Economic model predictive control for time-varying systems

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UNIVERSITÄT BAYREUTH

Outline

Introduction

Model predictive control

MPC convergence results

(Prospective) Application

Conclusion

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Setting

- Consider a discrete-time, time-varying system

$$x(k+1) = f(k, x(k), u(k)),\ x(0) = x_0$$

with $x(k) \in X$, $u(k) \in U$.
$\rightsquigarrow$ solution trajectory $x_u(\cdot, x_0)$

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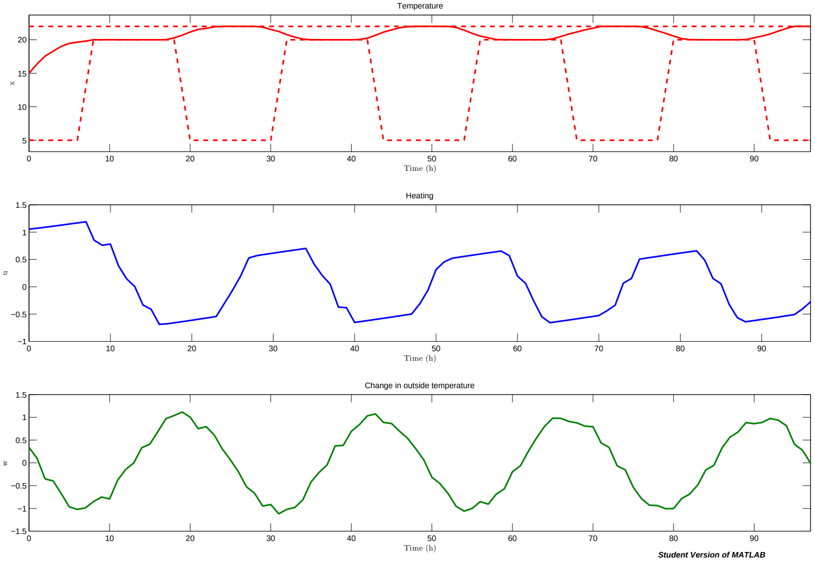
- Example: Simplified building model:

$$x(k+1) = \underbrace{x(k)}_{\text{inside temperature}} + \underbrace{u(k)}_{\text{heating}} + \underbrace{w(k)}_{\text{outside temperature}}$$

- Goal: Keep temperature (x) within a certain range $\mathbb{X}(k)$, using as little energy (u) as possible.

Example: optimal trajectory?



Setting

- Infinite horizon optimal control problem

$$\min_{u \in \mathbb{U}^\infty(k,x_0)} J_\infty(k, x_0, u) = \sum_{j=0}^{\infty} \ell(k+j, x_u(j, x_0), u(j)) \qquad (1)$$

with stage cost $\ell : \mathbb{N}_0 \times X \times U \to \mathbb{R}$, and where $\mathbb{U}^\infty(k, x_0)$ is the set of admissible control sequences.

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- In the example:
  Stage cost:

$$\ell(k, x, u) = u^2$$

What is optimal?

- $J_\infty(k, x_0, u)$ may not be finite for any u.
  $\rightsquigarrow J_\infty(k, x_0, u^*) \leq J_\infty(k, x_0, u)$ is not a meaningful definition of optimality.

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- (x^*, u^*) is called *overtaking optimal*[a] if

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x^*(k), u^*(k)) \geq 0 \qquad (2)$$

holds for all pairs $(x_u(\cdot, x_0), u(\cdot))$.
Note: We do not demand that $x^*(0) = x_0$.

[a] Joël Blot and Naïla Hayek. *Infinite-horizon optimal control in the discrete-time framework*. Springer, 2014.

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- $\rightsquigarrow$ Basic idea of MPC:
  Consider *finite horizon* optimization problems

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where ℓ is the stage cost and $N \in \mathbb{N}$ the *horizon length*.

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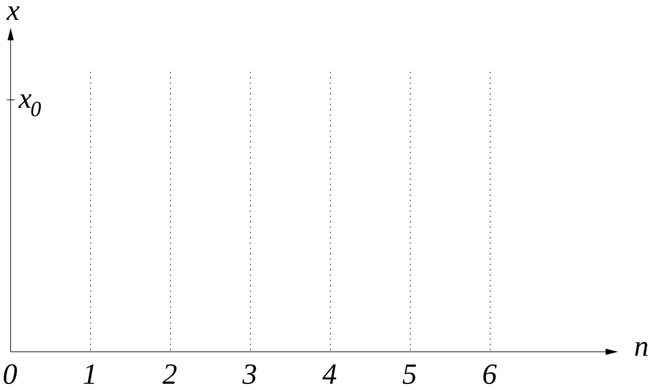
  where ℓ is the stage cost and $N \in \mathbb{N}$ the *horizon length*.
- Solve (3) instead $\rightsquigarrow u_N^*$.
- Apply $\mu_N(x_0) := u_N^*(0)$ as a feedback to the system.

