THE TRACTABILITY INDEX OF DIFFERENTIAL-ALGEBRAIC EQUATIONS: BACKGROUND AND SETTING

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Outline

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1. Introduction
2. Projector based decoupling of regular linear DAEs
3. Adjoint pairs and Lyapunov spectra
4. Nonlinear DAEs

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Differential-algebraic equations (DAEs)

arise in standard form and with properly involved derivative:

$$\mathfrak{f}(x'(t), x(t), t) = 0,$$

$$f(\frac{\mathrm{d}}{\mathrm{dt}}(\sigma(x(t), t)), x(t), t) = 0,$$

in which the partial derivatives $\mathfrak{f}_{x^1}(x^1, x, , t)$ and $f_y(y, x, t)$ are singular everywhere.

Most frequently applied special forms

$$\begin{aligned} x_1'(t) + g_1(x_1(t), x_2(t), t) &= 0, \\ g_2(x_1(t), x_2(t), t) &= 0, \end{aligned}$$

and

$$Ex'(t) + Fx(t) = q(t).$$

1971: Branin, Brayton, Hachtel, Gear, Gustavson,... (circuit simulation)

Simplest DAEs:

$$\begin{aligned} x_1'(t) + x_1(t) &= q_1(t) \\ x_2(t) &= q_2(t) \end{aligned}$$

$$\begin{aligned} x_2'(t) + x_1(t) &= q_1(t) \\ x_2(t) &= q_2(t) \end{aligned}$$

Matters of course in classical ODEs:

- Let $B(t) \in \mathbb{R}^{m\times m}$ be continuous in $t \in \mathcal{I}$.

Then, for each arbitrary $x_0 \in \mathbb{R}^m$ and $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, the IVP

$$x'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I}, \quad x(t_0) = x_0, \qquad (1)$$

is uniquely solvable and its solution depends continuously on x_0 and q.

- The dynamical behavior can be characterized by the Lyapunov spectrum $\alpha_1, \dots, \alpha_m$ of the ODE.

If B is a constant matrix with eigenvalues $\lambda_1, \dots, \lambda_m$, then the Lyapunov spectrum of the ODE (1) consists of $\alpha_i = -\mathrm{Re}\,\lambda_i$, $i = 1, \dots, m$.

Each solution pair x, y of the adjoint pair of explicit ODEs

$$x'(t) + B(t)x(t) = 0, \quad t \in \mathcal{I}, \tag{2}$$

$$-y'(t) + B(t)^*y(t) = 0, \quad t \in \mathcal{I}, \tag{3}$$

satisfies the Lagrange identity

$$\boxed{x(t)^*y(t) = < x(t), y(t) > = < x(0), y(0) >, \quad t \in \mathcal{I}.}$$

The fundamental solution matrices normalized at the same point satisfy

$$\boxed{X(t)^*Y(t) = I, \quad t \in \mathcal{I}.}$$

If $\alpha_1 \leq \cdots \leq \alpha_m$ and $\alpha_{*\,1} \geq \cdots \geq \alpha_{*\,m}$ denote the ordered complete Lyapunov spectra of (2) and (3), respectively, then the ODE (2) is Lyapunov regular if the Perron identity is valid, i.e.,

$$\boxed{\alpha_i + \alpha_{*\,i} = 0, \; i = 1, \ldots, m.}$$

Example (Ill-posed behavior)

$$\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix} x'(t) + \begin{bmatrix}-\alpha&-1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} x(t) = \begin{bmatrix}0\\0\\0\\0\\\gamma(t)\end{bmatrix},$$

$$\begin{bmatrix}1&0&0&0&0\end{bmatrix} x(0) = 0,$$

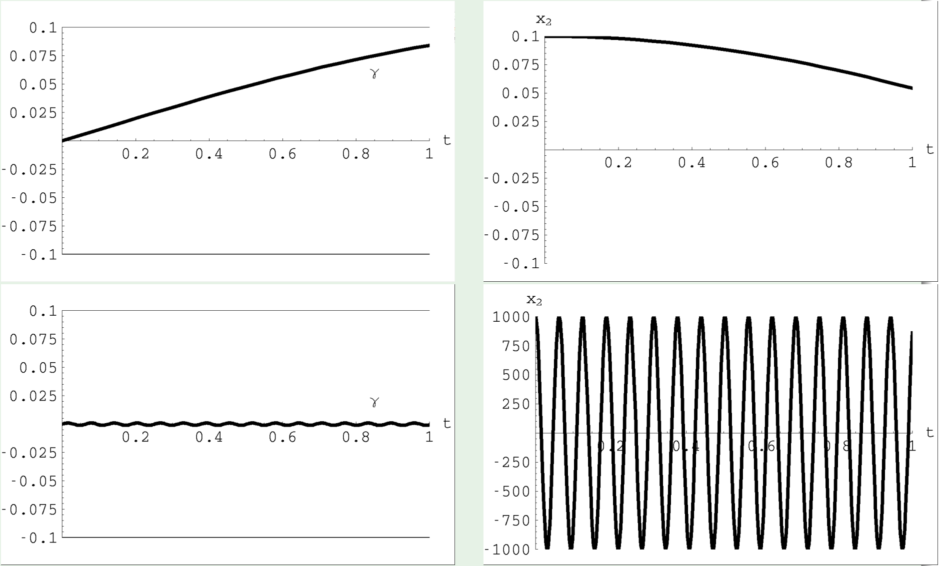
is uniquely solvable for every sufficiently smooth function γ.
Response to the small excitation $\gamma(t) = \varepsilon \frac{1}{n} \sin nt$, $n \in \mathbb{N}$, $\varepsilon = 0.1$:

$$x_1(t) = \varepsilon \int_0^t n^2 e^{\alpha(t-s)} \cos ns \mathrm{d}s, \quad x_2(t) = \varepsilon n^2 \cos nt,$$

$$x_3(t) = -\varepsilon n \sin nt, \quad x_4(t) = -\varepsilon \cos nt, \quad x_5(t) = \varepsilon \frac{1}{n} \sin nt.$$

Example (Continuation)

The figures show $\gamma(t) = \varepsilon \frac{1}{n} \sin nt$ and the responses x_2 for $n = 1$ and $n = 100$.



