Performance Estimates for Multiobjective Model Predictive Control Schemes

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11. Elgersburg Workshop

UNIVERSITÄT BAYREUTH

Where Does Multiobjective MPC Occur?

Cooperative handling of

- One system with multiple (conflicting) objectives.
- Multiple systems (players) with at least one objective (each).

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Example [Müller, Reble, Allgöwer '12]:

Six two-dimensional nonlinear systems (dynamically decoupled)

$$x_i^+ = \begin{pmatrix} 0.9 & 0.1 \\ -0.2 & 0.8 \end{pmatrix} x_i + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_i + 0.1 \begin{pmatrix} x_{i,2}^2 \\ x_{i,1}^2 \end{pmatrix},$$

$i \in \{1, \ldots, 6\}$, with state and control constraints, coupling constraints $\|x_3 - x_4\| \leq 4$ and coupled stage costs

$$\ell_i(x,u) = x_i^T Q_i x_i + u_i^T R_i u_i + \sum_{j \in \mathcal{N}_i} (C_i x_i - C_j x_j)^T Q_{ij} (C_i x_i - C_j x_j).$$

Task: Stabilize the origin.

Topics of the Talk

- Multiobjective optimization and optimality
- A multiobjective MPC algorithm
- Statements on the infinite-horizon performance of the MPC closed-loop for each objective
- Trajectory convergence

Setting

Control system in discrete time

$$x^{\mathbf{u}}(n+1) = f(x^{\mathbf{u}}(n), u(n)), \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ n \in \mathbb{N}_0,$$

admissible state and control spaces $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$.

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admissible state and control spaces $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$.

$$J_i^N(x_0, \mathbf{u}) := \sum_{k=0}^{N-1} \ell_i(x^{\mathbf{u}}(k, x_0), u(k)) + F_i(x^{\mathbf{u}}(N, x_0)),\ i \in \{1, \dots, s\},$$

stage costs $\ell_i : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, terminal costs $F_i : \mathbb{X}_0 \to \mathbb{R}$.

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stage costs $\ell_i : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, terminal costs $F_i : \mathbb{X}_0 \to \mathbb{R}$.

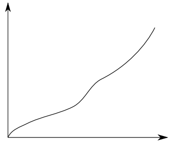
Task: $' \min_{\mathbf{u}} ' J^N(x_0, \mathbf{u}) = ' \min_{\mathbf{u}} ' \left(J_1^N(x_0, \mathbf{u}), \dots, J_s^N(x_0, \mathbf{u}) \right)$

$$\text{s.t.} \left. \begin{aligned} & x^{\mathbf{u}}(k, x_0) \in \mathbb{X},\ k = 0, \dots, N-1, \\ & x^{\mathbf{u}}(N, x_0) \in \mathbb{X}_0, \\ & u(k) \in \mathbb{U},\ k = 0, \dots, N-1. \end{aligned} \right\} \mathbf{u} \in \mathbb{U}^N(x_0)$$

Assumptions

Desired equilibrium $x_* = f(x_*, u_*)$.
For each i and all x:

$$\min_{u \in \mathbb{U}} \ell_i(x, u) \geq \alpha_{\ell,i}(\|x - x_*\|),\ \alpha_{\ell,i} \in \mathcal{K}.$$



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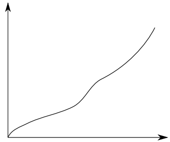

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The terminal constraint satisfies:

- $x_* \in \mathbb{X}_0$; $F_i(x) \geq 0\ \forall\ i, \forall\ x \in \mathbb{X}_0$.
- $\exists\ \kappa : \mathbb{X}_0 \to \mathbb{U} :\ f(x, \kappa(x)) \in \mathbb{X}_0\ \forall\ x \in \mathbb{X}_0$.
- $F_i(f(x,\kappa(x))) + \ell_i(x,\kappa(x)) \leq F_i(x)\ \forall\ i, \forall\ x \in \mathbb{X}_0$.

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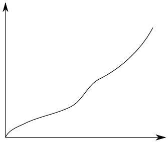

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- $\mathbb{U}$ compact, $\mathbb{X}$ and $\mathbb{X}_0$ closed.
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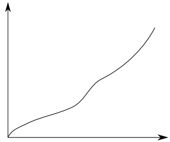

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No assumption on coupling structure.

Optimality in Multiobjective (MO) Optimization

What does $'\min'_{\mathbf{u}\in\mathbb{U}^N(x_0)} \left(J_1^N(x_0,\mathbf{u}),\ldots,J_s^N(x_0,\mathbf{u})\right)$ mean?