MPC-closed loop

The MPC principle for optimization horizon $N = 4$:



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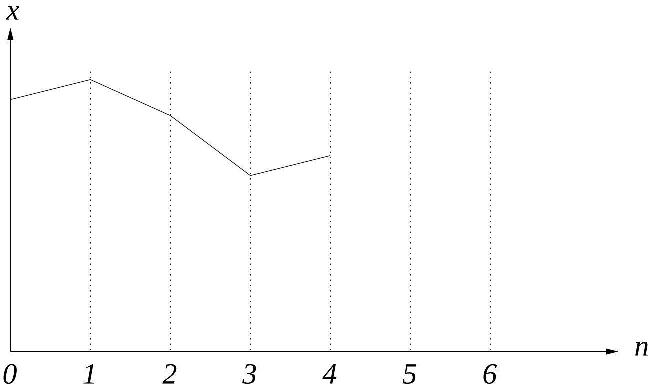

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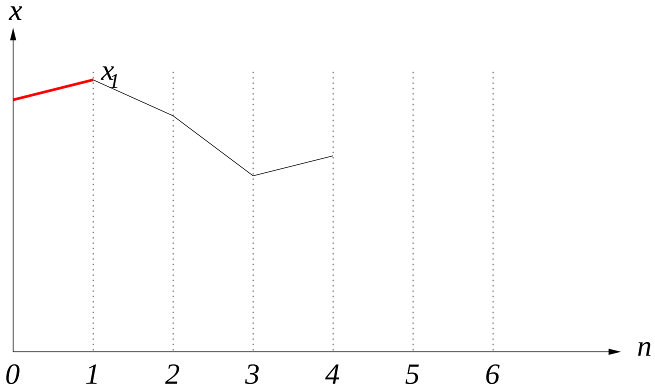

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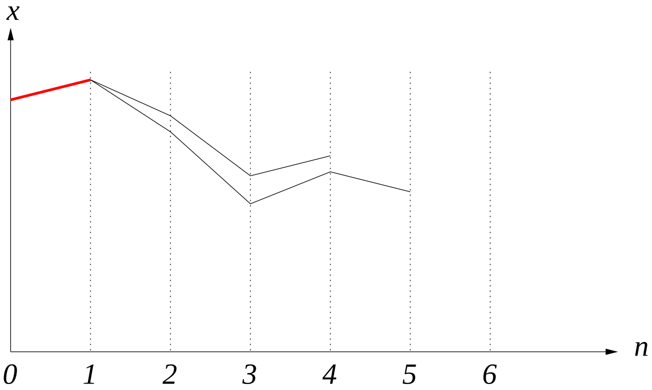

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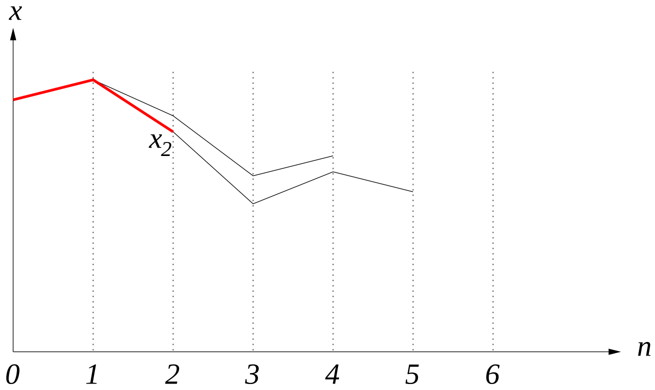

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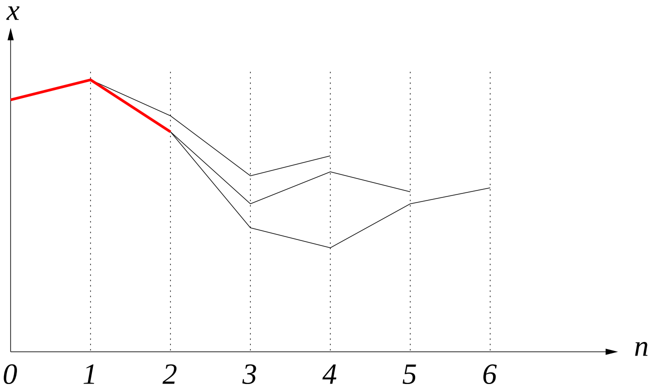