Outline

Consider the DAE,

$$\boxed{A(Dx)' + Bx = q,}$$

with continuous coefficients $A(t) \in \mathbb{R}^{m\times n}$, $D(t) \in \mathbb{R}^{n\times m}$, $B(t) \in \mathbb{R}^{m\times m}$, and properly involved derivative, that is, $\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n$, $t \in \mathcal{I}$.

The DAE in standard form $\boxed{Ex' + Fx = q}$ can be reformulated as $\boxed{A(Dx)' + (F + AD')x = q}$ by any proper factorization $E = AD$.

Definition (Solution)

A classical solution of the DAE is a function $x_* \in \mathcal{C}(\mathcal{I}_*, \mathbb{R}^m)$ such that $Dx \in \mathcal{C}^1(\mathcal{I}_*, \mathbb{R}^n)$ and the DAE is satisfied for all $t \in \mathcal{I}_*$.

A generalized solution of the DAE is, e.g., a function $x \in L^2(\mathcal{I}_x, \mathbb{R}^m)$ such that $Dx \in H^1(\mathcal{I}_x, \mathbb{R}^n)$ and the DAE is satisfied a.e. on $\mathcal{I}_x$.

Basic subspaces of $\boxed{A(Dx)' + Bx = q}$. Let $\bar{t} \in \mathcal{I}$.

$$S_{can}(\bar{t}) := \{z \in \mathbb{R}^m : \text{the IVP} \quad A(Dx)' + Bx = 0,\ x(\bar{t}) = z \quad \text{is solvable}\}$$

$N_{can}(\bar{t})$:= subspace $\bar{N} \subseteq \mathbb{R}^m$ of maximal possible dimension such that

$$A(Dx)' + Bx = 0, \quad x(\bar{t}) \in \bar{N} \quad \text{implies} \quad x = 0,$$

and the IVP $\quad A(Dx)' + Bx = q,\ x(\bar{t}) \in \bar{N} \quad$ is not affected by consistency conditions at $\bar{t}$,

- Regular ODE: $\quad S_{can}(\bar{t}) = \mathbb{R}^m, \ N_{can}(\bar{t}) = \{0\}$
- $\boxed{\begin{aligned} x_1'(t) + x_1(t) &= q_1(t) \\ x_2(t) &= q_2(t) \end{aligned}}$: $S_{can}(\bar{t}) = \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ N_{can}(\bar{t}) = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\boxed{\begin{aligned} x_2'(t) + x_1(t) &= q_1(t) \\ x_2(t) &= q_2(t) \end{aligned}}$: $S_{can}(\bar{t}) = \{0\}, \ N_{can}(\bar{t}) = \mathbb{R}^2$

DAE solutions: $\boxed{x = x_{hom} + x_q}$

$$A(Dx_{hom})' + Bx_{hom} = 0, \quad x_{hom}(\bar{t}) = z \in S_{can}(\bar{t}),$$
$$A(Dx_q)' + Bx_q = q, \quad x_q(\bar{t}) \in N_{can}(\bar{t}),$$

$$\boxed{A(Dx)' + Bx = q, \quad x(\bar{t}) - z \in N_{can}(\bar{t}).}$$

By construction: $S_{can}(t) \cap N_{can}(t) = \{0\}$.

For regular DAEs it holds that $\boxed{S_{can}(t) \oplus N_{can}(t) = \mathbb{R}^m}$, which defines the canonical projector function,

$$\Pi_{can} : \mathcal{I} \to \mathbb{R}^{m\times m},$$
$$\Pi_{can}(t)^2 = \Pi_{can}(t),\ \operatorname{im} \Pi_{can}(t) = S_{can}(t),\ \ker \Pi_{can}(t) = N_{can}(t),\ t \in \mathcal{I}.$$

Matrix function sequence (pointwise for $t \in \mathcal{I}$)

We set $G_0 := AD$, $B_0 := B$, Q_0 projector function: $\operatorname{im} Q_0 = \ker D$,
$P_0 := I - Q_0$, $\Pi_0 := P_0$, $r_0 = \operatorname{rank} G_0$,
and form, as long as the expressions exist, for $i \geq 0$,

$$G_{i+1} := G_i + B_i Q_i, \quad r_{i+1} := \operatorname{rank} G_{i+1}, \quad N_{i+1} := \ker G_{i+1},$$

Q_{i+1} projector function: $\operatorname{im} Q_{i+1} = N_{i+1}$, $N_0 + \cdots + N_i \subseteq \ker Q_{i+1}$,

$$P_{i+1} := I - Q_{i+1},$$

$$\Pi_{i+1} := \Pi_i P_{i+1},$$

$$B_{i+1} := B_i P_i - G_{i+1} D^- (D \Pi_{i+1} D^-)' D \Pi_i.$$

D^- is a generalized inverse of D such that $D^- D = P_0$.

The products Π_j, $\Pi_j Q_{j+1}$, $D\Pi_j D^-$, $D\Pi_j Q_{j+1} D^-$ are projector valued functions, too.