$\rightsquigarrow$ Concept of optimality:

A sequence $\mathbf{u}^\star \in \mathbb{U}^N(x_0)$ is called Pareto optimal if there is no $\mathbf{u} \in \mathbb{U}^N(x_0)$ such that

$$\forall\ i \in \{1,\ldots,s\}:\ J_i^N(x_0,\mathbf{u}) \leq J_i^N(x_0,\mathbf{u}^\star) \text{ and}$$
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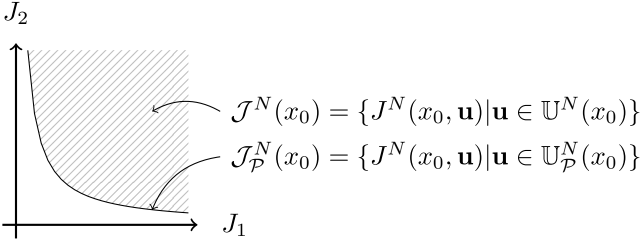

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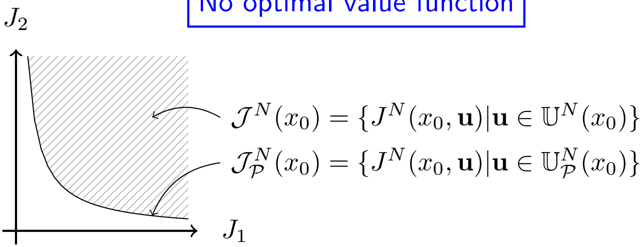
No optimal value function



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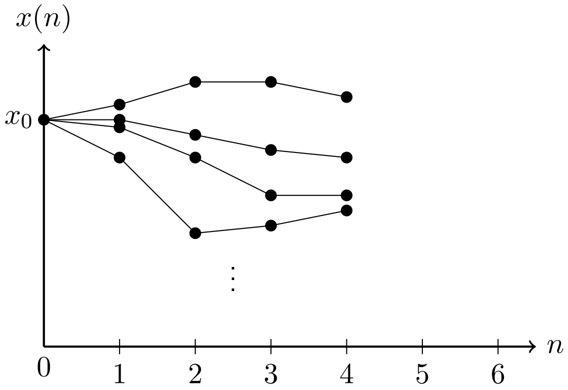

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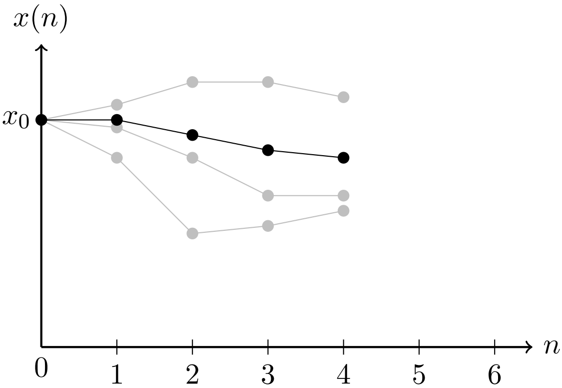

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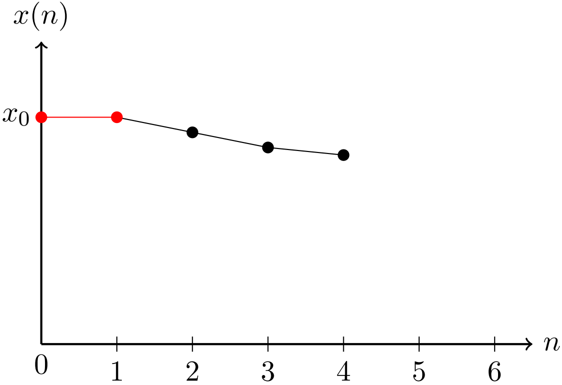

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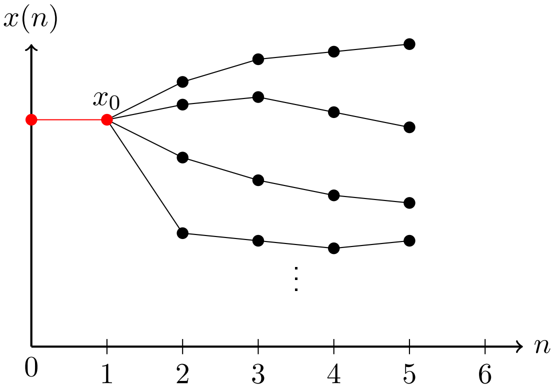

