Zur Äquivalenz von Dissipativität und der Turnpike-Eigenschaft

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basierend auf einer Arbeit mit Matthias A. Müller (Universität Stuttgart)

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Outline

- Dissipativity and strict dissipativity
- The turnpike property and its variants
- Known results
- New results and proof ideas

System class

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x$$

with $x_{\mathbf{u}}(n) \in X$, $\mathbf{u}(n) \in U$, $\quad X$, U normed spaces

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f = solution operator of a controlled ODE/PDE or of a discrete time model (or a numerical approximation of one of these)

Dissipativity and strict dissipativity

Dissipativity

$$x^+ = f(x, u)$$

Introduce functions $s : X \times U \to \mathbb{R}$ and $\lambda : X \to \mathbb{R}_0^+$

$s(x, u) \in \mathbb{R}$ supply rate, measuring the (possibly negative) amount of energy supplied to the system via the input u in the next time step

$\lambda(x) \geq 0$ storage function, measuring the amount of energy stored inside the system when the system is in state x

Dissipativity

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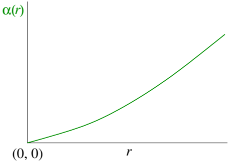
$\alpha \in \mathcal{K}$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$



Physical interpretation of dissipativity

$$\lambda(x^+) \le \lambda(x) + s(x,u) \quad \big[-\alpha(\|x - x^e\|) \big]$$

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dissipativity: energy can only be dissipated (=lost) but not be generated inside the system

strict dissipativity: a certain amount of energy, depending on $\|x - x^e\|$ must be dissipated

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Translation to discrete time systems is quite straightforward [Byrnes/Lin '94]

Strict dissipativity for LQ problems

Consider a linear quadratic finite dimensional discrete time problem with

$$x^+ = Ax + Bu, \quad s(x,u) = x^T Q x + u^T R u + s^T x + v^T u$$

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However, we still think this might be known ...

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If we can construct F with $s(x, F(x)) < 0$ for $x \neq x^*$, then

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Constructing F is particularly easy in case of passivity, because for $s(x, u) = \langle y, u \rangle$ it suffices to define the output feedback $F(y) := -y$

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In both cases, λ is a Lyapunov function

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Finally, strict dissipativity plays a major role in the analysis of so called economic model predictive control schemes

(Matthias' talk)

Available storage

Theorem [Willems '72, Byrnes/Lin '94] A system is [strictly] dissipative with supply rate s [and $\alpha \in \mathcal{K}$] if and only if

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The particular storage function defined above is called

"available storage"

The turnpike property

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We illustrate the property by two simple examples

Example 1: minimum energy control

Example: Keep the state of the system inside a given interval X minimising the quadratic control effort

$$\ell(x,u) = u^2$$

with dynamics

$$x^+ = 2x + u$$

and spaces $X = [-2, 2]$, $U = [-3, 3]$

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$\rightsquigarrow$ optimal trajectory should stay near $x^e = 0$

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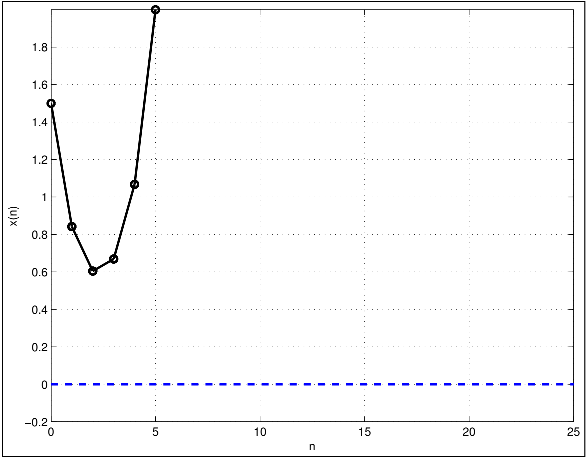

Optimal trajectory for $N = 5$

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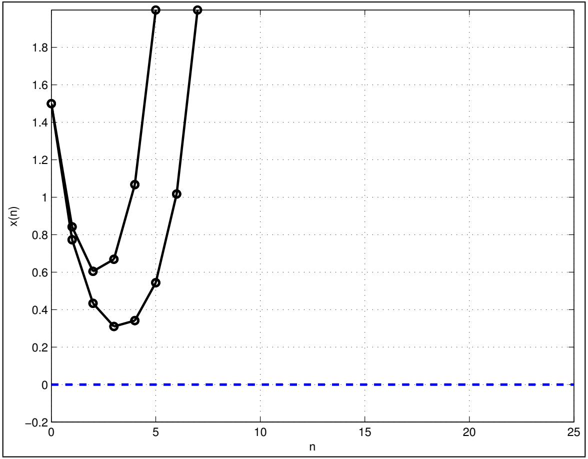

Optimal trajectories for $N = 5, \ldots, 7$

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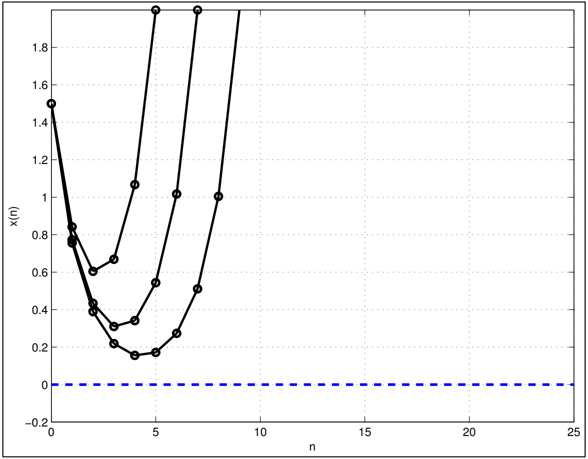

Optimal trajectories for $N = 5, \ldots, 9$

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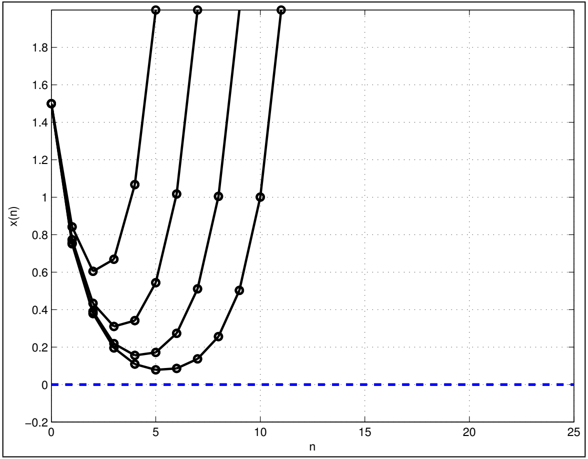

Optimal trajectories for $N = 5, \ldots, 11$

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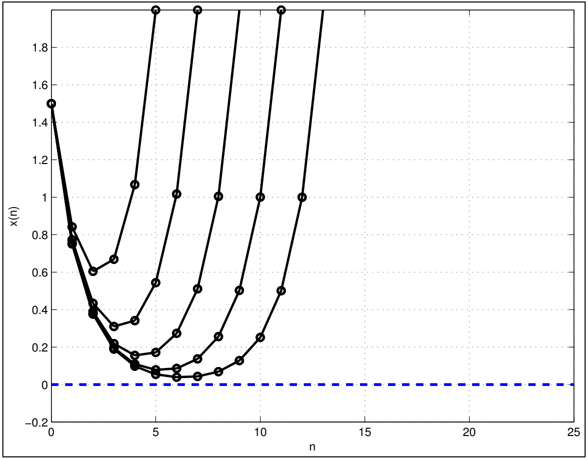

Optimal trajectories for $N = 5, \ldots, 13$

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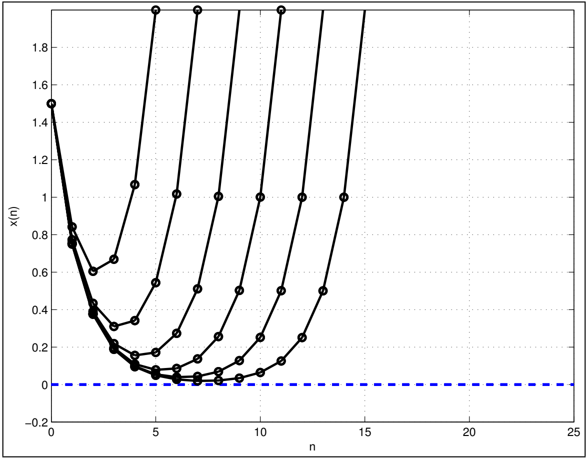

Optimal trajectories for $N = 5, \dots, 15$

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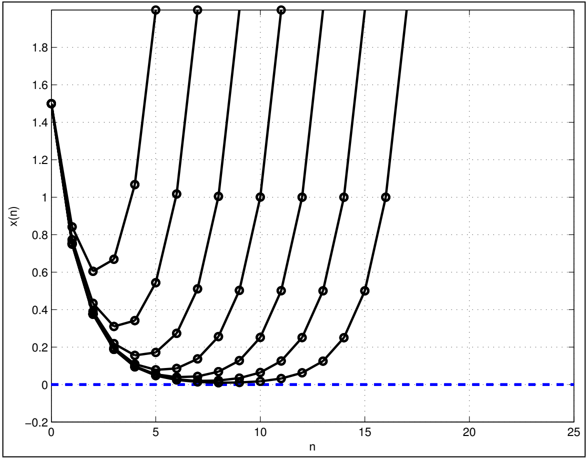