Definition (Admissible matrix function sequence)

The matrix function sequence is said to be admissible on $\mathcal{I}$ up to level $\kappa \in \mathbb{N}$, if

(a) the rank r_i is constant on $\mathcal{I}$, $i = 1, \ldots, \kappa$,

(b) the intersection $N_i \cap (N_0 + \ldots + N_{i-1})$ is trivial so that Q_i is well-defined, $i = 1, \ldots, \kappa$,

(c) the projector function Π_i is continuous, and $D\Pi_i D^-$ is continuously differentiable on $\mathcal{I}$, $\quad i = 0, \ldots, \kappa$.

It holds that

$$0 \le r_0 \le \ldots \le r_\kappa.$$

The associated sequences of projector functions $P_0, \ldots, P_\kappa$ and $Q_0, \ldots, Q_\kappa$ are also said to be admissible on $\mathcal{I}$..

Example: AD pure Jordan chain index 3, $B = I$

$$\begin{aligned} x_2'(t) + x_1(t) &= q_1(t), \\ x_3'(t) + x_2(t) &= q_2(t), \\ x_3(t) &= q_3(t). \end{aligned}$$

$$\begin{aligned} x_1(t) &= q_1(t) - (q_2 - q_3')'(t), \\ x_2(t) &= q_2(t) - q_3'(t), \\ x_3(t) &= q_3(t). \end{aligned}$$

$$G_0 = AD = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_0 = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Pi_0 Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \Pi_1 Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$r_0 = r_1 = r_2 = 2,\ r_3 = m = 3.$

Definition (Regular DAE)

The DAE is said to be regular on $\mathcal{I}$, if there are a number $\mu \in \mathbb{N}$, and an matrix function sequence being admissible up to level μ such that

$$0 \leq r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m.$$

The number μ is named tractability index, and the ranks $r_0, \ldots, r_{\mu-1}$ are said to be characteristic values of the DAE on $\mathcal{I}$.

$d := m - \sum_{j=0}^{\mu-1}(m - r_j)$ is the dynamical degree of freedom, and

$$\operatorname{rank} \Pi_{\mu-1} = \operatorname{rank} D\Pi_{\mu-1}D^{-} = d.$$

Decouple (or split or re-sort) the DAE. Index-1 case:

$$\boxed{A(Dx)' + Bx = q},$$

$$ADD^-(Dx)' + B(Q_0 + P_0)x - q = 0,$$

$$\{G_1 P_0 G_1^{-1} + G_1 Q_0 G_1^{-1}\}\{\underbrace{(AD + BQ_0)}_{=G_1}(D^-(Dx)' + Q_0 x) + BP_0 x - q\} = 0,$$

$$G_1 P_0\{D^-(Dx)' + P_0 G_1^{-1} BP_0 x - P_0 G_1^{-1} q\}$$
$$+ G_1 Q_0\{Q_0 x + Q_0 G_1^{-1} BP_0 x - Q_0 G_1^{-1} q\} = 0$$

$D^- D = P_0 = \Pi_0,\ DD^- = D\Pi_0 D^-.$

$$(Dx)' - (D\Pi_0 D^-)' Dx + DG_1^{-1} BD^- Dx - DG_1^{-1} q = 0,$$
$$Q_0 x + Q_0 G_1^{-1} BD^- Dx - Q_0 G_1^{-1} q = 0.$$

$$x = P_0 x + Q_0 x = D^- Dx + Q_0 x = D^- D\Pi_0 x + Q_0 x$$

Structural decoupling: general regular DAE

$$I = \Pi_{\mu-1} + \Pi_{\mu-2}Q_{\mu-1} + \cdots + \Pi_0 Q_1 + Q_0,$$
$$x = \underbrace{\Pi_{\mu-1}x}_{=D^- u} + \underbrace{\Pi_{\mu-2}Q_{\mu-1}x}_{=v_{\mu-1}} + \cdots + \underbrace{\Pi_0 Q_1 x}_{=v_1} + \underbrace{Q_0 x}_{v_0},$$

- $u = D\Pi_{\mu-1}x$ satisfies the IERODE residing in $\mathbb{R}^{r_0}$,
- $v_0 = Q_0 x,\ v_i = \Pi_{i-1}Q_i x,\ i = 1, \ldots, \mu-1$ satisfy a triangular subsystem involving several differentiations if $\mu > 1$

Inherent explicit regular ODE (IERODE)

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}D^-u = D\Pi_{\mu-1}G_\mu^{-1}q,$$

Triangular (nilpotent) subsystem

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (Dv_1)' \\ \vdots \\ (Dv_{\mu-1})' \end{bmatrix} +$$

$$\begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ \\ \vdots \\ v_{\mu-1} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_0 \\ \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q - \begin{bmatrix} \mathcal{H}_0 D^- u \\ \\ \vdots \\ \mathcal{H}_{\mu-1} D^- u \end{bmatrix}$$

decoupling by arbitrary admissible projector functions

Inherent explicit regular ODE (IERODE)

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}D^-u = D\Pi_{\mu-1}G_\mu^{-1}q,$$

Triangular (nilpotent) subsystem

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (Dv_1)' \\ \vdots \\ (Dv_{\mu-1})' \end{bmatrix} +$$

$$\begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ \\ \vdots \\ v_{\mu-1} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_0 \\ \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q$$

complete decoupling

All coefficients are continuous and explicitly given in terms of the matrix function sequence:

