{n_m} \) exists which converges to a point \( x \in S \)
- Continuity and compactness overly restrictive!
Examples of convex sets

Convex sets

Non-convex sets

Basic properties of convex sets

Theorem
Let $S$ and $T$ be convex. Then
- $\alpha S := \{ x \mid x = \alpha s, s \in S \}$ is convex
- $S + T := \{ x \mid x = s + t, s \in S, t \in T \}$ is convex
- closure of $S$ and interior of $S$ are convex
- $S \cap T := \{ x \mid x \in S \text{ and } x \in T \}$ is convex.

Examples of convex sets

• With $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$, the hyperplane
  $$\mathcal{H} = \{ x \in \mathbb{R}^n \mid a^T x = b \}$$
and the half-space
  $$\mathcal{H}_- = \{ x \in \mathbb{R}^n \mid a^T x \leq b \}$$
are convex.

• The intersection of finitely many hyperplanes and half-spaces is a polyhedron. Any polyhedron is convex and can be described as
  $$\{ x \in \mathbb{R}^n \mid Ax \leq b, Dx = e \}$$
for suitable matrices $A$ and $D$ and vectors $b, e$.

• A compact polyhedron is a polytope.

Examples of convex sets

The convex hull

Definition
The convex hull of a set $S \subseteq \mathcal{X}$ is
$$\text{conv}(S) := \cap \{ T \mid T \text{ is convex and } S \subseteq T \}$$

• $\text{conv}(S)$ is convex for any set $S$
• $\text{conv}(S)$ is set of all convex combinations of points of $S$
• The convex hull of finitely many points $\text{conv}(x_1, \ldots, x_n)$ is a polytope. Moreover, any polytope can be represented in this way!!

Latter property allows explicit representation of polytopes. For example $\{ x \in \mathbb{R}^n \mid a \leq x \leq b \}$ consists of $2n$ inequalities and requires $2^n$ generators for its representation as convex hull!
Convex functions

**Definition**

A function $f : S \to \mathbb{R}$ is convex if

- $S$ is convex and
- for all $x_1, x_2 \in S$, $\alpha \in (0, 1)$ there holds
  
  $$f(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha f(x_1) + (1 - \alpha) f(x_2)$$

We have

$$f : S \to \mathbb{R} \text{ convex } \implies \{ x \in S | f(x) \leq \gamma \} \text{ convex for all } \gamma \in \mathbb{R}$$

Derives convex sets from convex functions

- Converse $\iff$ is not true!
- $f$ is strictly convex if $<$ instead of $\leq$

Examples of convex functions

- $f(x) = ax^2 + bx + c$ convex if $a > 0$
- $f(x) = |x|$
- $f(x) = \| x \|$
- $f(x) = \sin x$ on $[\pi, 2\pi]$

Non-convex functions

- $f(x) = x^3$ on $\mathbb{R}$
- $f(x) = -|x|$
- $f(x) = \sqrt{x}$ on $\mathbb{R}_+$
- $f(x) = \sin x$ on $[0, \pi]$

Recap: Hermitian and symmetric matrices

**Definition**

For a real or complex matrix $A$ the inequality $A < 0$ means that $A$ is Hermitian and negative definite.

- $A$ is Hermitian if $A = A^*$ or $A = A^T$. If $A$ is real this amounts to $A = A^T$ and we call $A$ symmetric.
- All eigenvalues of Hermitian and symmetric matrices are real.
- By definition a Hermitian matrix $A$ is negative definite if
  
  $$u^* Au < 0 \quad \text{for all complex vectors } u \neq 0$$

A is negative definite if and only if all its eigenvalues are negative.
- $A \preceq B$, $A \succeq B$ and $A \succ B$ defined and characterized analogously.