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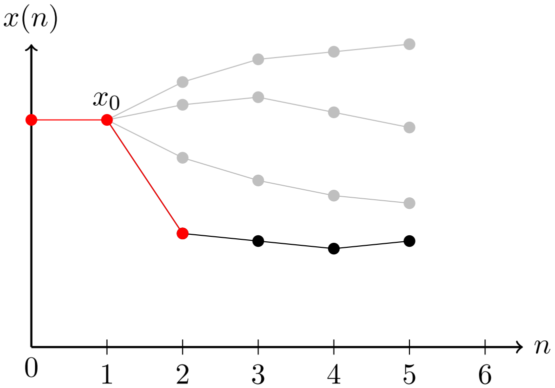

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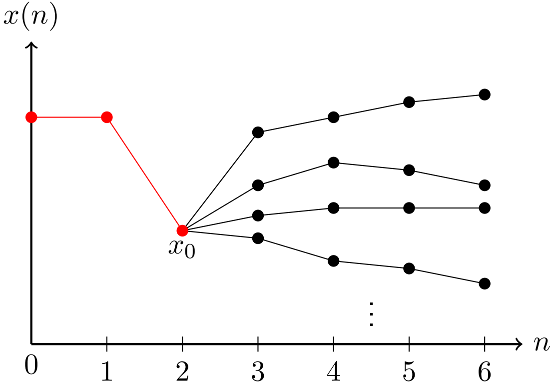

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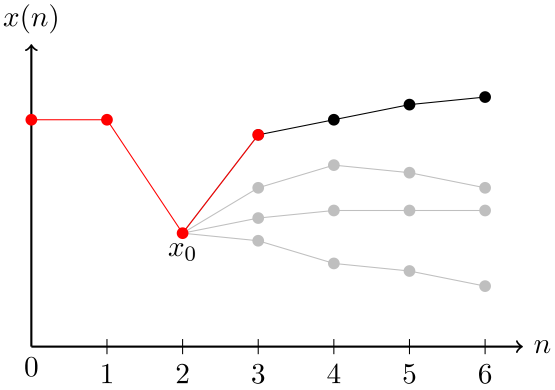

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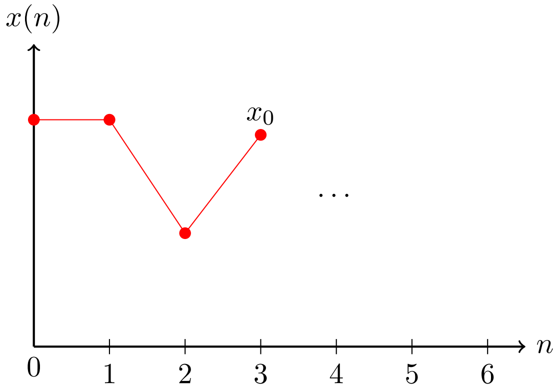

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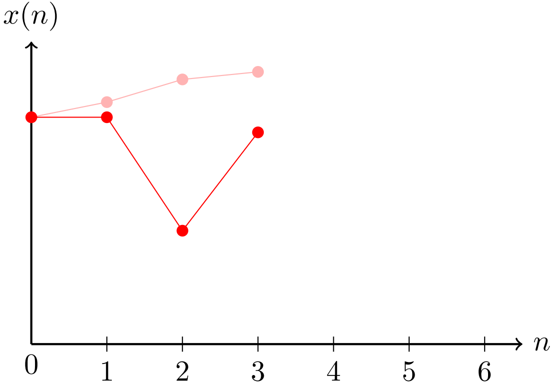

MO MPC with Terminal Conditions

Algorithm:

0) Measure $x(n)$ and choose $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$. Go to 2).

1) Measure $x(n)$. Choose $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ such that

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq J_i^N\left(x(n), \mathbf{u}^{N}_{x(n)}\right) \quad \forall\ i.$$

2) For $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N, x(n))$ set

$$\mathbf{u}^N_{x(n+1)} := \left(u^{\star,N}_{x(n)}(1), \ldots, u^{\star,N}_{x(n)}(N-1), \kappa(x)\right).$$

3) Apply the feedback $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, set $n = n+1$ and go to 1).

Performance Theorem for MO MPC

Theorem

For each objective function i the MPC-feedback defined in the algorithm has the following performance

$$J_i^\infty(x_0, \mu^N) := \lim_{K\to\infty} \sum_{k=0}^{K-1} \ell_i(x(k,x_0), \mu^N(x(k,x_0))) \leq J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N}),$$

in which $\mathbf{u}_{x_0}^{\star,N}$ is the PO chosen in step 0).