Optimal trajectories for $N = 5, \ldots, 17$

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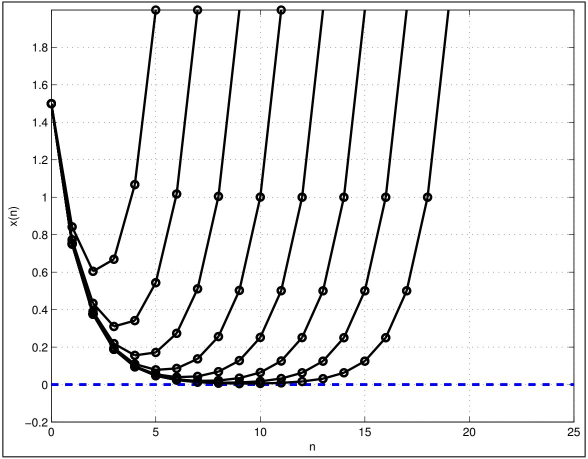

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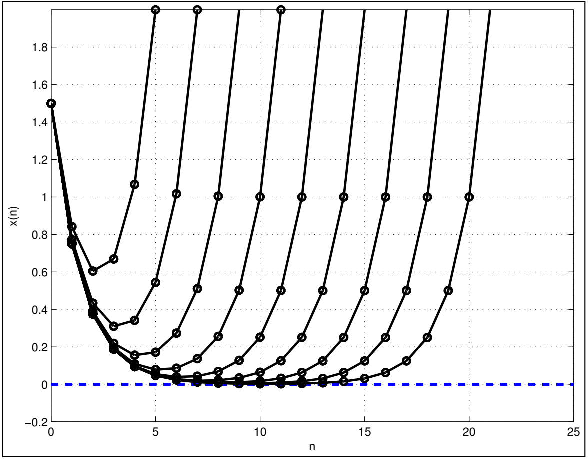

Optimal trajectories for $N = 5, \ldots, 21$

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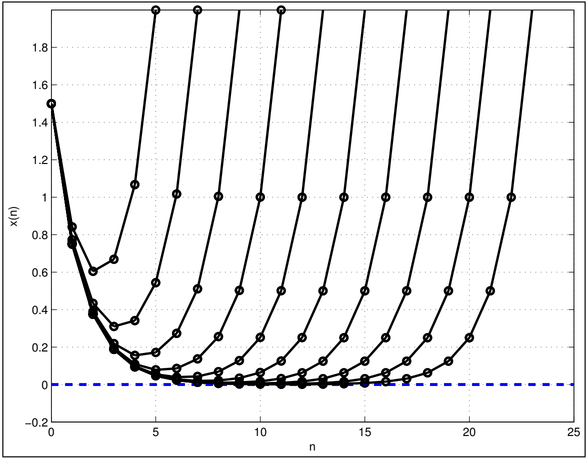

Optimal trajectories for $N = 5, \ldots, 23$

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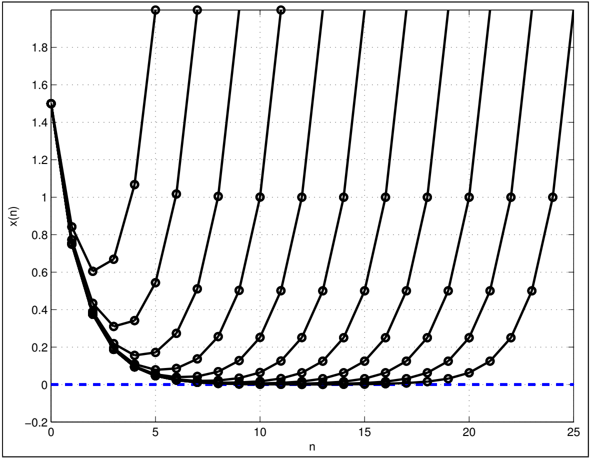

Optimal trajectories for $N = 5, \dots, 25$

Example 2: a macroeconomic model

The second example is a 1d macroeconomic model

[Brock/Mirman '72]

Minimise the finite horizon objective with

$$\ell(x,u) = -\ln(Ax^{\alpha} - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $\quad x^+ = u$

on $\quad X = U = [0, 10]$

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On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

Example 2: a macroeconomic model

The second example is a 1d macroeconomic model

[Brock/Mirman '72]

Minimise the finite horizon objective with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $\quad x^+ = u$

on $\quad X = U = [0, 10]$

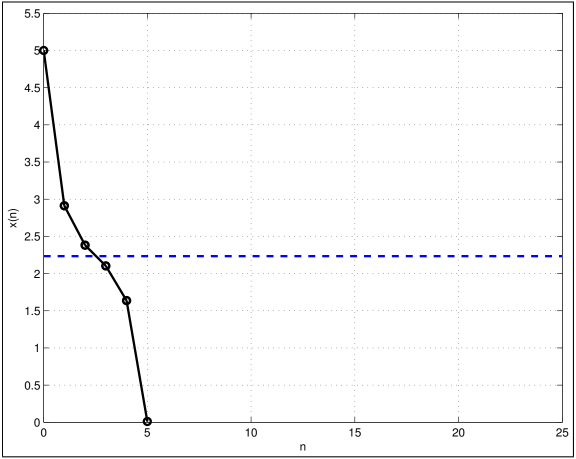
Here the optimal trajectories are less obvious

On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

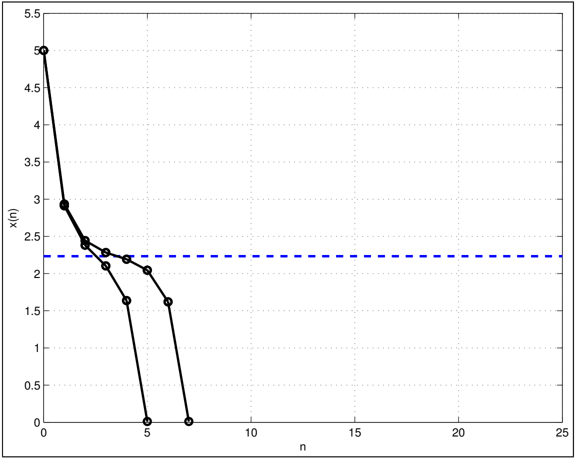
One may thus expect that finite horizon optimal trajectories also stay for a long time near that equilibrium

Example 2: optimal trajectories



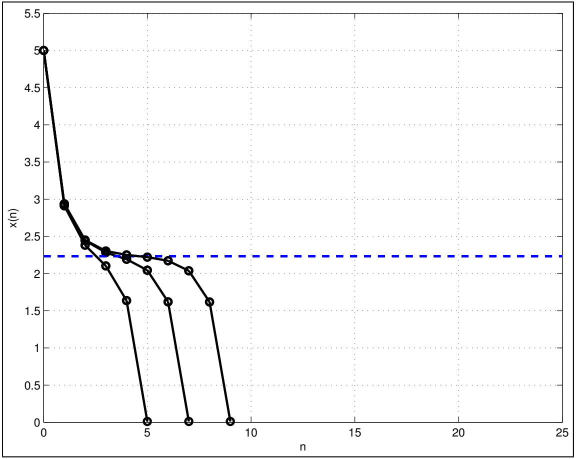
Optimal trajectory for $N = 5$

Example 2: optimal trajectories



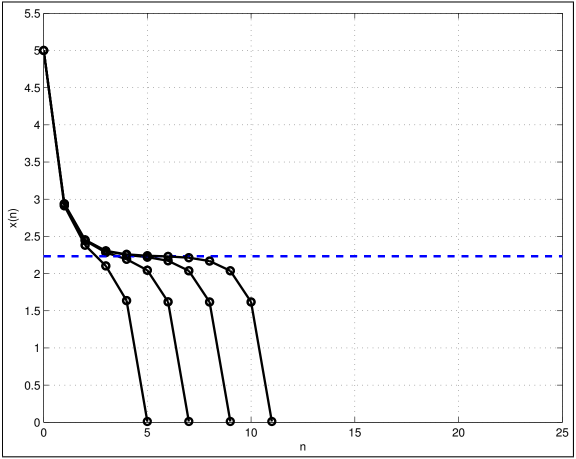
Optimal trajectories for $N = 5, \ldots, 7$

Example 2: optimal trajectories



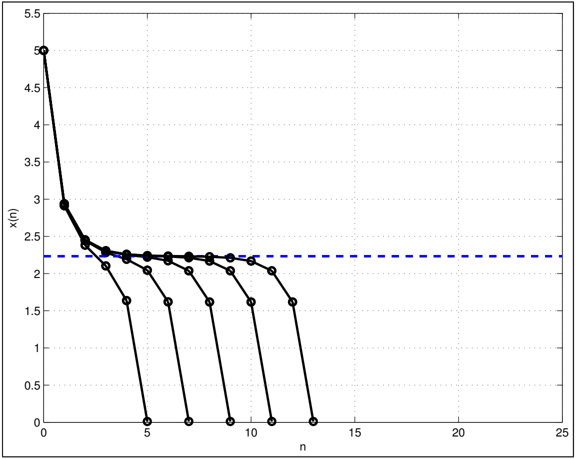
Optimal trajectories for $N = 5, \ldots, 9$