Convex matrix-valued functions

An interesting generalization:

- Define $\mathbb{H}$ and $\mathbb{S}$ to be the sets of Hermitian and Symmetric matrices, i.e., sets of square matrices $A$ for which
  
  $$A = A^* \quad \text{or} \quad A = A^T$$

- Define partial ordering on $\mathbb{H}$ and $\mathbb{S}$
  
  $$A_1 \prec A_2, \quad A_1 \preceq A_2, \quad A_1 \succeq A_2, \quad A_1 \succ A_2$$

by requiring $x^*(A_1 - A_2)x$ to be negative, non-positive, positive or non-positive for all $x \neq 0$.

**Definition**

A matrix-valued function $F : S \to \mathbb{H}$ is convex if $S$ is a convex set and

$$F(\alpha x_1 + (1 - \alpha) x_2) \preceq \alpha F(x_1) + (1 - \alpha) F(x_2)$$

for any $x_1, x_2 \in S$ and $0 < \alpha < 1$. 
Affine sets

**Definition**
A subset $S$ of a linear vector space is **affine** if $x = \alpha x_1 + (1 - \alpha) x_2$ belongs to $S$ for every $x_1, x_2 \in S$ and $\alpha \in \mathbb{R}$.

- Geometric idea: line through any two points belongs to set
- Every affine set is convex
- $S$ affine if and only if there exists $x_0$ such that $S = \{ x | x = x_0 + m, \ m \in M \}$
  with $M$ a linear subspace

Affine functions

**Definition**
A function $f : S \to T$ is **affine** if

$$f(\alpha x_1 + (1 - \alpha) x_2) = \alpha f(x_1) + (1 - \alpha) f(x_2)$$

for all $x_1, x_2 \in S$ and for all $\alpha \in \mathbb{R}$.

**Theorem**
If $S$ and $T$ are finite dimensional, then $f : S \to T$ is affine if and only if

$$f(x) = f_0 + T(x)$$

where $f_0 \in T$ and $T : S \to T$ a linear map (a matrix).

Cones and convexity

**Definition**
A **convex cone** is a set $K \subset \mathbb{R}^n$ with the property that

$$x_1, x_2 \in K \implies \alpha_1 x_1 + \alpha_2 x_2 \in K \quad \text{for all } \alpha_1, \alpha_2 \geq 0.$$

- Since $x_1, x_2 \in K$ implies $\alpha x_1 + (1 - \alpha) x_2 \in K$ for all $\alpha \in (0, 1)$ a convex cone is convex.
- If $S \subset \chi$ is an arbitrary set, then

$$K := \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0 \text{ for all } x \in S \}$$

is a cone. Also denoted $K = S^*$ and called **dual cone** of $S$.

- Cones are unbounded sets.
- If $K_1$ and $K_2$ are convex cones, then so are

$$\alpha K_1, K_1 \cap K_2, K_1 + K_2$$

for any $\alpha \in \mathbb{R}$.
- The intersection of a cone with a hyperplane is a **conic section**.
Why is convexity interesting ???

Reason 1: absence of local minima

Definition

\[ f : S \rightarrow \mathbb{R} \] Then \( x_0 \in S \) is a

- local minimum if \( \exists \varepsilon > 0 \) such that
  \[ f(x_0) \leq f(x) \quad \text{for all } x \in S \text{ with } ||x - x_0|| \leq \varepsilon \]
- global minimum if \( f(x_0) \leq f(x) \) for all \( x \in S \)

Theorem

If \( f : S \rightarrow \mathbb{R} \) is convex then every local minimum \( x_0 \) is a global minimum of \( f \). If \( f \) is strictly convex, then the global minimum \( x_0 \) is unique.

Reason 2: uniform bounds

Theorem

Suppose \( S = \text{conv}(S_0) \) and \( f : S \rightarrow \mathbb{R} \) is convex. Then equivalent are:

- \( f(x) \leq \gamma \) for all \( x \in S \)
- \( f(x) \leq \gamma \) for all \( x \in S_0 \)

Very interesting if \( S_0 \) consists of finite number of points, i.e, \( S_0 = \{x_1, \ldots, x_n\} \). A finite test!!