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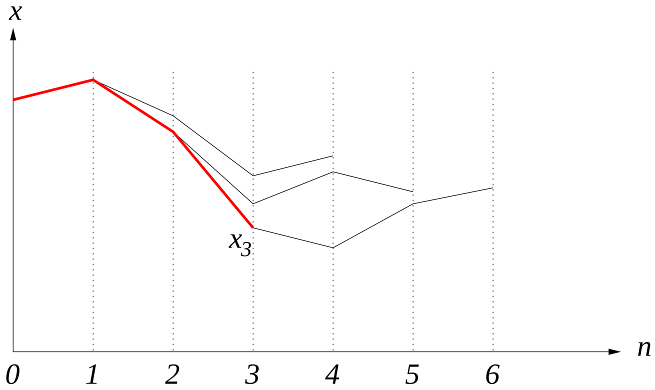

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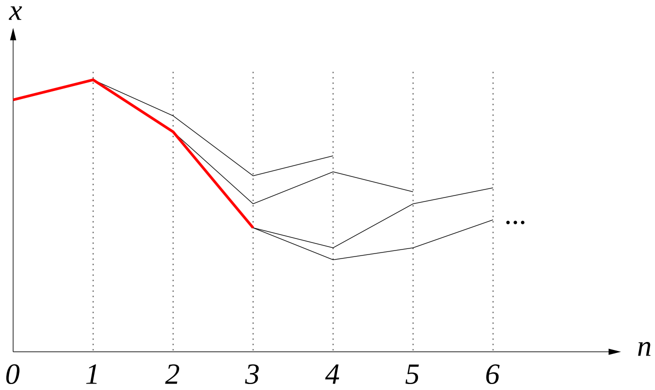

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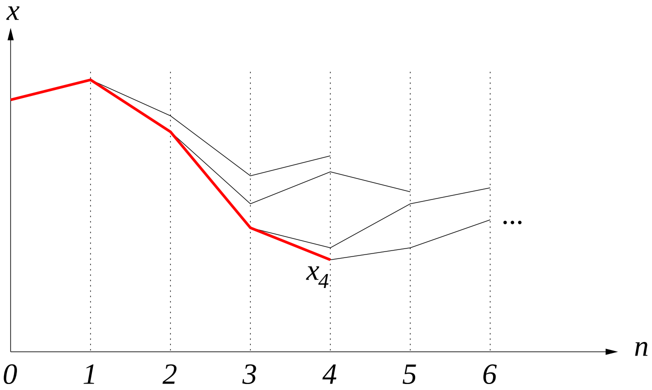

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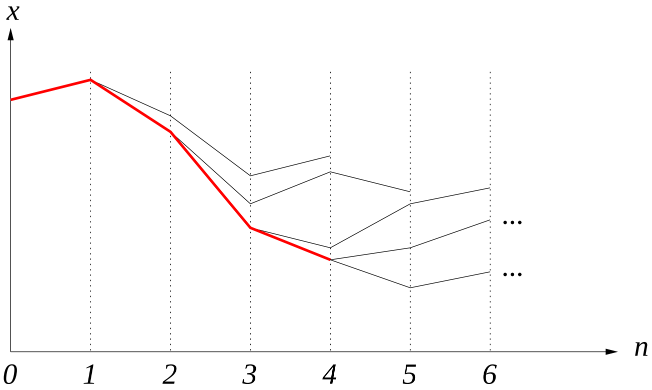

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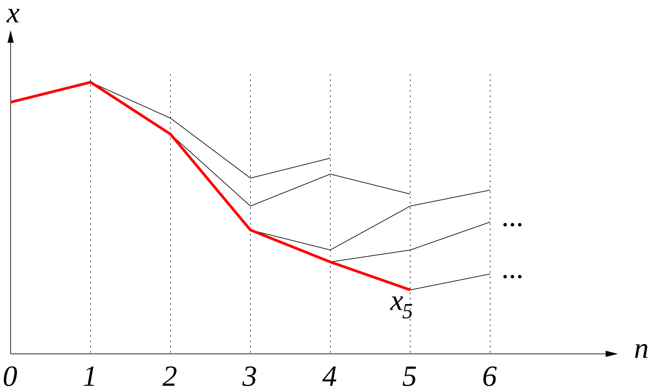