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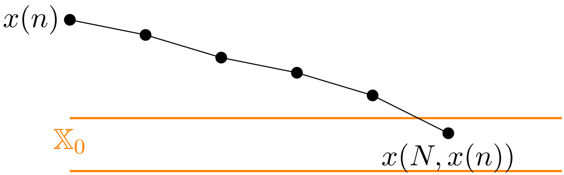
in which $\mathbf{u}_{x_0}^{\star,N}$ is the PO chosen in step 0).

- Upper bound for performance can be estimated a priori by a multiobjective optimization wrt horizon N.
- Approximate the whole Pareto set in step 0). Only calculate one solution in subsequent steps.

Proof

Step 1: $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$



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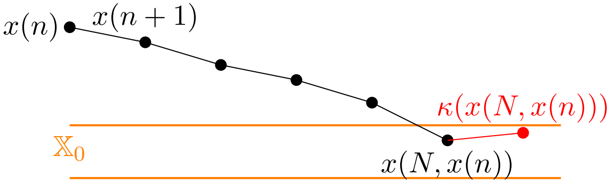
2) For $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N, x(n))$ set
$\mathbf{u}^N_{x(n+1)} := \left(u^{\star,N}_{x(n)}(1), \ldots, u^{\star,N}_{x(n)}(N-1), \kappa(x)\right)$.

Assumption:

$F_i(f(x,\kappa(x))) + \ell_i(x,\kappa(x)) \le F_i(x)$

Proof

Step 1: $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$, then $\mathbf{u}^{\star,N}_{x(n)}(\cdot+1) \in \mathbb{U}^{N-1}_{\mathcal{P}}(x(n+1))$



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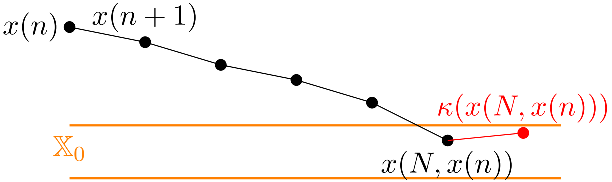

$$J_i^N\left(x(n+1), \mathbf{u}^N_{x(n+1)}\right) \leq J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right)$$

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$$\sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) = \sum_{k=0}^{K-1} \ell_i(x(k), u^{\star,N}_{x(k)}(0))$$

$$= \sum_{k=0}^{K-1} \left[J_i^N(x(k), \mathbf{u}^{\star,N}_{x(k)}) - J_i^{N-1}(x(k+1), \mathbf{u}^{\star,N}_{x(k)}(\cdot+1))\right]$$

$$\leq \sum_{k=0}^{K-1} \left[J_i^N(x(k), \mathbf{u}^{\star,N}_{x(k)}) - J_i^N(x(k+1), \mathbf{u}^N_{x(k+1)})\right] \leq J_i^N(x_0, \mathbf{u}^{\star,N}_{x_0})$$

1) Choose $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ such that $J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq J_i^N\left(x(n), \mathbf{u}^N_{x(n)}\right)$.

2) For $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N, x(n))$ set $\mathbf{u}^N_{x(n+1)} := \left(u^{\star,N}_{x(n)}(1), \ldots, u^{\star,N}_{x(n)}(N-1), \kappa(x)\right)$.

Assumption:

$F_i(f(x,\kappa(x))) + \ell_i(x,\kappa(x)) \leq F_i(x)$

Proof cont'd

Step 3: Do Pareto optima exist in general, and in particular, are there Pareto optima such that 1) is feasible?

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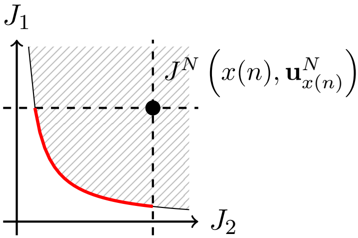

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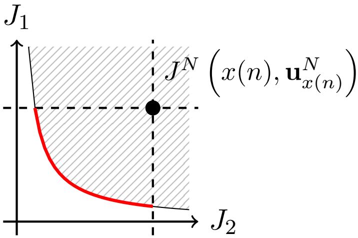

$\rightsquigarrow$ External stability ensured by the assumptions.