Example 2: optimal trajectories



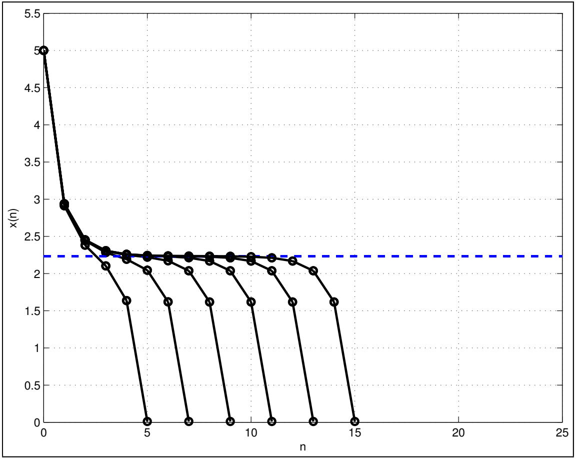
Optimal trajectories for $N = 5, \ldots, 11$

Example 2: optimal trajectories



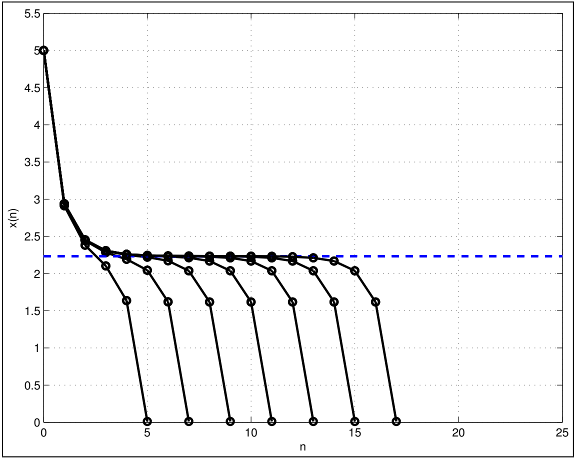
Optimal trajectories for $N = 5, \ldots, 13$

Example 2: optimal trajectories



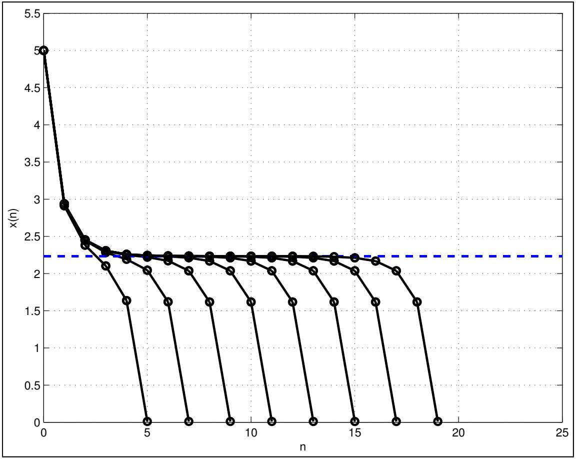
Optimal trajectories for $N = 5, \ldots, 15$

Example 2: optimal trajectories



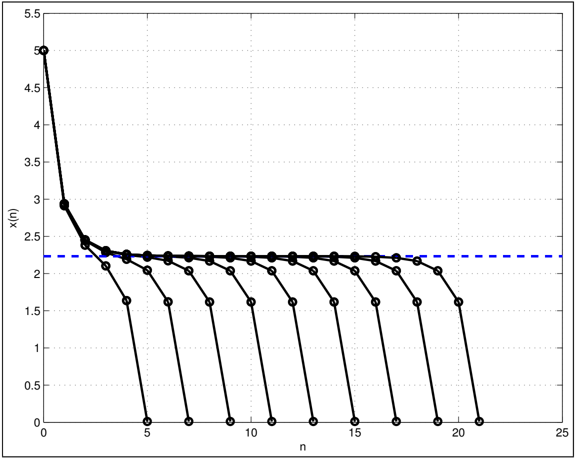
Optimal trajectories for $N = 5, \ldots, 17$

Example 2: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 19$

Example 2: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 21$

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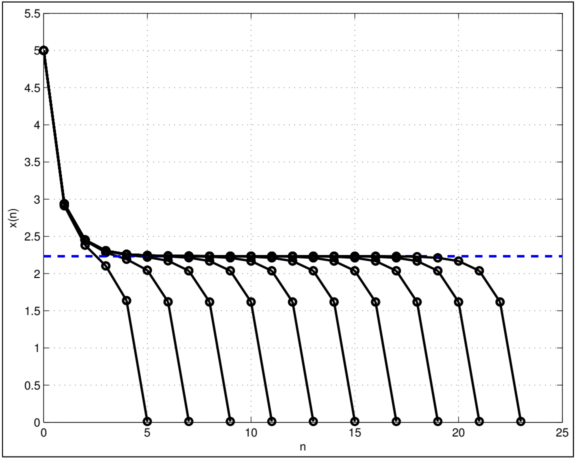

Optimal trajectories for $N = 5, \ldots, 23$

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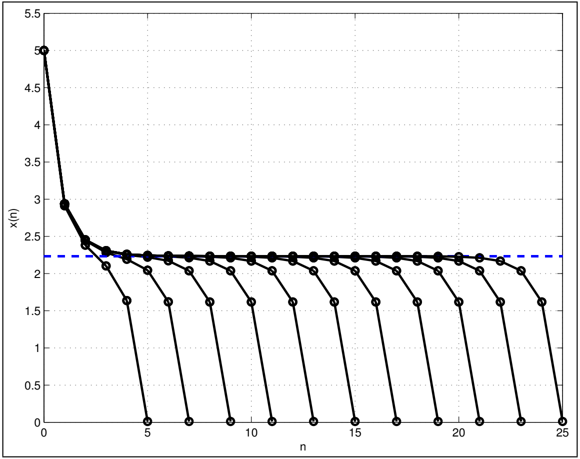
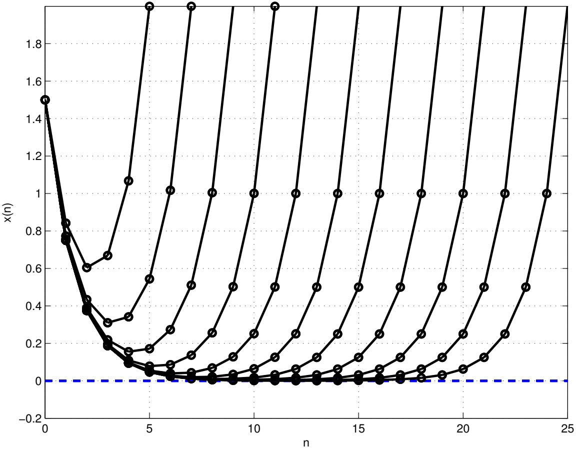


Optimal trajectories for $N = 5, \ldots, 25$

How to formalize the turnpike property?



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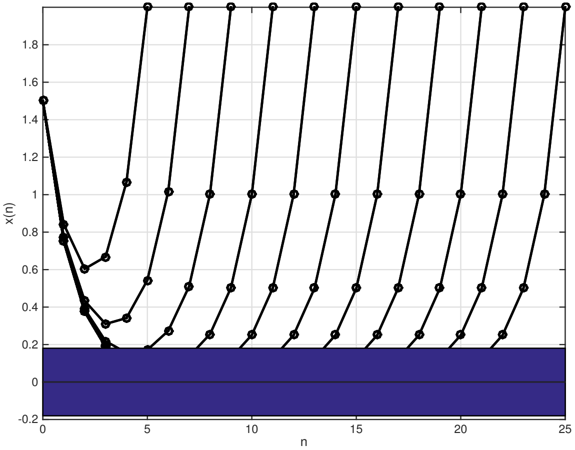

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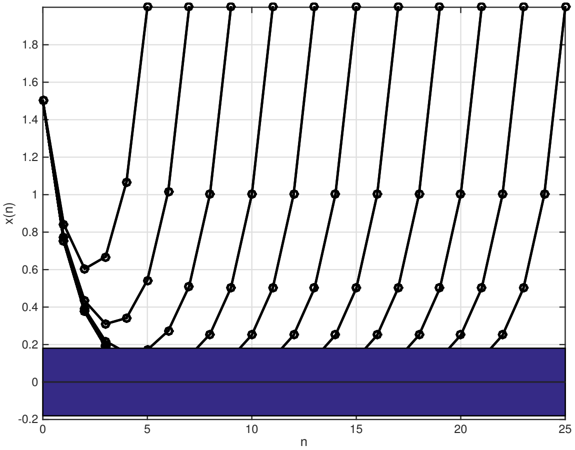

Number of points outside the blue neighbourhood is bounded by a number independent of N (here: by 8)

The turnpike property: formal definitions

Let x^e be an equilibrium, i.e., $f(x^e, u^e) = x^e$

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Turnpike property: For each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that for all $N \in \mathbb{N}$ all optimal trajectories $x^\star$ satisfy the inequality

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Near optimal turnpike property: For each $\varepsilon > 0$ and $\delta > 0$ there is $C_{\varepsilon,\delta} > 0$ such that for all $x \in X$ and $N \in \mathbb{N}$, all trajectories $x_{\mathbf{u}}$ with $x_{\mathbf{u}}(0) = x$ and $J_N(x, \mathbf{u}) \leq V_N(x) + \delta$ satisfy the inequality

$$\#\Big\{k \in \{0, \ldots, N-1\} \,\Big|\, \|x_{\mathbf{u}}(k) - x^e\| \geq \varepsilon\Big\} \leq C_{\varepsilon,\delta}$$

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- Apparently first described by [von Neumann 1945]

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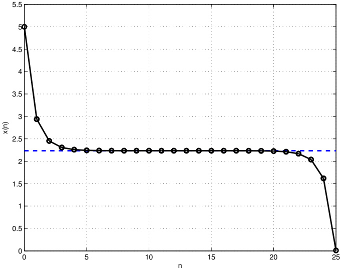

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- Extensively studied in the 1970s in mathematical economy, cf. survey [McKenzie 1983]
- Renewed interest in recent years [Zaslavski '14, Trélat/Zuazua '15, Faulwasser et al. '15, ...]