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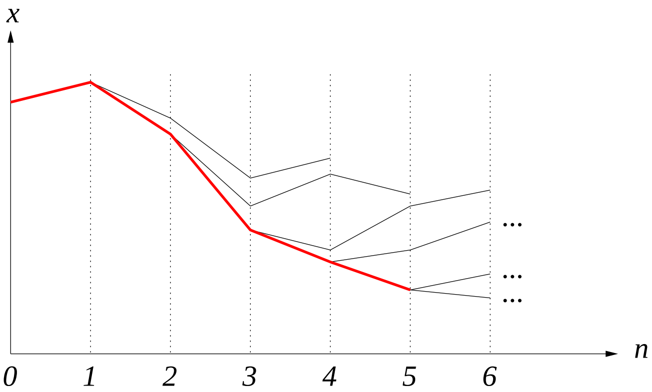

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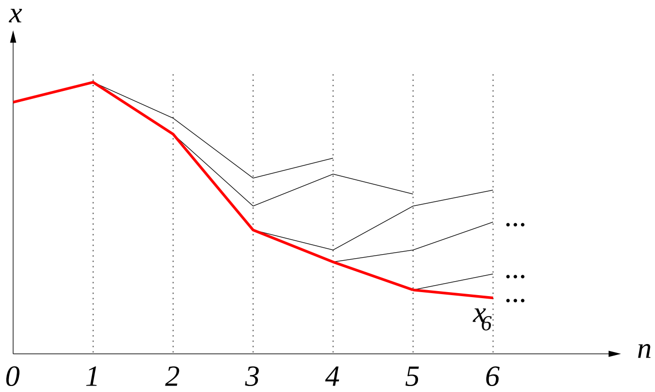

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Why use economic MPC?

- Most commonly used type of MPC: setpoint stabilization or tracking
  Stage cost penalizes distance to (good) reference trajectory:

$$\ell(k,x,u) = \|x - x^*(k)\|^2 + \|u - u^*(k)\|^2$$

[a] Lars Grüne and Marleen Stieler. "Asymptotic stability and transient optimality of economic MPC without terminal conditions". In: *SIAM Journal on Control and Optimization* 52.3 (2014), pp. 1935–1957.

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- Reference trajectories:
  Simplest case: $(x^*(k), u^*(k)) = (x^e, u^e), k \in \mathbb{N}$ optimal equilibrium
  Also possible: periodic orbit as in Matthias' talk.

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- Two problems:
  - No way to compute optimal reference a priori.
  - Tracking does not lead to optimal performance $\rightsquigarrow$ [3][a]
- Solution: Use economic criterion in stage cost function of MPC.
  In the example $\rightsquigarrow \ell(k, x, u) = u^2$

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- Look at relation between MPC closed loop cost

$$J_L^{\mathsf{cl}}(k, x_0, \mu_N) = \sum_{j=0}^{L-1} \ell(k+j, x_{\mu_N}(j, x_0), \mu_N(x_{\mu_N}(j, x_0)))$$

and some optimal value "$V_\infty(k, x_0) = \inf_{u \in \mathbb{U}^\infty(k, x_0)} J_\infty(k, x_0, u)$".

Modified cost function

To avoid the possibly infinite cost we introduce

$$\hat{\ell}(k, x, u) = \ell(k, x, u) - \ell(k, x^*(k), u^*(k)) \quad (4)$$

and the corresponding cost

$$\hat{J}_N(k, x_0, u) = \sum_{j=0}^{N-1} \hat{\ell}(k + j, x_u(j, x_0), u(j)) \quad (5)$$

for $N \in \mathbb{N}_0 \cup \{\infty\}$.