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0 Q_1 D^- \\
\mathcal{N}_{0j} &:= -Q_0 P_1 \cdots P_{j-1} Q_j D^-, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1} Q_i Q_{i+1} D^-, \\
\mathcal{N}_{ij} &:= -\Pi_{i-1} Q_i P_{i+1} \cdots P_{j-1} Q_j D^-, \\
\mathcal{M}_{0j} &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{M}_j D \Pi_{j-1} Q_j, \\
\mathcal{M}_{ij} &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} \mathcal{M}_j D \Pi_{j-1} Q_j, \\
\mathcal{L}_0 &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1}, \\
\mathcal{L}_i &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1}, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1},
\end{aligned}$$

$$\mathcal{M}_j := \sum_{k=0}^{j-1} (I - \Pi_k)\{P_k D^-(D\Pi_k D^-)' - Q_{k+1} D^-(D\Pi_{k+1} D^-)'\} D\Pi_{j-1} Q_l D^-$$

Canonical projector function

Complete decoupling yields $\boxed{\Pi_{\mu-1} = \Pi_{can}}$,

$S_{can} = \operatorname{im} \Pi_{\mu-1}$ and $N_{can} = \ker \Pi_{\mu-1} = N_0 + \cdots + N_{\mu-1}$.

Maximal-size fundamental solution matrix $X(t, \bar{t})$

$$A(DX)' + BX = 0, \; X(\bar{t}) = \Pi_{can}(\bar{t}) \Longrightarrow$$
$$\operatorname{im} X(t, \bar{t}) = S_{can}(t), \quad \ker X(t, \bar{t}) = N_{can}(\bar{t}).$$

Consistent values

$$\mathcal{M}_{can,\, q}(t) = \{z + v_q(t) : z \in S_{can}(t)\}, \quad v_q(t) := \sum_{i=0}^{\mu-1} v_i(t) \in N_{can}(t)$$

Choose condensing matrix functions $\Gamma_d, \Gamma_i,\ i = 0, \dots, \mu - 1$ and generalized inverses such that

$$\operatorname{im} \Gamma_d = \mathbb{R}^d,\ \ker \Gamma_d = \ker D\Pi_{\mu-1}D^-, \quad \Gamma_d^- \Gamma_d = D\Pi_{\mu-1}D^-, \quad \Gamma_d \Gamma_d^- = I_d,$$
$$\operatorname{im} \Gamma_i = \mathbb{R}^{m-r_i},\ \ker \Gamma_i = \ker D\Pi_{i-1}Q_iD^-,\ \Gamma_i^- \Gamma_i = D\Pi_{i-1}Q_iD^-,\ \Gamma_i \Gamma_i^- = I_{m-}$$
$$\operatorname{im} \Gamma_0 = \mathbb{R}^{m-r_0},\ \ker \Gamma_0 = \ker Q_0,\ \Gamma_0^- \Gamma_0 = Q_0,\ \Gamma_0 \Gamma_0^- = I_{m-r_0}.$$

Γ_d^* forms a basis of $\operatorname{im} (D\Pi_{\mu-1}D^-)^*$ ...

Then the $m \times m$ -matrix function $K = \begin{bmatrix} \Gamma_d D\Pi_{\mu-1} \\ \Gamma_0 Q_0 \\ \Gamma_1 D\Pi_0 Q_1 \\ \vdots \\ \Gamma_{\mu-1} D\Pi_{\mu-2} Q_{\mu-1} \end{bmatrix}$ is nonsingular.

Condensing/transformation to T-canonical form/SCF

$$\boxed{A(Dx)' + Bx = q}$$

$\Downarrow$ $Kx = \tilde{x}$, refactorization, and multiplication by L

$$\begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix} \left(\begin{bmatrix} I_d & 0 & 0 \\ 0 & 0_{m-r_0} & 0 \\ 0 & 0 & I_{r_0-d} \end{bmatrix} \tilde{x} \right)' = \begin{bmatrix} W & 0 \\ 0 & I_{m-d} \end{bmatrix} \tilde{x} = Lq.$$

$$L = \begin{bmatrix} I_d & 0 \\ 0 & (I+M)^{-1} \end{bmatrix} K G_\mu^{-1}$$

N and M are blockwise upper triangular, N is nilpotent with index μ.
$M = 0$, if the involved projector functions are constant.

M. 1996, 2004; Ngai Wong, 2009 (Passivity test),

N. Banagaaya and W. Schilders, 2014,2017 (Index-aware model order reduction)

Set $Kx = \tilde{x} =: \begin{bmatrix} \eta \\ \star \end{bmatrix}, Lq =: \begin{bmatrix} p \\ \star \end{bmatrix}$.

Definition (Essential underlying ODE of the DAE (EUODE))

The regular ODE $\boxed{\eta' + W\eta = p}$ is named an EUODE of the given DAE.

U. Ascher and L. Petold, 1991, K. Balla and Vu Hoang Linh, Vu Hoang Linh and M. 2015

$$\boxed{u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}D^- u = D\Pi_{\mu-1}G_\mu^{-1}q,}$$

$$\Downarrow\ \eta = \Gamma_d u \qquad \Uparrow\ u = \Gamma_d^- \eta$$

$$\boxed{\eta' \underbrace{-\Gamma_d'\Gamma_d^-\eta + \Gamma_d D\Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}D^-\Gamma_d^-\eta}_{W\eta} = \underbrace{\Gamma_d D\Pi_{\mu-1}G_\mu^{-1}q}_{p}.}$$