Reason 3: subgradients

Definition

A vector \( g = g(x_0) \in \mathbb{R}^n \) is a subgradient of \( f \) at \( x_0 \) if

\[ f(x) \geq f(x_0) + \langle g, x - x_0 \rangle \]

for all \( x \in S \)

Geometric idea: graph of affine function \( x \mapsto f(x_0) + \langle g, x - x_0 \rangle \) tangent to graph of \( f \) at \( (x_0, f(x_0)) \).

Theorem

A convex function \( f : S \rightarrow \mathbb{R} \) has a subgradient at every interior point \( x_0 \) of \( S \).
Examples and properties of subgradients

- If \( f \) differentiable, then \( g = g(x_0) = \nabla f(x_0) \) is subgradient.
  So, for differentiable functions every gradient is a subgradient.

- The non-differentiable function \( f(x) = |x| \) has any real number \( g \in [-1,1] \) as its subgradient at \( x_0 = 0 \).

- \( f(x_0) \) is global minimum of \( f \) if and only if \( 0 \) is subgradient of \( f \) at \( x_0 \).

- Since \( \langle g, x - x_0 \rangle > 0 = \Rightarrow f(x) > f(x_0) \),
  all points in half space \( H := \{ x \mid \langle g, x - x_0 \rangle \leq 0 \} \) can be discarded in searching for minimum of \( f \).

  Used explicitly in ellipsoidal algorithm

Ellipsoidal algorithm

**Aim:** Minimize convex function \( f : \mathbb{R}^n \to \mathbb{R} \)

- **Step 0** Let \( x_0 \in \mathbb{R}^n \) and \( P_0 > 0 \) such that all minimizers of \( f \) are located in the ellipsoid
  \( \mathcal{E}_0 := \{ x \in \mathbb{R}^n \mid (x - x_0)^T P_0^{-1} (x - x_0) \leq 1 \} \).

  Set \( k = 0 \).

- **Step 1** Compute a subgradient \( g_k \) of \( f \) at \( x_k \). If \( g_k = 0 \) then stop,
  otherwise proceed to Step 2.

- **Step 2** All minimizers are contained in
  \( \mathcal{H}_k := \mathcal{E}_k \cap \{ x \mid \langle g_k, x - x_k \rangle \leq 0 \} \).

- **Step 3** Compute \( x_{k+1} \in \mathbb{R}^n \) and \( P_{k+1} > 0 \) with minimal determinant \( \det P_{k+1} \) such that
  \( \mathcal{E}_{k+1} := \{ x \in \mathbb{R}^n \mid (x - x_{k+1})^T P_{k+1}^{-1} (x - x_{k+1}) \leq 1 \} \)
  contains \( \mathcal{H}_k \).

- **Step 4** Set \( k \) to \( k + 1 \) and return to Step 1.

Remarks on ellipsoidal algorithm:

- Convergence \( f(x_k) \to \inf_x f(x) \).

- Exist explicit equations for \( x_k, P_k, \mathcal{E}_k \) such that volume of \( \mathcal{E}_k \) decreases with factor \( e^{-1/2n} \) at each step.
  (See lecture notes).

- Simple, robust, reliable, easy to implement, but slow convergence.

Why is convexity interesting ???

**Reason 4: Duality and convex programs**

Set of feasible decisions often described by equality and inequality constraints:

\[
S = \{ x \in X \mid g_k(x) \leq 0, \ k = 1, \ldots, K, \ h_\ell(x) = 0, \ \ell = 1, \ldots, L \}
\]

**Primal optimization:**

\[
P_{\text{opt}} = \inf_{x \in S} f(x)
\]

- One of index sets \( K \) or \( L \) infinite: semi-infinite optimization

- Both index sets \( K \) and \( L \) finite: nonlinear program
Why is convexity interesting ??