Optimal value function

- Define the optimal value function

$$\hat{V}_N(k, x_0) := \inf_{u \in \mathbb{U}^N(k, x_0)} \hat{J}_N(k, x_0, u) \tag{6}$$

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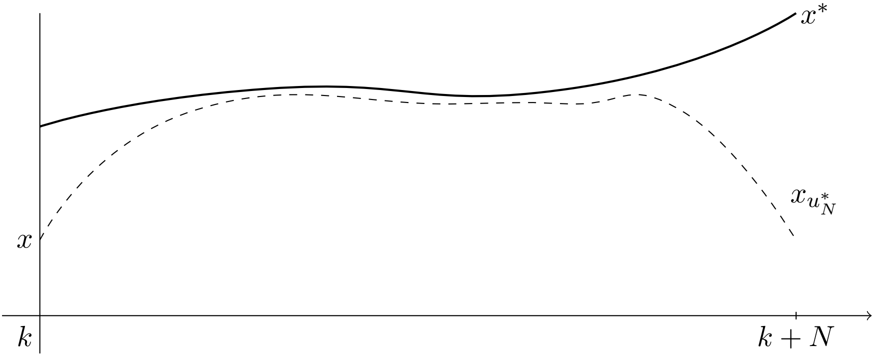
- For $N = \infty$: $\hat{V}_\infty(k, x^*(k)) = 0$.
- Key assumptions:
  - Turnpike property
  - Continuity of optimal value function

Turnpike property (finite horizon)

The turnpike property holds if optimal trajectories satisfy

$$|(x_{u_N^*}(j,x), u_N^*(j))|_{(x^*(k+j),u^*(k+j))} \leq \sigma(P) \quad \text{for all } j \in \{0,\dots,N\} \text{ with } j \notin \mathcal{Q}(k,x,P,N). \tag{7}$$

for some $\sigma \in \mathcal{L}$.



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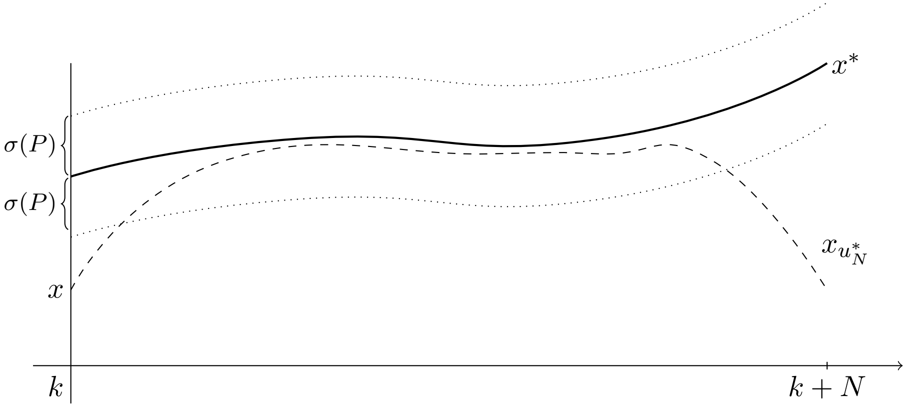

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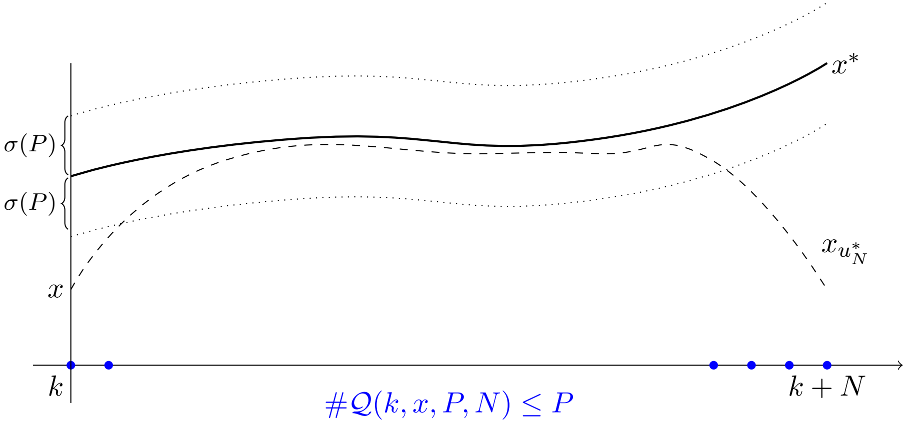