Example

The DAE

$$\begin{aligned} x_1'(t) - \alpha x_1(t) - x_2(t) &= q_1(t), \\ x_3'(t) + x_2(t) &= q_2(t), \\ x_4'(t) + x_3(t) &= q_3(t), \\ x_5'(t) + x_4(t) &= q_4(t), \\ x_5(t) &= q_5(t), \end{aligned}$$

can be written with properly involved derivative as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x\right)' + \begin{bmatrix} -\alpha & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = q$$

Example (continuation)

This yields ([LMT ,p. 34/35]) $r_0 = r_1 = r_2 = r_3 = 4$, $r_4 = 5$, $\mu = 4$, and $d = 1$,

$$\Pi_{can} = \begin{bmatrix} 1 & 0 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D\Pi_{can} = \begin{bmatrix} 1 & 0 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D\Pi_{can}D^- = \begin{bmatrix} 1 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Gamma_d^* = \begin{bmatrix} 1 \\ 1 \\ -\alpha \\ \alpha^2 \end{bmatrix} \quad \Gamma_d^- = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example (continuation)

The IERODE reads

$$u_1' - \alpha u_1 = (D\Pi_{can} G_\mu^{-1} q)_1,$$
$$0 = 0,$$
$$0 = 0,$$
$$0 = 0.$$

The associated EUODE coincides with the first row.
The only nonzero row of the IERODE in detail is:

$$(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)' - \alpha(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5) = q_1 + q_2 - \alpha q_3 + \alpha^2 q_4$$

In contrast, the ODE for the component x_1:

$$x_1' - \alpha x_1 = q_1 + q_2 - (q_3 - (q_4 - q_5')')'.$$

Theorem (Vu Hoang Linh and M. 2015, 2016)

(1) *Each EUODE is a condensed IERODE by means of* $\eta = \Gamma_d u, \; u = \Gamma_d^- \eta.$

(2) *The component* $u = D\Pi_{can}x$ *is the only genuine solution part which satisfies an regular ODE being untroubled by derivatives of* q.

(3) *There exists an EUODE featuring the same Lyapunov spectrum as the DAE itself.*[a]

[a] If there is no restriction for Γ_d^* to represent an ONB!

$\Longrightarrow$ spectrum preserving EUODEs, Lyapunov regular DAEs.

The Lyapunov spectrum of a DAE is defined via minimal size fundamental solution matrices

Nguyen Dinh Cong and Hoang Nam, 2004; Vu Hoang Linh and V. Mehrmann, 2012; Vu Hoang Linh and M., 2015

Outline

The above assertions – and also the following ones as well – apply to standard form DAEs and their adjoints,

$$\boxed{Ex' + Fx = 0,} \quad \boxed{-(E^*y)' + F^*y = 0,}$$

by means of proper factorizations $E = AD$, and reformulations

$$\boxed{A(Dx)' + \underbrace{(F - AD')}_{=B} x = 0,} \quad \boxed{-D^*(A^*y)' + \underbrace{(F^* - D^{*\prime} A^*)}_{=B^*} y = 0,}$$

which does not affect the canonical projectors Π_{can} and $\Pi_{*\,can}$, and the entire flows.

However, for standard form DAEs,

- the IERODE depends on the chosen factorization,
- the EUODE depends on the chosen factorization, and, additionally, on the chosen basis Γ_d^* of $\operatorname{im}(D\Pi_{can}D^-)^*$.

Consider the regular DAE and its adjoint,

$$\boxed{A(Dx)' + Bx = 0,} \qquad \boxed{-D^*(A^*y)' + B^*y = 0.}$$

Theorem (Vu Hoang Linh and M. 2015)

1. *The adjoint DAE is also regular and shows the same characteristic values and index.*
2. $A^* \Pi_{*\,can} A^{*\,-} = (D \Pi_{can} D^-)^*$.
3. *The IERODEs of the DAE and the adjoint DAE are adjoint to each other exactly if $D\Pi_{can}D^-$ is time-invariant.*
4. *Each solution pair $x = \Pi_{can} D^- u$ and $y = \Pi_{*\,can} A^{*\,-} v$ satisfies the generalized Lagrange identity*

$$\boxed{< D(t)x(t), A(t)^* y(t) >=< u(t), v(t) >= \quad constant}$$

5. *The EUODEs of the DAE and the adjoint DAE built by consistent condensing transformations $\Gamma_{*\,d} = \Gamma_d^{-\,*}$ are adjoint to each other.*

Theorem (Vu Hoang Linh and M. 2015)

If $AD\Pi_{can}$ and $\Pi_{can}(AD)^-$ are bounded and there is a Γ_d so that $\Gamma_d D\Pi_{can}$ and $\Pi_{can}D^-\Gamma_d^-$ are bounded, too, then the DAE and its adjoint are Lyapunov regular at the same time.
In this case, the Perron identity is given, that is,

$$\boxed{\alpha_i + \alpha_{*\,i} = 0, \quad i = 1, \ldots, d.}$$

Example

Let $\gamma \in \mathbb{R}$ and β be a smooth function with no zeros. We study the DAE and its adjoint

$$\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix} x\right)' + \begin{bmatrix}\gamma & 0\\ 1 & -\frac{1}{\beta}\end{bmatrix} x = 0, \tag{4}$$

$$-\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix} y\right)' + \begin{bmatrix}\gamma & 1\\ 0 & -\frac{1}{\beta}\end{bmatrix} y = 0. \tag{5}$$

The flows are given by

$$x(t) = \begin{bmatrix}1\\ \beta(t)\end{bmatrix} \mathrm{e}^{-\gamma t} c_x, \qquad y(t) = \begin{bmatrix}1\\0\end{bmatrix} \mathrm{e}^{\gamma t} c_y, \quad t \in [0, \infty).$$