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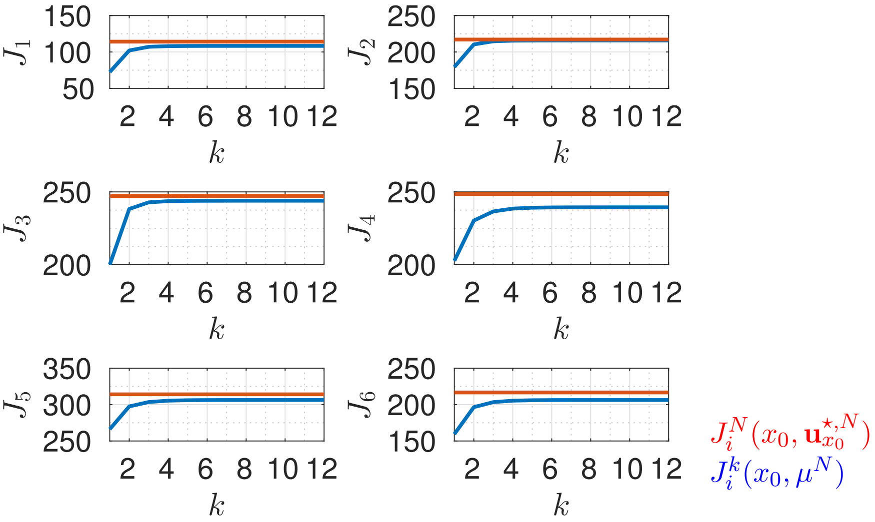
Trajectory Behaviour

Corollary

Any closed-loop trajectory resulting from the MO MPC algorithm converges to x_*.

No stability statement

Example [Müller et al. '12]



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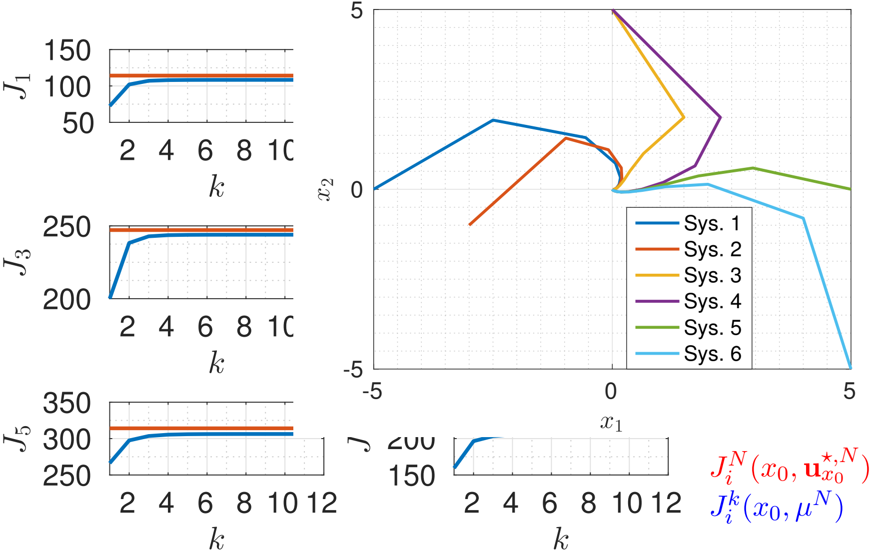

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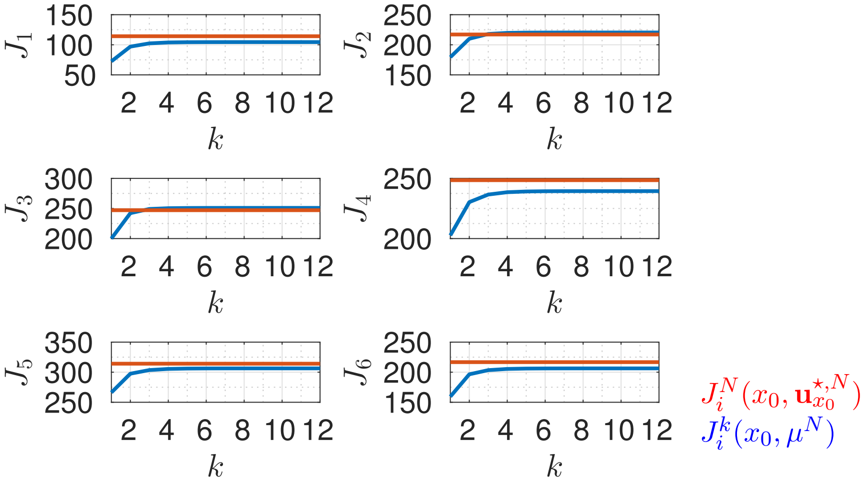