Applications

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Under suitable assumption these two properties are equivalent [Gr./Kellett/Weller '16]

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Ideas of this type can be found, e.g., in [Anderson/Kokotovic '87]

Application: Model predictive control

Turnpike properties are also pivotal for analysing economic Model Predictive Control (MPC) schemes

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MPC is a method in which an optimal control problem on an infinite horizon

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_\infty(x, \mathbf{u}) = \sum_{n=0}^{\infty} \ell(x_{\mathbf{u}}(n), \mathbf{u}(n))$$

is approximated by the iterative solution of finite horizon problems

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_N(x, \mathbf{u}) = \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k))$$

with fixed $N \in \mathbb{N}$

MPC from the trajectory point of view

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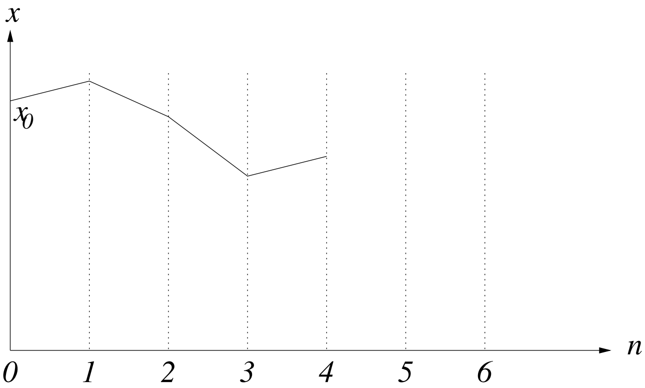

black = predictions (open loop optimization)

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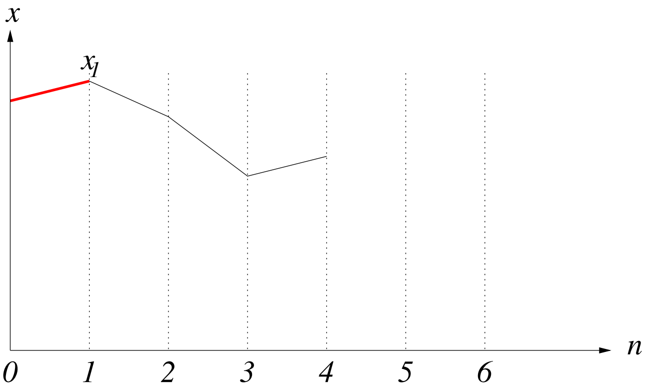

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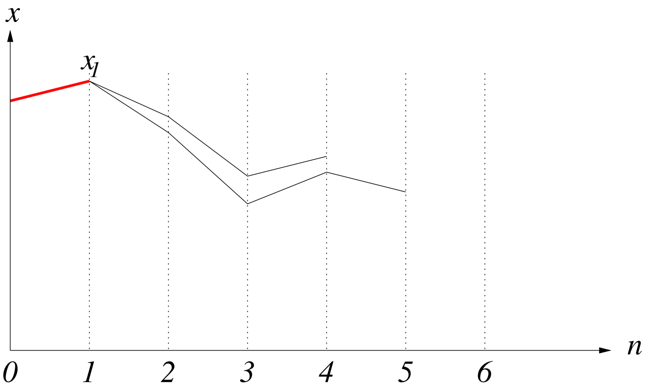

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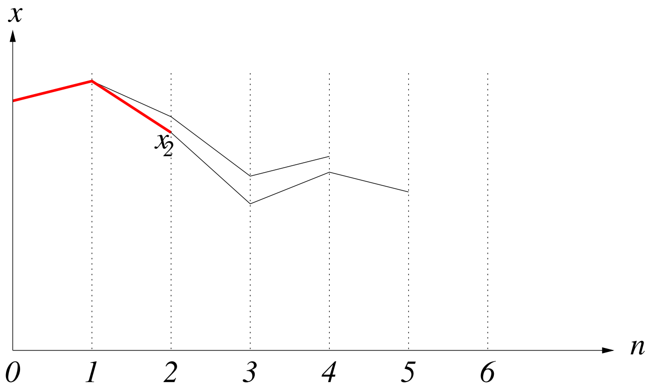

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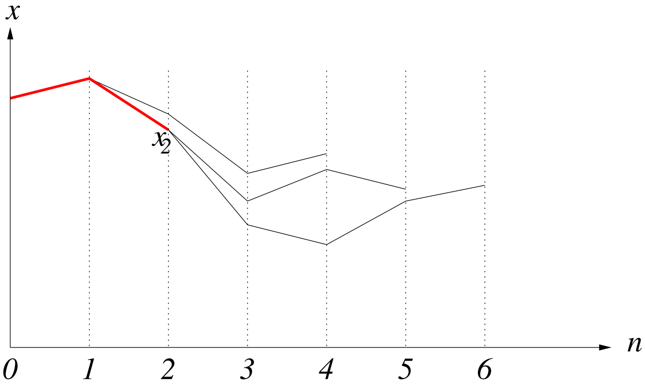

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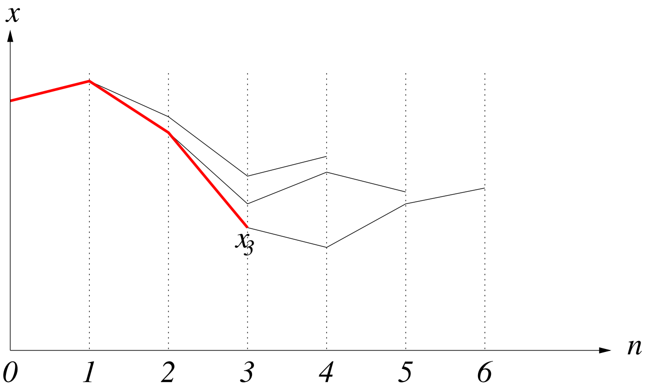

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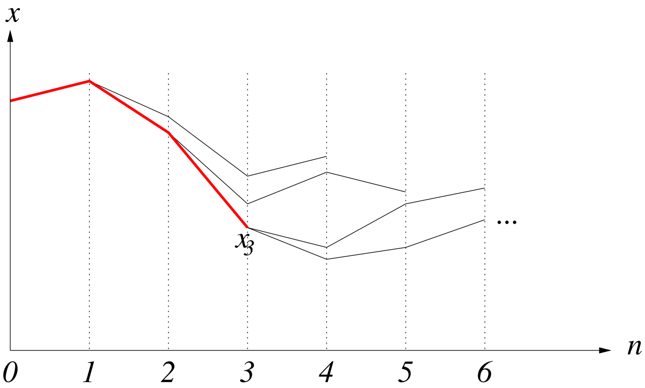

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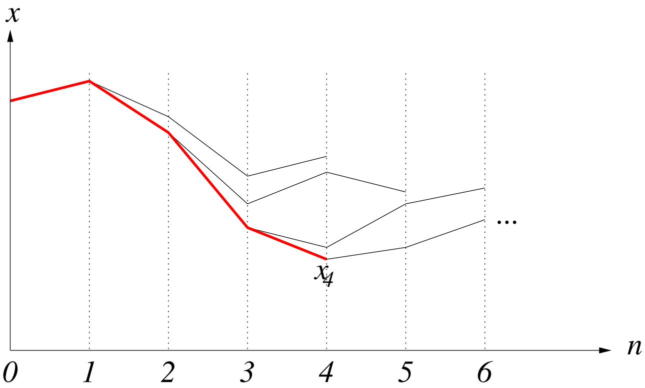