With $f := (1 + \beta^2)^{\frac{1}{2}}$, the Lyapunov exponents of (4) and (5) are

$$\boxed{\alpha_1 = \chi[x] = \chi[f] - \gamma, \quad \alpha_{*1} = \chi[y] = \gamma.}$$

Example (Continuation)

We have $\boxed{\alpha_1 = \chi[x] = \chi[f] - \gamma, \quad \alpha_{*1} = \chi[y] = \gamma}$ and

$$\Pi_{can} = \begin{bmatrix} 1 & 0 \\ \beta & 0 \end{bmatrix}, \ D\Pi_{can}D^- = 1, \quad \Pi_{*\,can} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A^*\Pi_{*\,can}A^{*\,-} = 1$$

The IERODEs are

$$u' + \gamma u = 0, \quad v' - \gamma v = 0.$$

With $\Gamma_d = 1$, the EUODEs coincide with the IERODEs.
With $\Gamma_d = f$ and $\Gamma_{*\,d} = \frac{1}{f}$, one obtains the EUODEs

$$\eta' + (\gamma - \frac{f'}{f})\eta = 0, \quad \eta(t) = f(t)\mathrm{e}^{-\gamma t}c_\eta, \quad \chi[\eta] = \chi[f] - \gamma,$$
$$\zeta' - (\gamma - \frac{f'}{f})\zeta = 0, \quad \zeta(t) = f(t)\mathrm{e}^{\gamma t}c_\zeta, \quad \chi[\zeta] = \chi[f] + \gamma.$$

Outline

$$f\left(\frac{\mathrm{d}}{\mathrm{dt}}(\sigma(x(t),t)), x(t), t\right) = 0,$$

$f(y,x,t) \in \mathbb{R}^m$, $\sigma(x,t) \in \mathbb{R}^n$ for $y \in \mathbb{R}^n,\ x \in \mathcal{D}_f \subseteq \mathbb{R}^m,\ t \in \mathcal{I}_f \subseteq \mathbb{R}$,
with partial derivatives $f_y, f_x, \sigma_x, \sigma_t$.
$f_y(y,x,t)$ is singular everywhere, and $\ker f_y(y,x,t) \oplus \operatorname{im} \sigma_x(x,t) = \mathbb{R}^n$.

Theorem (Local solvability)

Let $\bar{y} \in \mathbb{R}^n,\ \bar{x} \in \mathcal{D}_f,\ \bar{t} \in \mathcal{I}_f$ be such that $f(\bar{y},\bar{x},\bar{t}) = 0$ and the matrix

$$f_y(\bar{y},\bar{x},\bar{t})\sigma_x(\bar{x},\bar{t}) + f_x(\bar{y},\bar{x},\bar{t})Q_0(\bar{x},\bar{t})$$

is nonsingular. Then there is a unique continuous function $x_* : \mathcal{I}_* \subseteq \mathcal{I}_f \to \mathcal{D}_f$ such that $\sigma(x_*(\cdot),\cdot)$ belongs to $\mathcal{C}^1$ and the DAE is satisfied for all $t \in \mathcal{I}_*$

$Q_0(x,t)$ is a projector matrix onto $\ker \sigma_x(x,t)$

Matrix function sequence (built pointwise for $x \in \mathcal{D}_f$, $t \in \mathcal{I}_f$,...)

We set $D := \sigma_x$,
$G_0 := f_y D$, $B_0 := f_x$, Q_0 projector function such that $\operatorname{im} Q_0 = \ker D$,
$P_0 := I - Q_0$, $\Pi_0 := P_0$ $r_0 = \operatorname{rank} G_0$,
and form, as long as the expressions exist, for $i \geq 0$,

$$G_{i+1} := G_i + B_i Q_i, \quad r_{i+1} := \operatorname{rank} G_{i+1}, \quad N_{i+1} := \ker G_{i+1},$$

Q_{i+1} projector function: $\operatorname{im} Q_{i+1} = N_{i+1}$, $N_0 + \cdots + N_i \subseteq \ker Q_{i+1}$,

$$P_{i+1} := I - Q_{i+1},$$

$$\Pi_{i+1} := \Pi_i P_{i+1},$$

$$B_{i+1} := B_i P_i - G_{i+1} D^- \underbrace{(D \Pi_{i+1} D^-)'}_{\text{total derivative in jet variables}} D \Pi_i.$$

D^- is a generalized inverse of D with $D^- D = P_0$.

Definition (Regular DAE)

The DAE is said to be regular on the open set $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$, if there are a number $\mu \in \mathbb{N}$, and an matrix function sequence being admissible up to level μ such that

$$0 \leq r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m.$$

The number μ is named tractability index, and the ranks $r_0, \ldots, r_{\mu-1}$ are said to be characteristic values of the DAE on $\mathcal{G}$.

- $\mathcal{G}$ is named regularity region. Each open subset $\tilde{\mathcal{G}} \subset \mathcal{G}$ is also a regularity region. It inherits the characteristics of $\mathcal{G}$.

- Maximal regularity region: On the borders some of the ranks r_i change.
- In general, we do not expect a DAE to be regular on its entire definition domain. It is rather generic that the definition domain decomposes in several (maximal) regularity regions $\mathcal{G}_1, \mathcal{G}_2, \ldots$. Solutions may cross the borders of these regularity regions, and, in particular, undergo bifurcations.