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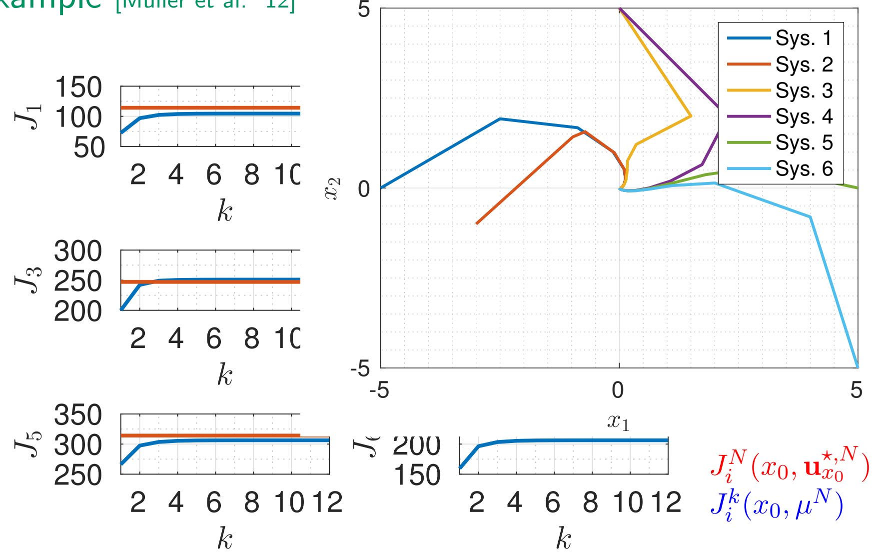

Relation to Infinite-Horizon Pareto Optima

We have

$$J_i^\infty(x_0, \mu^N) \le J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N}).$$

How about a relation to $J_i^\infty(x_0, \mathbf{u}_{x_0}^{\star,\infty})$, in which $\mathbf{u}_{x_0}^{\star,\infty}$ is a solution to

Infinite-horizon MO optimal control problem

$$\min_{\mathbf{u}\in\mathbb{U}^\infty} (J_1^\infty(x_0,\mathbf{u}), \ldots, J_s^\infty(x_0,\mathbf{u}))$$

with $J_i^\infty(x_0,\mathbf{u}) = \sum_{k=0}^{\infty} \ell_i(x(k,x_0), u(k))$

$$\text{s.t.} \left.\begin{array}{l} x(k+1,x_0) = f(x(k,x_0),u(k)), \quad k \in \mathbb{N}_0 \\ x(k,x_0) \in \mathbb{X}, \quad k \in \mathbb{N}. \end{array}\right\} \mathbf{u} \in \mathbb{U}^\infty(x_0)$$

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Unfortunately, $J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N}) \le J_i^\infty(x_0, \mathbf{u}_{x_0}^{\star,\infty})$ does not hold!
But...

Approximate Infinite-Horizon Optimality

Some more assumptions:

- $\forall\ i \in \{1, \ldots, s\}\ \exists\ \sigma_i \in \mathcal{K}$ such that $F_i(x) \leq \sigma_i(\|x - x_*\|)\ \forall\ x \in \mathbb{X}_0$.

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Theorem

Given $\mathbf{u}^{\star,\infty} \in \mathbb{U}^{\infty}_{\mathcal{P}}(x_0)$ with $J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty}) \le C\ \forall\ i$. Then, for each $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that $\forall\ N \ge N_0$ there is $\mathbf{u}^{\star,N} \in \mathbb{U}^{N}_{\mathcal{P}}(x_0)$ satisfying

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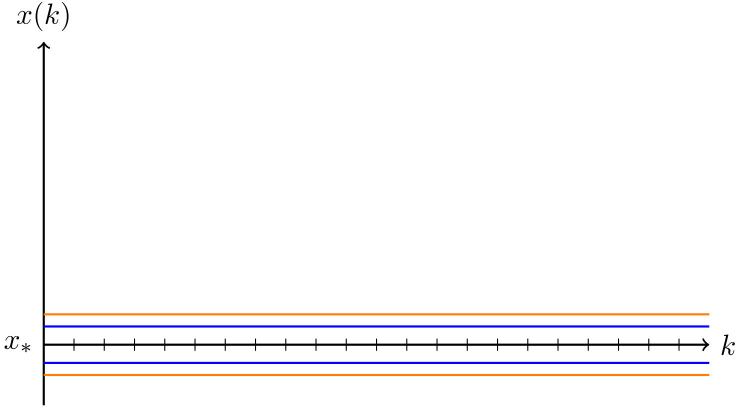
$\rightsquigarrow$ Optimal infinite-horizon performance can be approximated arbitrarily well by MPC.