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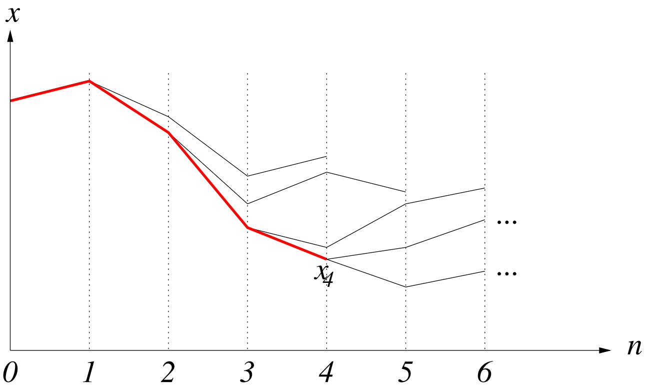

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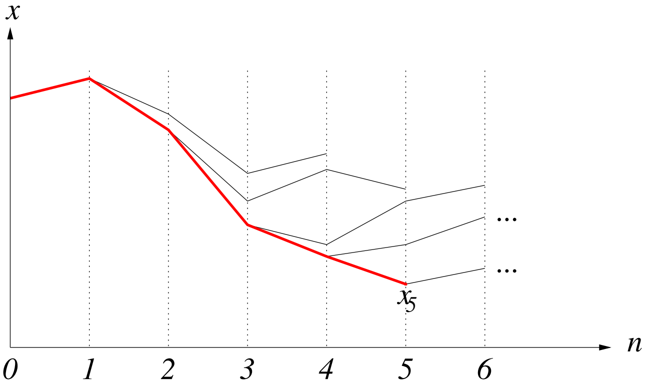

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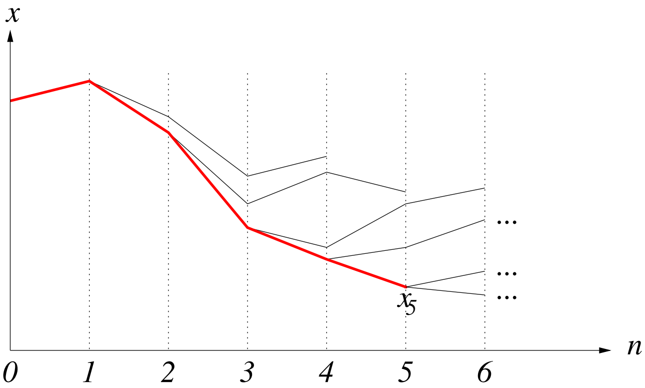

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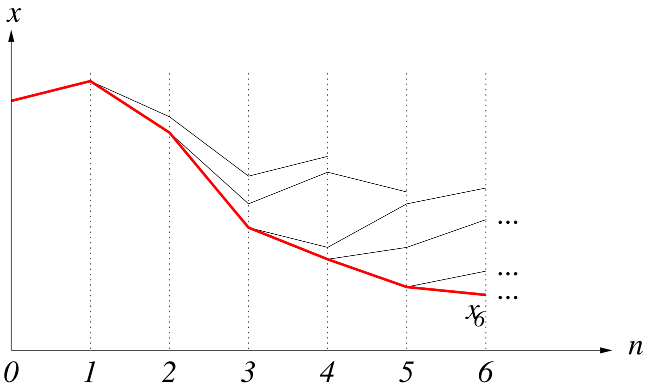

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Approximation result for MPC

If the finite horizon problems have the turnpike property, then a rigorous approximation result can be proved

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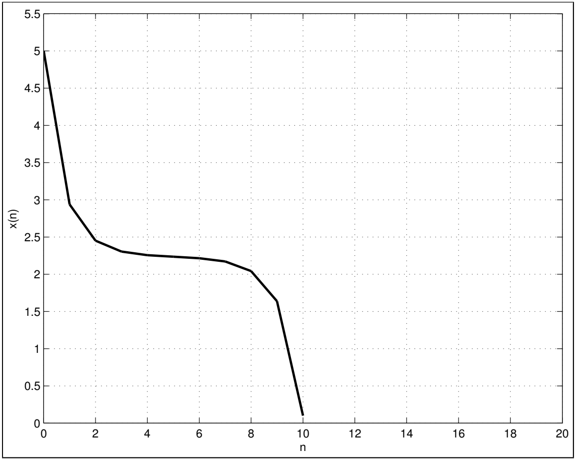
The result exploits that the **red** closed loop trajectory approximately follows the first part of the **black** predictions up to the equilibrium

We illustrate this behaviour by our second example for $N = 10$

MPC for Example 2



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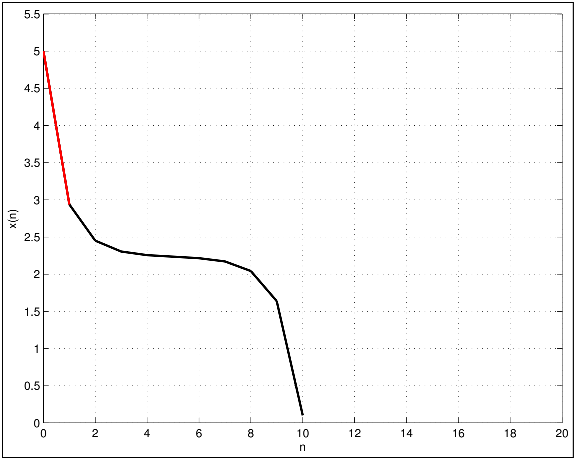

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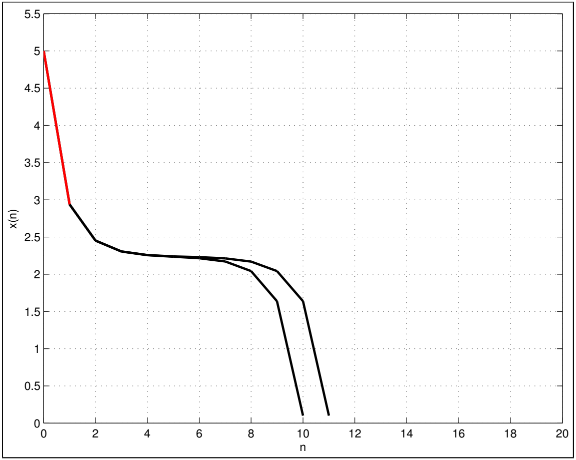

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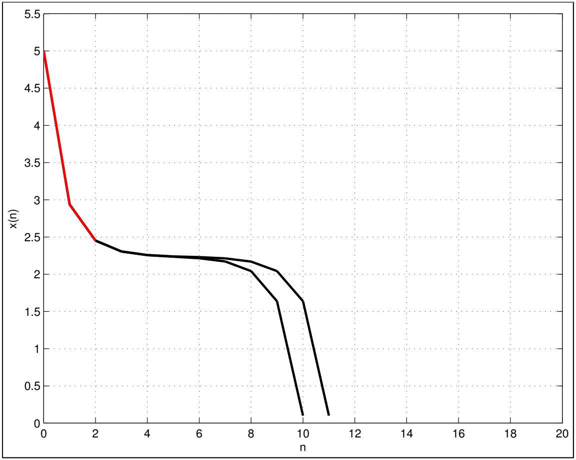

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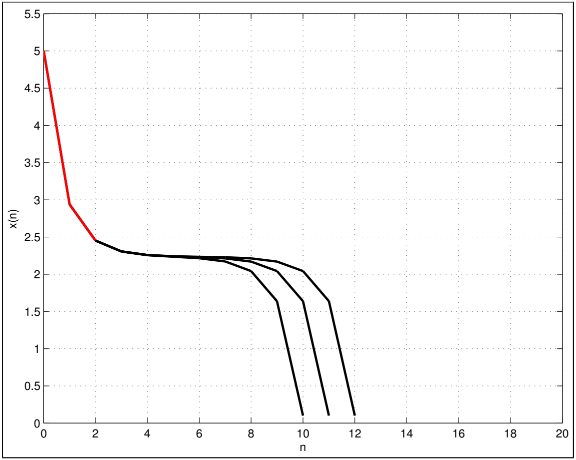

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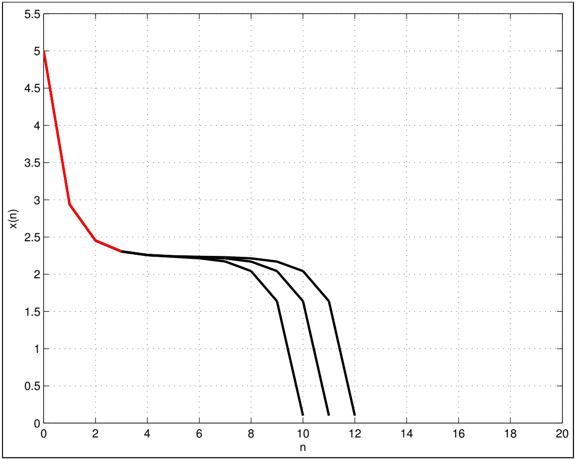

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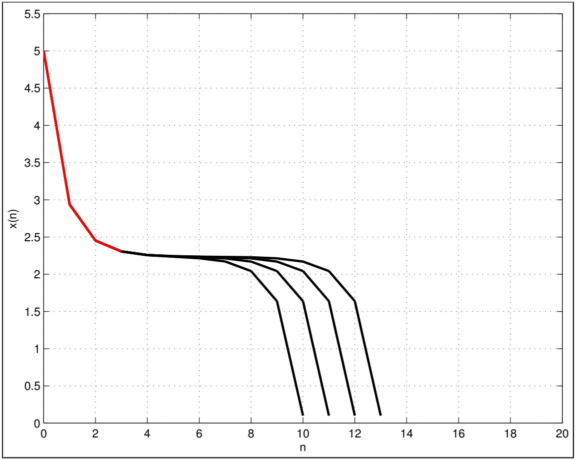