Example (Several regularity region)

The semi-explicit DAE

$$\begin{aligned} x_1'(t) - x_3(t) &= 0, \\ x_2(t)(1 - x_2(t)) - \gamma(t) &= 0, \\ x_1(t)x_2(t) + x_3(t)(1 - x_2(t)) - t &= 0, \end{aligned}$$

with $m = 3$, $n = 1$, $\sigma(x, t) = x_1$, $\gamma(t) = \frac{1}{4} - t^2$, and

$$f(y, x, t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -x_3 \\ x_2(1 - x_2) - \gamma(t) \\ x_1x_2 + x_3(1 - x_2) - t \end{bmatrix}, \quad y \in \mathbb{R},\ x \in \mathbb{R}^3,\ t \in \mathbb{R},$$

yields $\quad \det G_1(x, t) = (1 - 2x_2)(1 - x_2)$.

Example (Continuation)

$\det G_1(x,t) = (1-2x_2)(1-x_2)$. Owing to the zeros $x_2 = \frac{1}{2}$ and $x_2 = 1$, the definition domain $\mathcal{D}_f \times \mathcal{I}_f = \mathbb{R}^3 \times \mathbb{R}$ decomposes into the three open sets

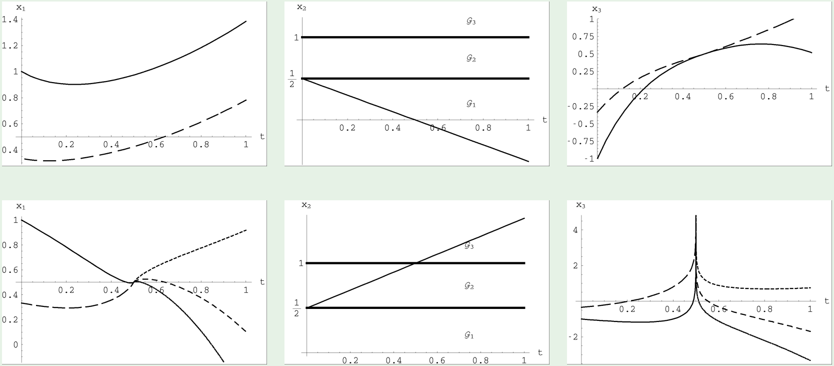
$$\begin{aligned}
\mathcal{G}_1 &:= \Big\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ x_2 < \frac{1}{2}\Big\}, \\
\mathcal{G}_2 &:= \Big\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ \frac{1}{2} < x_2 < 1\Big\}, \\
\mathcal{G}_3 &:= \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ 1 < x_2\},
\end{aligned}$$

such that $\mathcal{D}_f \times \mathcal{I}_f$ is the closure of $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

The DAE is regular with tractability index one on each region $\mathcal{G}_\ell$, $\ell = 1, 2, 3$.

Example (Continuation)

Two solutions starting in $(1, \frac{1}{2}, -1)$ (solid line), and two solutions starting in $(\frac{1}{3}, \frac{1}{2}, -\frac{1}{3})$ (dashed line):



$$f(\frac{\mathrm{d}}{\mathrm{dt}}(\sigma(x(t),t)),x(t),t)=0$$

For each function $x_* \in \mathcal{C}(\mathcal{I}_*,\mathbb{R}^m)$, $\mathcal{I}_* \subseteq \mathcal{I}_f$, with $(x_*(t),t) \in \mathcal{D}_f \times \mathcal{I}_*$, such that $\sigma(x_*(\cdot),\cdot) \in \mathcal{C}^1(\mathcal{I}_*,\mathbb{R}^n)$, we may consider the linear DAE

$$A_*(t)(D_*(t)x(t))' + B_*(t)x(t) = q(t), \quad t \in \mathcal{I}_*,$$

the continuous coefficients of which are given by

$$\begin{aligned} A_*(t) &:= f_y(\frac{\mathrm{d}}{\mathrm{dt}}(\sigma(x_*(t),t)),x_*(t),t), \\ D_*(t) &:= d_x(x_*(t),t), \\ B_*(t) &:= f_x(\frac{\mathrm{d}}{\mathrm{dt}}(\sigma(x_*(t),t)),x_*(t),t), \quad t \in \mathcal{I}_*. \end{aligned}$$

We stress that the linearization function x_* is not necessarily a solution!

Definition (Linearization)

The linear DAE is called the linearization of the nonlinear DAE along x_*.

The question concerning the structure of the nonlinear DAE can be traced back to linear problems in the following sense:

Theorem (Linearization)

Let $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be open and connected. The following three assertions are equivalent:

1. *The nonlinear DAE is regular on $\mathcal{G}$.*
2. *Each linear DAE representing a linearization along a sufficiently smooth function with graph in $\mathcal{G}$ is regular.*
3. *All linear DAEs arising as linearizations along sufficiently smooth functions with graph in $\mathcal{G}$ are regular with uniform tractability index and characteristics.*

Outline

• Each regular DAE can be completely decoupled according to its structure and also transformed into standard canonical form (SCF). The projector based analysis provides the canonical projector function Π_{can} associated with the decomposition,

$$S_{can}(t) \oplus N_{can}(t) = \mathbb{R}^m, \quad t \in \mathcal{I},$$

$$\boxed{\Pi_{can} = P_0 \cdots P_{\mu-1}, \quad N_{can} = N_0 + \cdots + N_{\mu-1}}$$

• The IERODE of a regular DAE and its condensed version EUODE represent the only genuine regular ODE-part of the DAE, which remains unaffected by derivatives of the inhomogenuity.

• Nonetheless DAEs are different! Open questions concerning solvability of nonlinear DAEs and in view of the numerical treatment!

• Rank conditions: Are they a matter of limits or a matter of capability?

