Optimal control of regular differential-algebraic equations

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Differentiability of the input

"Nilpotent" part of the DAE:

$$\frac{\mathrm{d}}{\mathrm{d}t} N x_N(t) = x_N(t) + b_N u(t)$$

$$\boldsymbol{u} \in \mathcal{W}^{\boldsymbol{n_N-1,1}}_{\mathbf{loc}} \quad \implies \; x_N \overset{\text{ae}}{=} -\sum_{i=0}^{n_N-1} N^i b_N u^{(i)}$$

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Index of a regular DAE

$$\nu = \operatorname{nil\,ind} N = \min\{i \in \mathbb{N} \mid N^i = 0\} \leq n_N$$

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***Input* index of a regular DAE**

$$\omega_u := \min\{i \in \mathbb{N}_0 \mid N^i b_N = 0\} \leq \nu$$

Differentiability of the input

"Nilpotent" part of the DAE:

$$\frac{\mathrm{d}}{\mathrm{d}t} N x_N(t) = x_N(t) + b_N u(t)$$

$$\implies x_N \overset{\text{ae}}{=} -\sum_{i=0}^{\omega_u - 1} N^i b_N u^{(i)}$$

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Differentiability condition

Input index of a regular DAE

$$\omega_u := \min\{i \in \mathbb{N}_0 \mid N^i b_N = 0\}$$

In this presentation, we will only consider systems with $\omega_u \geq 2$

Theorem

$$(x, u) \in \mathfrak{B}_{[E,A,b]}$$
$$\implies u \in \mathcal{W}^{\omega_u - 1, 1}_{loc} \text{ [i. e. } u \text{ is } (\omega_u - 1) \text{ times differentiable a. e.]}$$
$$x_N \overset{ae}{=} -\sum_{i=0}^{\omega_u - 1} N^i b_N u^{(i)}$$

Construction of an augmented system

Idea

u must be differentiable $(\omega_u - 1)$ times a. e.
⇝ "true" input is not u, but $u^{(\omega_u-1)}$
⇝ introduce $u, \ldots, u^{(\omega_u-2)}$ as new state variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_{n_J} & 0 \\ 0 & N \end{bmatrix}\begin{pmatrix} x_J \\ x_N \end{pmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_N} \end{bmatrix}\begin{pmatrix} x_J \\ x_N \end{pmatrix} + \begin{pmatrix} b_J \\ b_N \end{pmatrix} u$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ \dot{u} \\ \vdots \\ u^{(\omega_u-2)} \end{pmatrix} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}\begin{pmatrix} u \\ \dot{u} \\ \vdots \\ u^{(\omega_u-2)} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u^{(\omega_u-1)}$$

The augmented system

Augmented system $[\widehat{E}, \widehat{A}, \widehat{b}]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & I_{\omega_u-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}\widehat{x} = \begin{bmatrix} J & b_J & 0 & 0 \\ 0 & 0 & I_{\omega_u-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_N & Nb_N, \dots, N^{\omega_u-1}b_N & I_{n_N} \end{bmatrix}\widehat{x} + \begin{pmatrix} 0_{n_J} \\ 0_{\omega_u-2} \\ 1 \\ N^{\omega_u-1}b_N \end{pmatrix}\widehat{u}$$

Proposition

$$\left(\begin{pmatrix} x_J \\ x_N \end{pmatrix}, u\right) \in \mathfrak{B}_{[E,A,b]} \iff \left(\underbrace{\begin{pmatrix} x_J \\ u \\ \dot{u} \\ \vdots \\ u^{(\omega_u-2)} \\ x_N \end{pmatrix}}_{=:\widehat{x}}, u^{(\omega_u-1)}\right) \in \mathfrak{B}_{[\widehat{E},\widehat{A},\widehat{b}]}$$

Example

Nominal DAE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x_J \\ x_{N,1} \\ x_{N,2} \\ x_{N,3} \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x_J \\ x_{N,1} \\ x_{N,2} \\ x_{N,3} \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} u$$

Input index $\omega_u = 3$, augmented system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x_J \\ u \\ \dot{u} \\ x_{N,1} \\ x_{N,2} \\ x_{N,3} \end{pmatrix} = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x_J \\ u \\ \dot{u} \\ x_{N,1} \\ x_{N,2} \\ x_{N,3} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} \widehat{u}$$

Optimal control problem (OCP)

Performance index

$$J(x,u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top \begin{bmatrix} Q & h \\ h^\top & r \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t$$