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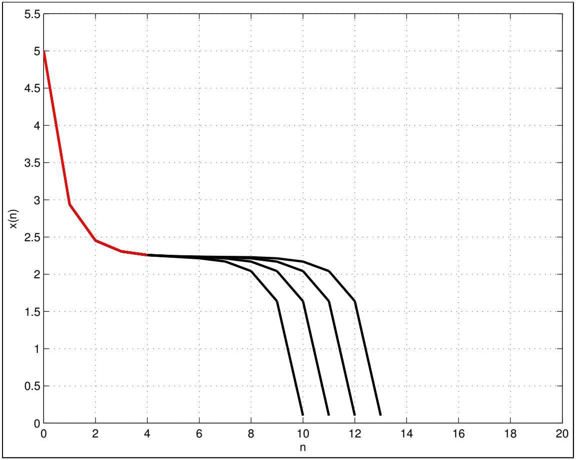

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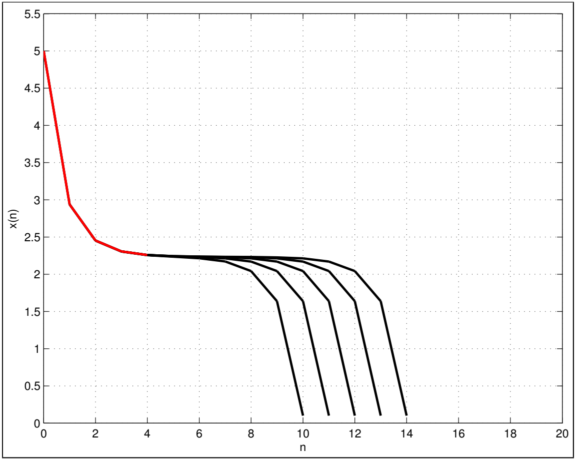

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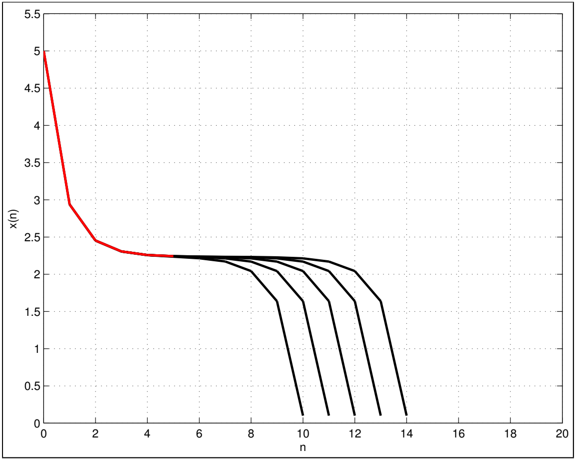

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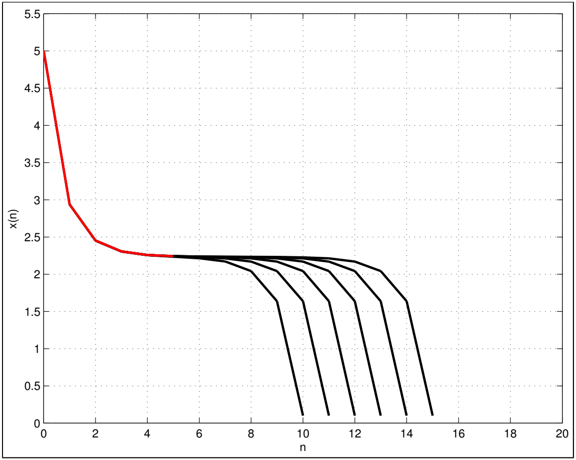

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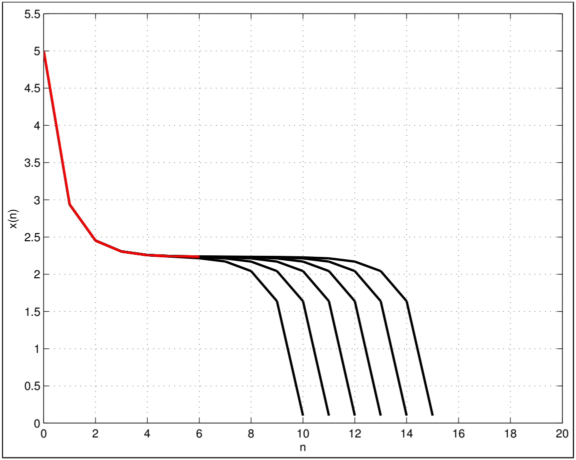

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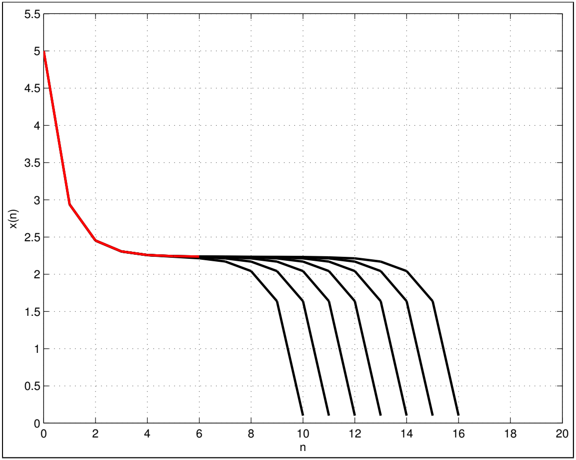

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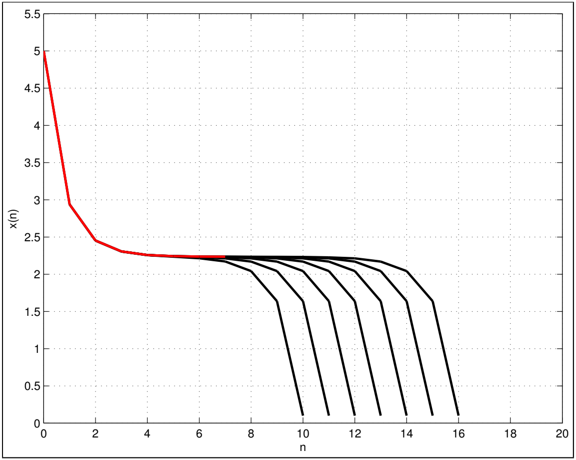

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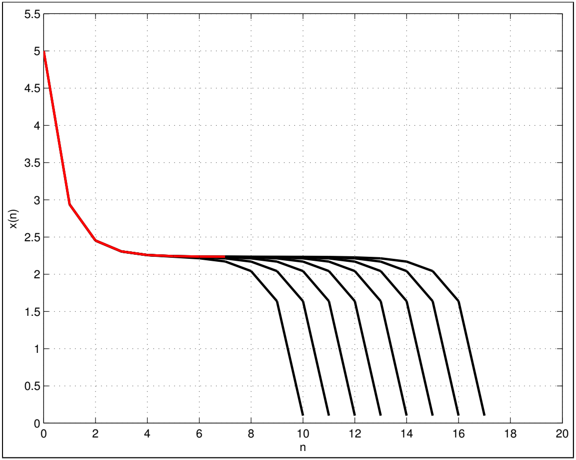

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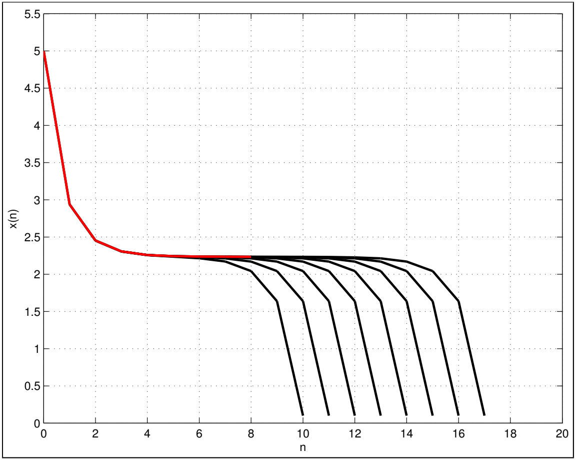

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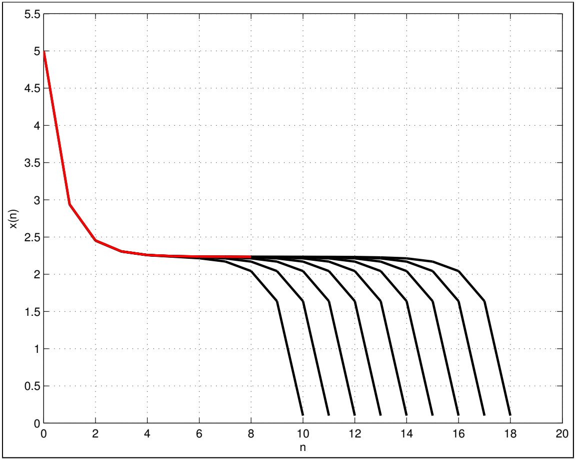