Sketch of Proof $(J_i^N(x_0,\mathbf{u}^{\star,N}) \le J_i^\infty(x_0,\mathbf{u}^{\star,\infty}) + \varepsilon)$

Fix $\varepsilon > 0$, choose δ such that $\mathcal{B}_\delta(x_*) \subseteq \mathbb{X}_0$ and $\sigma_i(\delta) \le \varepsilon$. Let N be sufficiently large.



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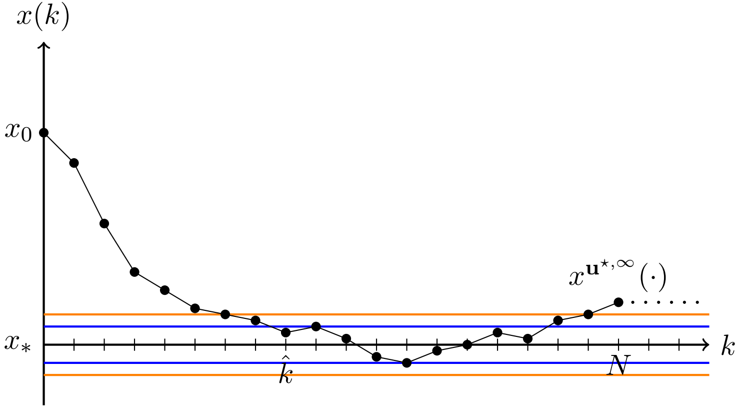

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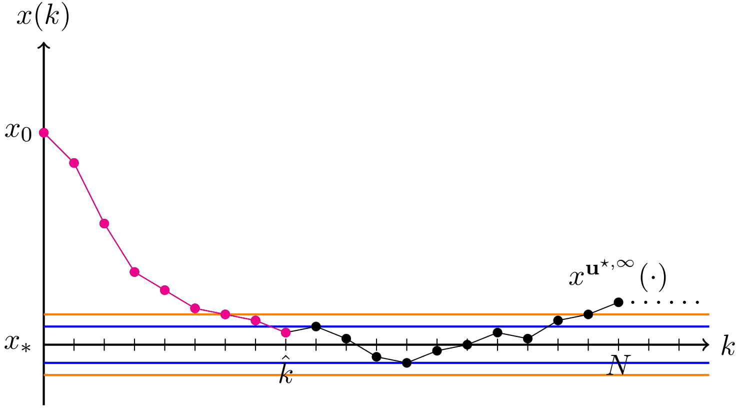

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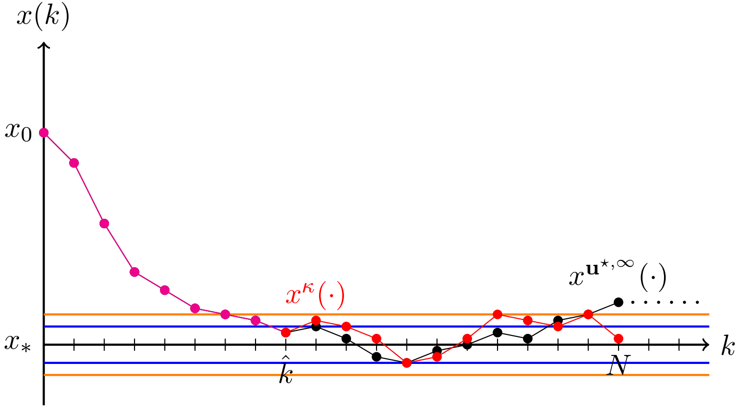

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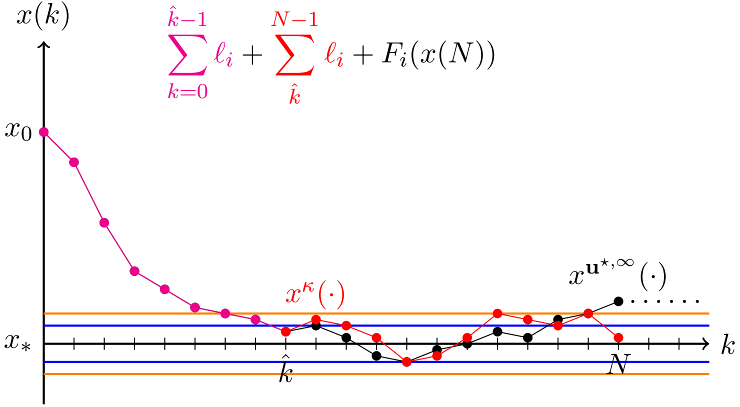