- Examples: saturation constraints, safety margins, constitutive and balance equations all assume form \( S = \{ x \mid g(x) \leq 0, h(x) = 0 \} \).
- semi-definite program:
  
  \[
  \text{minimize } f(x) \text{ subject to } g(x) \leq 0, h(x) = 0
  \]
  
  \( f, g \) and \( h \) affine.
- quadratic program:
  
  \[
  \text{minimize } x^\top Q x + 2s^\top x + r \text{ subject to } g(x) \leq 0, h(x) = 0
  \]
  
  with \( g \) and \( h \) affine.
- quadratically constraint quadratic program:
  
  \[
  f(x) = x^\top Q x + 2s^\top x + r, g_j(x) = x^\top Q_j x + 2s_j^\top x + r_j, h(x) = h_0 + H x
  \]
  
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Upper and lower bounds for convex programs

Primal optimization problem

\[
P_{\text{opt}} = \inf_{x \in \mathcal{X}} f(x) \quad \text{subject to } g(x) \leq 0, h(x) = 0
\]

Can we obtain bounds on optimal value \( P_{\text{opt}} \)?

- Upper bound on optimal value
  
  For any \( x_0 \in S \) we have
  
  \[
P_{\text{opt}} \leq f(x_0)
  \]
  
  which defines an upper bound on \( P_{\text{opt}} \).

- Lower bound on optimal value
  
  Let \( x \in S \). Then for arbitrary \( y \succeq 0 \) and \( z \) we have
  
  \[
  L(x, y, z) := f(x) + (y, g(x)) + (z, h(x)) \leq f(x)
  \]
  
  and, in particular,
  
  \[
  \ell(y, z) := \inf_{x \in \mathcal{X}} L(x, y, z) \leq \inf_{x \in S} L(x, y, z) \leq \inf_{x \in S} f(x) = P_{\text{opt}}.
  \]
  
  so that
  
  \[
  D_{\text{opt}} := \sup_{y \succeq 0, z} \ell(y, z) = \sup_{y \succeq 0, z} \inf_{x \in \mathcal{X}} L(x, y, z) \leq P_{\text{opt}}
  \]
  
  defines a lower bound for \( P_{\text{opt}} \).

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Duality and convex programs

Some terminology:

- Lagrange function: \( L(x, y, z) \)
- Lagrange dual cost: \( \ell(y, z) \)
- Lagrange dual optimization problem:
  
  \[
  D_{\text{opt}} := \sup_{y \succeq 0, z} \ell(y, z)
  \]
  
  Remarks:

  - \( \ell(y, z) \) computed by solving an unconstrained optimization problem. Is concave function.
  - Dual problem is concave maximization problem. Constraints are simpler than in primal problem
  - Main question: when is \( D_{\text{opt}} = P_{\text{opt}} \)?

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Example of duality

- **Primal Linear Program**
  \[ P_{\text{opt}} = \inf_x c^T x \]
  subject to \( x \geq 0, b - Ax = 0 \)

  Lagrange dual cost
  \[ \ell(y, z) = \inf_x c^T x - y^T x + z^T (b - Ax) \]
  \[ = \begin{cases} 
  b^T z & \text{if } c - A^T z - y = 0 \\
  -\infty & \text{otherwise}
  \end{cases} \]

- **Dual Linear Program**
  \[ D_{\text{opt}} = \sup_z b^T z \]
  subject to \( y = c - A^T z \geq 0 \)

Karush-Kuhn-Tucker and duality

**Theorem (Karush-Kuhn-Tucker)**
If \((g, h)\) satisfies the constraint qualification, then we have strong duality:
\[ D_{\text{opt}} = P_{\text{opt}}. \]

There exist \( y_{\text{opt}} \succeq 0 \) and \( z_{\text{opt}} \), such that \( D_{\text{opt}} = \ell(y_{\text{opt}}, z_{\text{opt}}) \).
Moreover, \( x_{\text{opt}} \) is an optimal solution of the primal optimization problem and \((y_{\text{opt}}, z_{\text{opt}})\) is an optimal solution of the dual optimization problem, if and only if
1. \( g(x_{\text{opt}}) \leq 0, h(x_{\text{opt}}) = 0 \),
2. \( y_{\text{opt}} \succeq 0 \) and \( x_{\text{opt}} \) minimizes \( L(x, y_{\text{opt}}, z_{\text{opt}}) \) over all \( x \in X \) and
3. \( \langle y_{\text{opt}}, g(x_{\text{opt}}) \rangle = 0 \).