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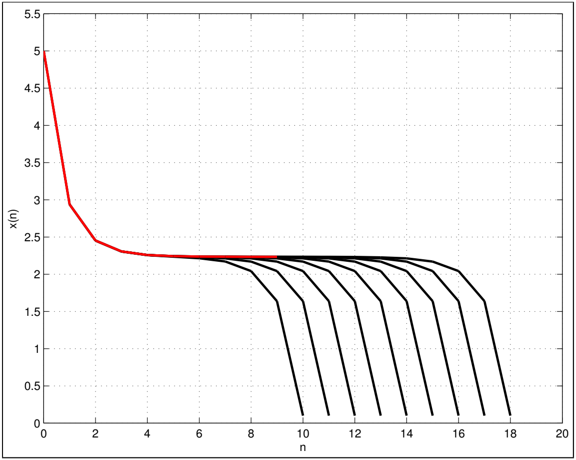

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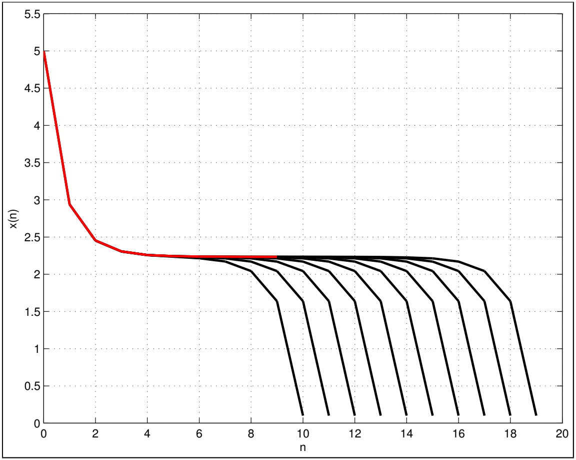

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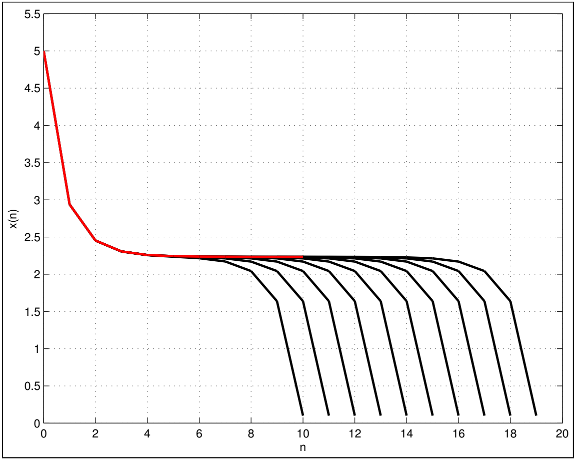

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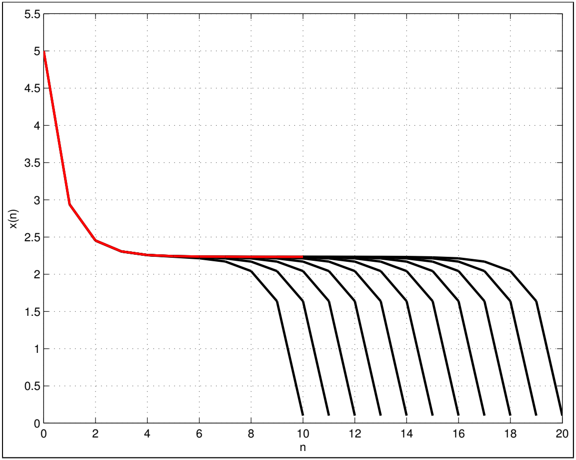

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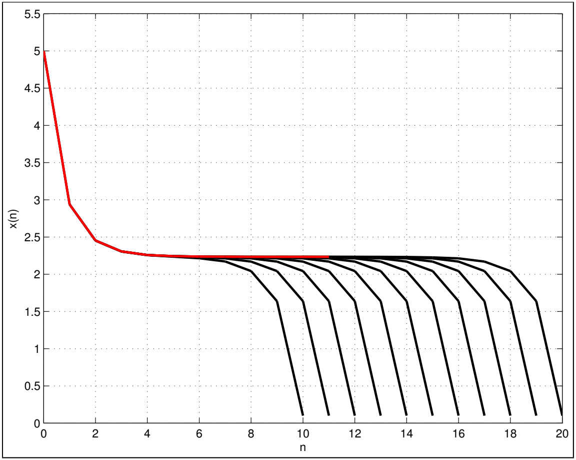

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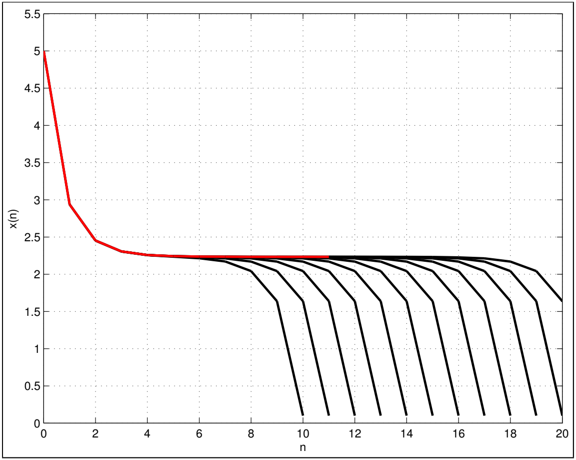

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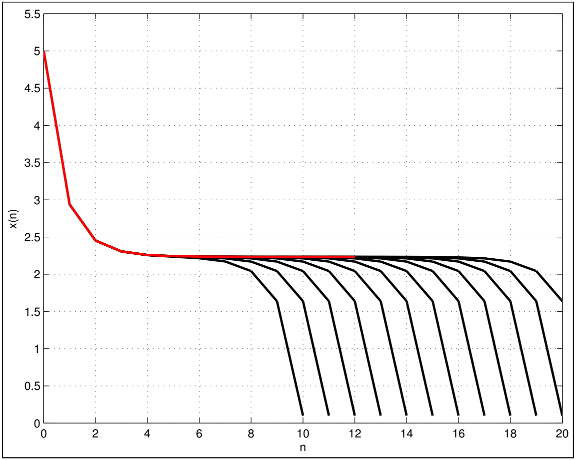

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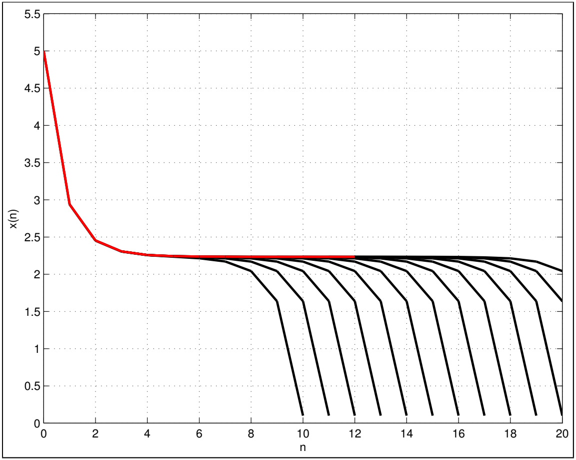

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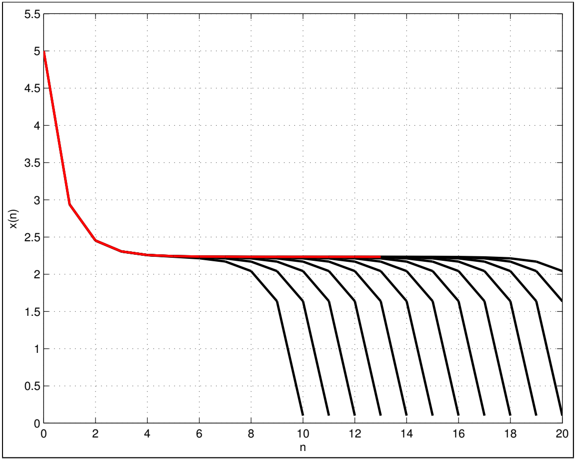
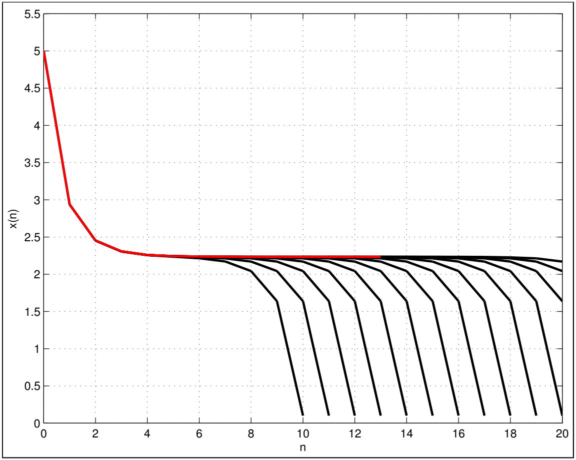