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$$\sum_{k=0}^{\hat{k}-1} \ell_i + \sum_{\hat{k}}^{N-1} \ell_i + F_i(x(N))$$

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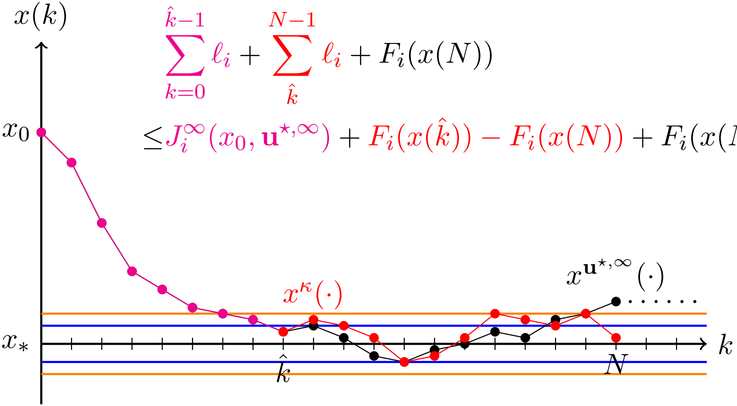

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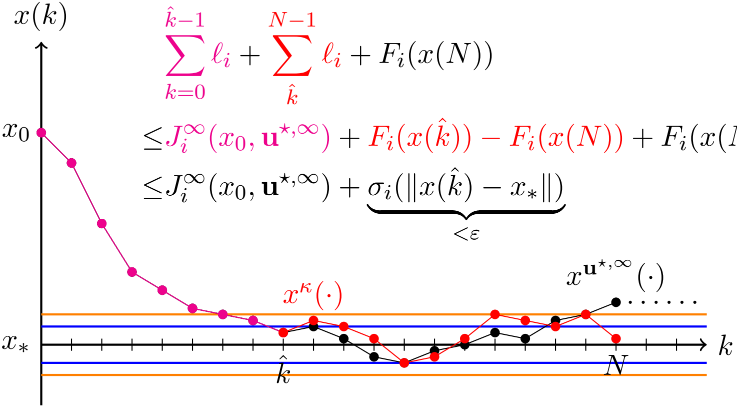

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$$J_i^N(x, \mathbf{u}^{\star,N})$$
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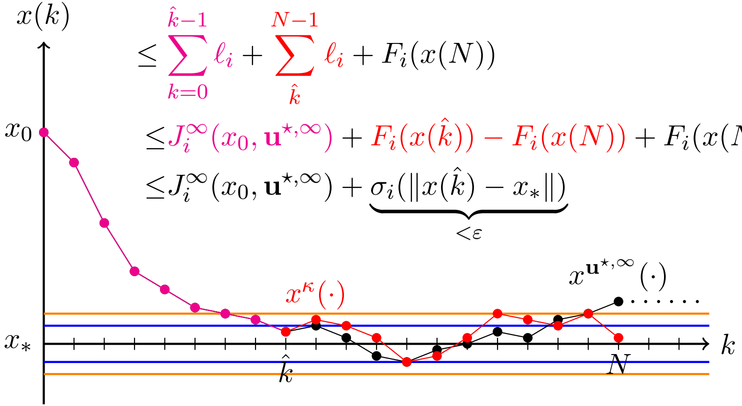

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Conclusion

- Framework for MO MPC, that does not depend on a specific coupling
- Performance guarantees for all objectives
- MO MPC approximates optimal solutions on the infinite horizon arbitrarily well
- Convergence of closed-loop trajectories
- Concept of Lyapunov functions cannot be used

- MO MPC also works without terminal conditions (see my talk at the GAMM annual meeting, S20)

Research supported by DFG

Methods in MO Optimization

How can we calculate/find Pareto-optimal solutions?

- Scalarization approaches, e.g.
  - weighted sum
  - ε-constraint method
  - distance to ideal point (method of the global criterion)
  - normal boundary intersection
- Evolutionary algorithms, e.g.
  - elitist non-dominated sorting genetic algorithm (NSGA-II)
  - ant colony optimization
- Hierarchical methods, e.g., lexicographic ordering

$\vdots$