Example: Least-squares collocation/ linear time-varying index-3 DAE

The index-3 DAE system

$$\begin{aligned} x_2'(t) + x_1(t) &= q_1(t), \\ t\eta x_2'(t) + x_3'(t) + (\eta+1)x_2(t) &= q_2(t), \\ t\eta x_2(t) + x_3(t) &= q_3(t), \quad t \in [0,1]. \end{aligned}$$

can be cast into the proper form by setting

$$A = \begin{bmatrix} 1 & 0 \\ t\eta & 1 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\eta & 0 \\ 0 & t\eta & 1 \end{bmatrix}.$$

$x(t) = \begin{bmatrix} e^{-t}\sin t, \\ e^{-2t}\sin t, \\ e^{-t}\cos t, \end{bmatrix}$ serves as exact solution and this determines q.

Example

The most sensible component concerning numerical computations is x_1.

Table : Collocation results, $\eta = -2$, $N = 3$

n	Standard: $\|x_1 - p_1\|_\infty$	Least-squares: $\|x_1 - p_1\|_\infty$
20	3.74e+006	3.26e–4
40	9.84e+016	7.52e–5
80	3.51e+038	1.81e–5
160	2.04e+082	4.42e–6
320	2.98e+170	1.11e–6
640	3.06e+307	1.06e–6

✓

Thank you for your attention !

Literature:

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▼ R. März: *New answers to an old question in the theory of differential-algebraic equations: Essential underlying ODE versus inherent ODE.* J. Comput. Appl. Math. 316, 271-286, 2017.

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Example: Singular type 0 critical point

The IERODE of the DAE

$$\begin{bmatrix} 0 & \beta(t) \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) \right)' + \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} x(t) = q(t)$$

reads

$$x_2{}'(t) = \frac{-2}{\beta(t)} x_2(t) + \frac{1}{\beta(t)} (q_1(t) - 2q_2(t)),$$
$$x_1(t) = x_2(t) + q_2(t).$$

In the particular case of $\beta(t) = t$, this ODE has a singularity of the first kind at $t = 0$, and all solution grow unboundedly if t approaches to zero. Here is an associated matrix function sequence:

$$G_0 = \begin{bmatrix} 0 & \beta(t) \\ 0 & 0 \end{bmatrix}, Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 2 & \beta(t) \\ 1 & 0 \end{bmatrix}.$$

Example: Harmless type 0 critical point

The DAE

$$\begin{bmatrix}0 & \beta(t)\\ 0 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}x(t)\right)' + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t) = q(t)$$

has the harmless solution

$$x_1(t) = -\beta(t){x_2}'(t) + q_1(t), \qquad x_2(t) = q_2(t).$$

Here is an (quasi-)admissible matrix function sequence:

$$G_0 = \begin{bmatrix}0 & \beta(t)\\ 0 & 0\end{bmatrix}, Q_0 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, G_1 = \begin{bmatrix}1 & \beta(t)\\ 0 & 0\end{bmatrix}, Q_1 = \begin{bmatrix}0 & \beta(t)\\ 0 & 1\end{bmatrix},$$

$$G_2 = \begin{bmatrix}1 & \beta(t)\\ 0 & 1\end{bmatrix}.$$

Example: Type 1-B and 2-A critical points

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}x(t)\right)' + \begin{bmatrix}0 & 0 & \beta(t)\\1 & 1 & 0\\\gamma(t) & 0 & 0\end{bmatrix}x(t) = q(t)$$

This DAE means in detail

$$x_1(t) = \frac{1}{\gamma(t)}q_3(t),$$

$$x_2{}'(t) = -x_2(t) + q_2(t) - \frac{1}{\gamma(t)}q_3(t),$$

$$\beta(t)x_3(t) = q_1(t) - \left(\frac{1}{\gamma(t)}q_3(t)\right)'.$$

The matrix function sequence shows: $r_0 = r_1 = 2$,
$N_0(t) \cap N_1(t) = \{z \in \mathbb{R}^3 : z_1 = 0, z_2 = 0, \beta(t)z_3 = 0\} = \{0\} \leftrightarrow \beta(t) \neq 0$,
$\det G_2(t) = -\beta(t)\gamma(t)$.

Example (Smooth projector function, but global bases do not exist)

$$M(x) = [x_1\, x_2\, x_3], \quad x \in \mathbb{R}^3 \setminus \{0\},$$

$$\ker M(x) = \{z \in \mathbb{R}^3 : x_1 z_1 + x_2 z_2 + x_3 z_3 = 0\},$$

$$Q(x) = \frac{1}{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{bmatrix},$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} -\frac{x_2}{x_1} & -\frac{x_3}{x_1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{if } x_1 \neq 0,$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{x_3}{x_2} \\ 0 & 1 \end{bmatrix} \quad \text{if } x_1 = 0,\ x_2 \neq 0,$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{if } x_1 = 0,\ x_2 = 0,\ x_3 \neq 0,$$

Definition

The ODE $\boxed{x'(t) - B(t)x(t) = 0}$ is Lyapunov regular $\overset{def}{:}$ if

$$\alpha_1, \dots, \alpha_m \in \mathbb{R}, \quad \sum_{i=1}^{m} \alpha_i = \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{trace}(-B(s))\mathrm{d}s$$

Example

The ODE $\boxed{x'(t) - 2t\,x(t) = 0}$ fails to be Lyapunov regular:

$$m = 1,\ x(t) = \mathrm{e}^{t^2} c,$$

$$c \neq 0 \Longrightarrow \alpha_1 = \chi(x) = \limsup_{t\to\infty} \frac{1}{t} \ln|x(t)| = \infty$$