MPC for Example 2



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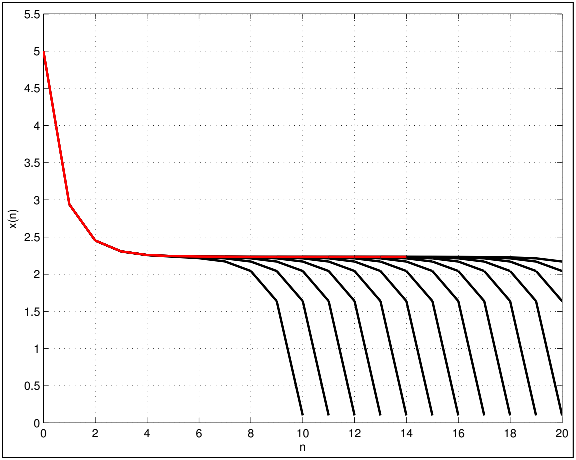

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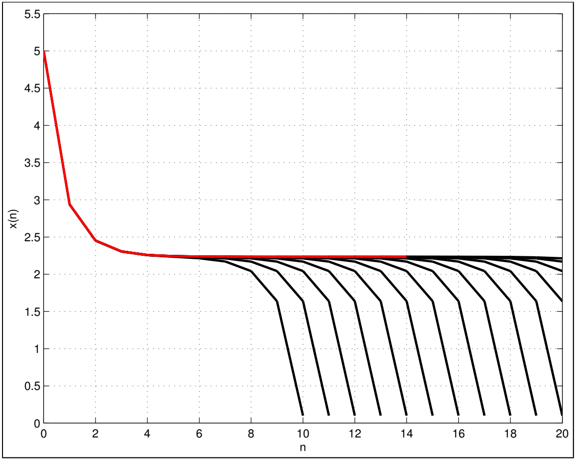

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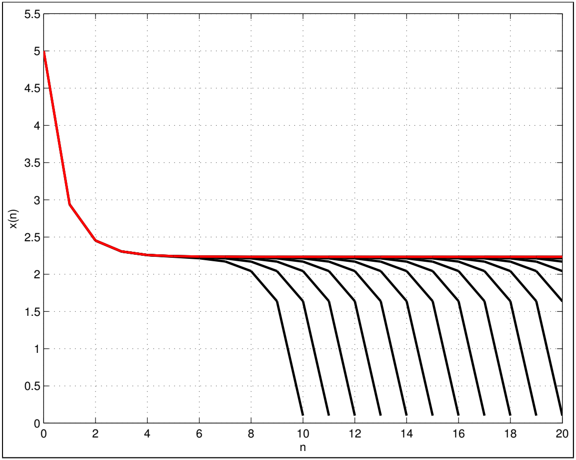

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We say that x^e is cheaply reachable if there is $E > 0$ such that $V_N(x) \leq N\ell(x^e, u^e) + E$ for all $x \in X$, $N \in \mathbb{N}$

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(In fact, similar statements can be found in earlier papers and monographs, e.g. in [Carlson/Haurie/Leizarowitz '91])

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$\Rightarrow$ if $\|x_{\mathbf{u}}(k) - x^e\| \geq \varepsilon$ for K times $k \in \{0,\dots,N-1\}$, then $J_N(x,\mathbf{u}) - N\ell(x^e,u^e) \to \infty$ as $K, N \to \infty$, contradicting near optimality

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In fact, the theorem just presented relies on another theorem which does not require cheap reachability

Known results

Near equilibrium turnpike property: For each $\varepsilon > 0$ and $\delta > 0$ there is $C_{\varepsilon,\delta} > 0$ such that for all $x \in X$ and $N \in \mathbb{N}$ all trajectories $x_{\mathbf{u}}$ with $x_{\mathbf{u}} = x$ and $J_N(x, \mathbf{u}) \le N\ell(x^e, u^e) + \delta$ satisfy the inequality

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(c) The near equilibrium turnpike property holds and the system is dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and bounded storage function

In other words, the near equilibrium turnpike property exactly closes the gap between dissipativity and strict dissipativity

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Proof idea

We need to prove the equivalences of

(a) strict dissipativity

(b) near equilibrium turnpike and uniform near optim. of x^e

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Question: Can we get rid of dissipativity in (c)?

Yes, if we use that the near equilibrium turnpike property induces an averaged form of optimality of x^e which under additional conditions implies dissipativity

[Müller '14, Müller/Angeli/Allgöwer '13]

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New result II

Corollary: Assume X is closed and U is compact, ℓ is continuous and bounded from below, x^e is an equilibrium around which the system is locally controllable and $u^e \in \operatorname{argmin}\{\ell(x^e, u) \,|\, x^e \in X,\ f(x^e, u) = x^e\}$

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Question: Is local controllability really needed?

Example

$$x^+ = \frac{1}{2}x \qquad \text{and} \qquad \ell(x,u) = u^2 + \frac{\log 2}{\log |x|}$$

The system has (all three kinds of) the turnpike property at $x^e = 0$, because all solutions converge to 0

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However, one computes that the available storage satisfies

$$\lambda(x) := \sup_{K,u} \sum_{k=0}^{K-1} -\Big(\ell(x(k),u(k)) - \ell(x^e,u^e)\Big) = \infty,$$

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Since all other assumptions of the previous corollary are satisfied, it is the lack of controllability which makes its statement fail

Towards new result III

Recall the first two equivalences in the first theorem:

Theorem: The following statements are equivalent

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Can we replace the near equilibrium turnpike property by the (more intuitive and "classical") near optimal turnpike property?

Yes, but again we need additional assumptions

New result III

Theorem: Assume ℓ is bounded and the system is locally controllable around x^e

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Note: the implication "(b) $\Rightarrow$ strict dissipativity" also holds without assuming local controllability

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There is an obvious analogy between the equivalences

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In contrast, the existence of a Lyapunov function yields asymptotic stability for all trajectories for which its defining inequality holds

Application: shape of turnpike trajectories

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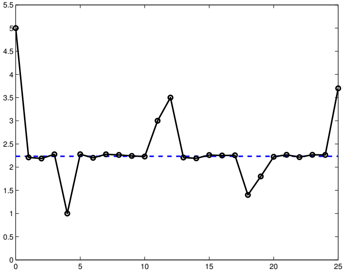

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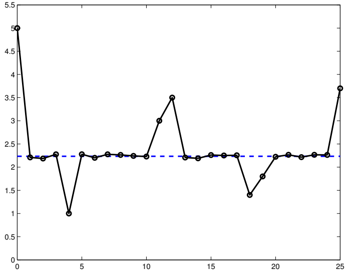
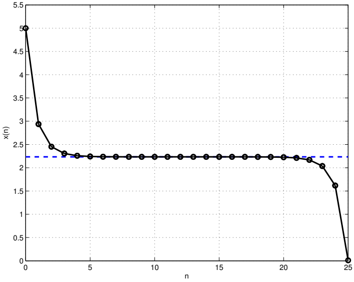


However, in practice in many examples turnpike trajectories look like this



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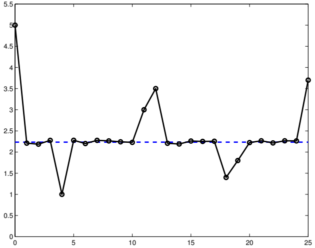

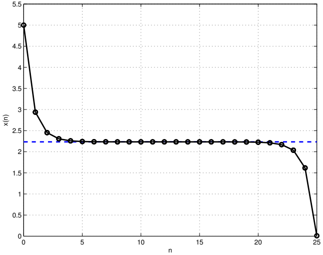

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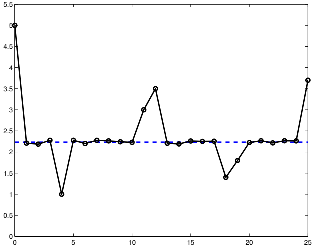

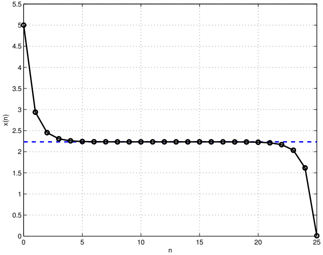

This is because under the stated conditions the turnpike property implies strict dissipativity, which in turn implies stability of the optimal trajectories, in the sense that if $x^\star(k) \approx x^e$ then $x^\star(k+p) \approx x^e$ for k, p sufficiently small relative to N [Gr. 13]

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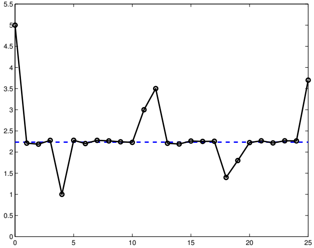

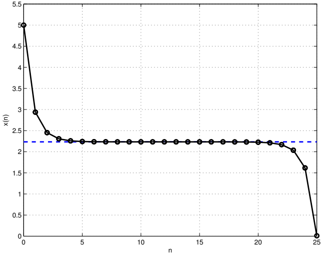

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This excludes excursions from x^e except at the end of the optimal trajecory

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Conclusions

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- The results precisely describe the gap between strict dissipativity and turnpike properties
- As a consequence, assuming strict dissipativity for ensuring the turnpike property does not seem overly conservative

References

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