Karush-Kuhn-Tucker and duality

**Definition**
Suppose \( f, g \) convex and \( h \) affine. \((g, h)\) satisfy the constraint qualification if \( \exists x_0 \) in the interior of \( X \) with \( g(x_0) \preceq 0, h(x_0) = 0 \) such that \( g_j(x_0) < 0 \) for all component functions \( g_j \) that are not affine.

**Example:** \((g, h)\) satisfies constraint qualification if \( g \) and \( h \) are affine.

**Remarks:**
- Very general result, strong tool in convex optimization
- Dual problem simpler to solve, \((y_{\text{opt}}, z_{\text{opt}})\) called Kuhn-Tucker point.
- The triple \((x_{\text{opt}}, y_{\text{opt}}, z_{\text{opt}})\) exist if and only if it defines a saddle point of the Lagrangian \( L \) in that
  \[ L(x_{\text{opt}}, y, z) \leq L(x_{\text{opt}}, y_{\text{opt}}, z_{\text{opt}}) \leq L(x, y_{\text{opt}}, z_{\text{opt}}) \]
  \[ = P_{\text{opt}} = D_{\text{opt}} \]
  for all \( x, y \succeq 0 \) and \( z \).
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Linear Matrix Inequalities

Definition

A linear matrix inequality (LMI) is an expression

\[ F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n < 0 \]

where

- \( x = \text{col}(x_1, \ldots, x_n) \) is a vector of real decision variables,
- \( F_i = F_i^T \) are real symmetric matrices and
- \( \prec 0 \) means negative definite, i.e.,

\[ F(x) \prec 0 \iff u^T F(x) u < 0 \text{ for all } u \neq 0 \]
\[ \iff \text{all eigenvalues of } F(x) \text{ are negative} \]
\[ \iff \lambda_{\max}(F(x)) < 0 \]

- \( F \) is affine function of decision variables

Simple examples of LMI’s

- \( 1 + x < 0 \)
- \( 1 + x_1 + 2x_2 < 0 \)
- \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} < 0. \)

All the same with \( \succ 0, \preceq 0 \) and \( \succeq 0 \).

Only very simple cases can be treated analytically.

Need to resort to numerical techniques!

Main LMI problems

1. **LMI feasibility problem**: Test whether there exists \( x_1, \ldots, x_n \) such that \( F(x) < 0 \).
2. **LMI optimization problem**: Minimize \( f(x) \) over all \( x \) for which the LMI \( F(x) < 0 \) is satisfied.

**How is this solved?**

\( F(x) < 0 \) is feasible iff \( \min_x \lambda_{\max}(F(x)) < 0 \) and therefore involves minimizing the function

\[ x \mapsto \lambda_{\max}(F(x)) \]

- Possible because this function is convex!
- There exist efficient algorithms (Interior point, ellipsoid).
Why are LMI’s interesting?

**Reason 1:** LMI’s define convex constraints on $x$, i.e., $S := \{ x \mid F(x) \prec 0 \}$ is convex.

Indeed, $F(\alpha x_1 + (1 - \alpha)x_2) = \alpha F(x_1) + (1 - \alpha)F(x_2) \prec 0$.

**Reason 2:** Solution set of multiple LMI’s

\[ F_1(x) \prec 0, \ldots, F_k(x) \prec 0 \]

is convex and representable as one single LMI

\[ F(x) = \begin{pmatrix} F_1(x) & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & 0 & F_k(x) \end{pmatrix} \prec 0 \]

Allows to combine LMI’s!

**Reason 3:** Incorporate affine constraints such as $F(x) \prec 0$ and $Ax = b$

$F(x) \prec 0$ and $x = Ay + b$ for some $y$

$F(x) \prec 0$ and $x \in S$ with $S$ an affine set.