Optimal value (free endpoint)

$$J^*(Ex^0) = \inf\{J(x,u) \mid (x,u) \in \mathfrak{B}_{[E,A,b]}, (Ex)(0) = Ex^0\}$$

Solution of the optimal control problem

Idea

Use explicit solution for x_N and ODE part of the augmented system to derive an equivalent **ODE** OCP

Theorem

$$J^*(Ex^0) = \inf \int_0^\infty \begin{pmatrix} \widehat{x}_1(t) \\ \widehat{u}(t) \end{pmatrix}^\top \begin{bmatrix} \widehat{Q} & \widehat{h} \\ \widehat{h}^\top & \widehat{r} \end{bmatrix} \begin{pmatrix} \widehat{x}_1(t) \\ \widehat{u}(t) \end{pmatrix} \mathrm{d}t$$

$$\text{s. t. } \dot{\widehat{x}}_1(t) = \widehat{A}_{11}\widehat{x}_1(t) + \widehat{b}_1\widehat{u}(t), \; \widehat{x}_1(0) = F^{-1}Ex^0$$

$\widehat{Q}$, $\widehat{h}$, $\widehat{r}$, F calculated algebraically from the DAE OCP

The augmented system (rep.)

Augmented system $[\widehat{E}, \widehat{A}, \widehat{b}]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & I_{\omega_u-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}\widehat{x} = \begin{bmatrix} J & b_J & 0 & 0 \\ 0 & 0 & I_{\omega_u-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_N & Nb_N, \dots, N^{\omega_u-1}b_N & I_{n_N} \end{bmatrix}\widehat{x} + \begin{pmatrix} 0_{n_J} \\ 0_{\omega_u-2} \\ 1 \\ N^{\omega_u-1}b_N \end{pmatrix}\widehat{u}$$

Proposition

$$\left(\begin{pmatrix} x_J \\ x_N \end{pmatrix}, u\right) \in \mathfrak{B}_{[E,A,b]} \iff \left(\underbrace{\begin{pmatrix} x_J \\ u \\ \dot{u} \\ \vdots \\ u^{(\omega_u-2)} \\ x_N \end{pmatrix}}_{=:\widehat{x}}, u^{(\omega_u-1)}\right) \in \mathfrak{B}_{[\widehat{E},\widehat{A},\widehat{b}]}$$

Example

Nominal DAE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{pmatrix}x_{N,1}\\ x_{N,2}\end{pmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_{N,1}\\ x_{N,2}\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} u$$

Performance index:

$$J(x,u) = \int_0^\infty \begin{pmatrix}x_{N,1}\\ x_{N,2}\\ u\end{pmatrix}^\top \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{pmatrix}x_{N,1}\\ x_{N,2}\\ u\end{pmatrix} \mathrm{d}t$$

Input index $\omega_u = 2$, augmented system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boxed{1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{pmatrix}\boxed{u}\\ x_{N,1}\\ x_{N,2}\end{pmatrix} = \begin{bmatrix}\boxed{0} & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{pmatrix}\boxed{u}\\ x_{N,1}\\ x_{N,2}\end{pmatrix} + \begin{pmatrix}\boxed{1}\\ 1\\ 0\end{pmatrix}\widehat{u}$$

Example (cont.)

Performance index of the augmented system:

$$\widehat{J}(u, \widehat{u}) = \int_0^\infty \begin{pmatrix} u(t) \\ \widehat{u}(t) \end{pmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u(t) \\ \widehat{u}(t) \end{pmatrix} \mathrm{d}t$$

$$\dot{u}(t) = \widehat{u}(t), \; u(0) = -x_2^0$$

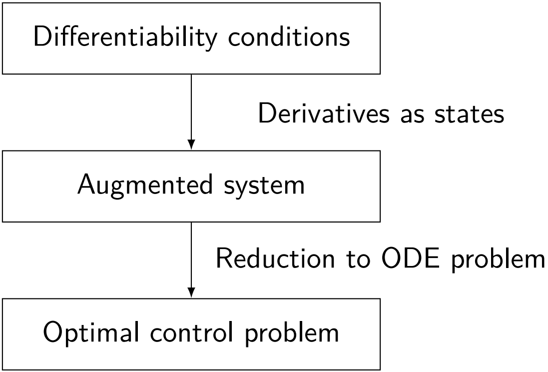
Optimal control:

$$\dot{u}^*(t) = -u^*(t), \quad u^*(0) = -x_2^0$$

Optimal feedback for the original DAE system:

$$u(\cdot) = x_{N,1}(\cdot)$$

Summary


Differentiability conditions
Derivatives as states
Augmented system
Reduction to ODE problem
Optimal control problem