**Reason 4:** Conversion nonlinear constraints to linear ones

**Theorem (Schur complement)**

Let $F$ be an affine function with

\[ F(x) = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix}, \quad F_{11}(x) \text{ is square.} \]

Then

\[ F(x) \prec 0 \iff \begin{cases} F_{11}(x) \prec 0 \\ F_{22}(x) - F_{21}(x) [F_{11}(x)]^{-1} F_{12}(x) \prec 0. \end{cases} \]

\[ \iff \begin{cases} F_{22}(x) \prec 0 \\ F_{11}(x) - F_{12}(x) [F_{22}(x)]^{-1} F_{21}(x) \prec 0. \end{cases} \]
First examples in control

Example 1: Stability
Consider autonomous system
\[ \dot{x} = Ax. \]
Verify stability through feasibility. We have:
\[ \dot{x} = Ax \text{ asymptotically stable} \iff \left( -X \atop 0 \right) \left( -X \atop 0 \right) + A^\top X + XA < 0 \text{ feasible} \]
Here \( X = X^\top \) defines a Lyapunov function
\[ V(x) := x^\top X x \]
for the flow \( \dot{x} = Ax \).

Example 2: Joint stabilization
Given \( (A_1, B_1), \ldots, (A_k, B_k) \), find \( F \) such that \( (A_1 + B_1 F), \ldots, (A_k + B_k F) \) are asymptotically stable.
Equivalent to finding \( F, X_1, \ldots, X_k \) such that for \( j = 1, \ldots, k \):
\[ \left( -X_j \atop 0 \right) \left( -X_j \atop 0 \right) + X_j (A_j + B_j F) \left( -X_j \atop 0 \right) + X_j (A_j + B_j F)^\top < 0 \]
Sufficient condition: \( X = X_1 = \ldots = X_k, K = FX \), yields
\[ \left( -X \atop 0 \right) \left( -X \atop 0 \right) + A_j X + X A_j^\top + B_j K + K^\top B_j^\top < 0 \text{ an LMI!!} \]
Set feedback \( F = KX^{-1} \).

Example 3: Eigenvalue problem
Given \( F : \mathbb{V} \to \mathbb{S} \) affine, minimize over all \( x \)
\[ f(x) = \lambda_{\max}(F(x)). \]
Observe that, with \( \gamma > 0 \), and using Schur complement:
\[ \lambda_{\max}(F(x)F(x)) < \gamma^2 \iff \frac{1}{\gamma} F(x)F(x) - \gamma I \prec 0 \iff \left( \begin{array}{cc} -\gamma I & F(x) \\ F(x)^\top & -\gamma I \end{array} \right) < 0 \]
We can define
\[ y := \begin{pmatrix} x \\ \gamma \end{pmatrix}; \quad G(y) := \begin{pmatrix} -\gamma I & F(x) \\ F(x)^\top & -\gamma I \end{pmatrix}; \quad g(y) := \gamma \]
then \( G \) is affine in \( y \) and \( \min_x f(x) = \min_{y, \gamma > 0} g(y) \). This is a LP!
Truss topology design

Problem features:
- Connect nodes by $N$ bars of length $\ell = \text{col}(\ell_1, \ldots, \ell_N)$ (fixed) and cross sections $s = \text{col}(s_1, \ldots, s_N)$ (to be designed).
- Impose bounds on cross sections $a_k \leq s_k \leq b_k$ and total volume $\ell^T s \leq v$ (and hence an upperbound on total weight of the truss).
- Let $a = \text{col}(a_1, \ldots, a_N)$ and $b = \text{col}(b_1, \ldots, b_N)$.
- Distinguish fixed and free nodes.
- Apply external forces $f = \text{col}(f_1, \ldots, f_M)$ to some free nodes. These result in a node displacements $d = \text{col}(d_1, \ldots, d_M)$.

Mechanical model defines relation $A(s)d = f$ where $A(s) \succ 0$ is the stiffness matrix which depends linearly on $s$.

Goal:

Maximize stiffness or, equivalently, minimize elastic energy $f^T d$

Truss topology design

Problem

Find $s \in \mathbb{R}^N$ which minimizes elastic energy $f^T d$ subject to the constraints

\[ A(s) \succ 0, \quad A(s)d = f, \quad a \leq s \leq b, \quad \ell^T s \leq v \]

Data:
- Total volume $v > 0$, node forces $f$, bounds $a, b$, lengths $\ell$ and symmetric matrices $A_1, \ldots, A_N$ that define the linear stiffness matrix $A(s) = s_1 A_1 + \ldots + s_N A_N$.

Decision variables:
- Cross sections $s$ and displacements $d$ (both vectors).

Cost function:
- Stored elastic energy $d \mapsto f^T d$.

Constraints:
- Semi-definite constraint: $A(s) \succ 0$
- Non-linear equality constraint: $A(s)d = f$
- Linear inequality constraints: $a \leq s \leq b$ and $\ell^T s \leq v$. 

Trusses

- Trusses consist of straight members (‘bars’) connected at joints.
- One distinguishes free and fixed joints.
- Connections at the joints can rotate.
- The loads (or the weights) are assumed to be applied at the free joints.
- This implies that all internal forces are directed along the members, (so no bending forces occur).
- Construction reacts based on principle of statics: the sum of the forces in any direction, or the moments of the forces about any joint, are zero.
- This results in a displacement of the joints and a new tension distribution in the truss.

Many applications (roofs, cranes, bridges, space structures, . . . ) !!

Design your own bridge
From truss topology design to LMI’s

• First eliminate affine equality constraint \( A(s)d = f \):
  
  \[
  \begin{align*}
  \text{minimize} & \quad f^T (A(s))^{-1} f \\
  \text{subject to} & \quad A(s) > 0, \quad \ell^T s \leq v, \quad a \leq s \leq b
  \end{align*}
  \]

• Push objective to constraints with auxiliary variable \( \gamma \):
  
  \[
  \begin{align*}
  \text{minimize} & \quad \gamma \\
  \text{subject to} & \quad \gamma > f^T (A(s))^{-1} f, \quad A(s) > 0, \quad \ell^T s \leq v, \quad a \leq s \leq b
  \end{align*}
  \]

• Apply Schur lemma to linearize
  
  \[
  \begin{align*}
  \text{minimize} & \quad \gamma \quad f^T (A(s))^{-1} f, \quad A(s) > 0, \quad \ell^T s \leq v, \quad a \leq s \leq b
  \end{align*}
  \]

Note that the latter is an LMI optimization problem as all constraints on \( s \) are formulated as LMI’s!!

Yalmip coding for LMI optimization problem

Equivalent LMI optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \gamma \\
\text{subject to} & \quad \left( \begin{array}{c}
\gamma \\
\ell^T A(s)
\end{array} \right) > 0, \quad \ell^T s \leq v, \quad a \leq s \leq b
\end{align*}
\]

The following YALMIP code solves this problem:

```matlab
gamma=sdpvar(1,1); x=sdpvar(N,1,'full');
lmi=set([gamma f'; f A*diag(x)*A']);
lmi=lmi+set(l'*x<=v);
lmi=lmi+set(a<=x<=b);
options=sdpsettings('solver','csdp');
solvesdp(lmi,gamma,options);
s=double(x);
```

Result: optimal truss

Useful software:

General purpose MATLAB interface Yalmip

• Free code developed by J. Löfberg accessible here

  Get Yalmip now
  
  Run yalmipdemo.m for a comprehensive introduction.
  Run yalmiptest.m to test settings.

• Yalmip uses the usual Matlab syntax to define optimization problems.
  Basic commands sdpvar, set, sdpsettings and solvesdp.
  Truely easy to use!!!

• Yalmip needs to be connected to solver for semi-definite programming.
  There exist many solvers:

  SeDuMi  PENOPT  OOQP
  DSDP  CSDP  MOSEK

• Alternative Matlab’s LMI toolbox for dedicated control applications.
Joseph-Louis Lagrange (1736)  Aleksandr Mikhailovich Lyapunov (1857)

to next class