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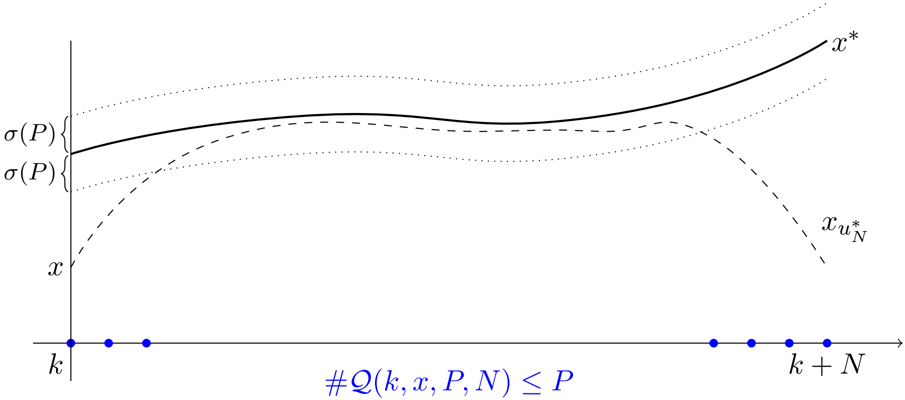

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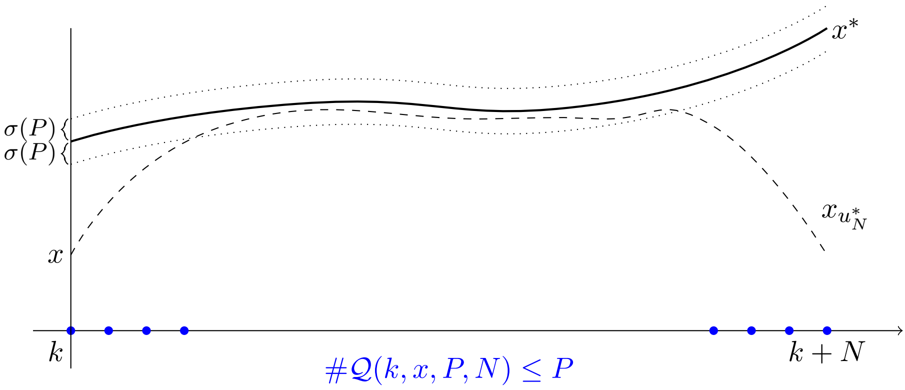

Continuity of $\hat{V}_N$ and $\hat{V}_\infty$

- We assume that the optimal value functions $\hat{V}_N$ and $\hat{V}_\infty$ are continuous in x^*, i.e. there is
  - $\varepsilon > 0$
  - $\gamma_V : \mathbb{N} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\gamma_V(N, r) \to 0$ if both $N \to \infty$ and $r \to 0$, and $\gamma_V(\cdot, r)$, $\gamma_V(N, \cdot)$ monotonous.

  such that for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$

$$|\hat{V}_N(k, x) - \hat{V}_N(k, x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|) \tag{8}$$

  for all $x \in \mathcal{B}_\varepsilon(x^*(k))$.

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  for all $x \in \mathcal{B}_\varepsilon(x^*(k))$.
- For $N = \infty$: $\rightsquigarrow$ $\hat{V}_\infty(k, x^*(k)) = 0$
  $|\hat{V}_\infty(k, x)| \leq \omega_V(\|x - x^*(k)\|)$ with $\omega_V \in \mathcal{K}_\infty$, i.e. $\hat{V}_\infty$ is bounded on $\mathcal{B}_\varepsilon(x^*(k))$.

Lemma 1

If the turnpike property holds and $\hat{V}_N$ and $\hat{V}_\infty$ are continuous in x^* then the following equations hold:

$$\hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + R_1(k,x,K)$$

with $|R_1(k,x,K)| \leq \omega_V(\sigma(P))$ for all $k \in \mathbb{N}_0$, for all $x \in \mathbb{X}(k)$, all $N \in \mathbb{N}$, all sufficiently large $P \in \mathbb{N}$ and all $K \notin \mathcal{Q}(k,x,P,\infty)$.

Lemma 2

If the turnpike property holds and $\hat{V}_N$ and $\hat{V}_\infty$ are continuous in x^* then the equation

$$\hat{J}_K(k, x, u^*_\infty) = \hat{J}_K(k, x, u^*_N) + R_3(k, x, K, N) \quad (9)$$

holds with $|R_3(k, x, K, N)| \leq 2(\gamma_V(N - K, \sigma(P)) + \omega_V(\sigma(P)))$ for all $k \in \mathbb{N}_0$, all $N \in \mathbb{N}$, all sufficiently large $P \in \mathbb{N}$, all $x \in \mathbb{X}(k)$ and all $K \in \{0, \ldots, N\} \setminus (\mathcal{Q}(k, x, P, N) \cup \mathcal{Q}(k, x, P, \infty))$.

Closed-loop cost estimate

Theorem 1

If the turnpike property holds and $\hat{V}_N$ and $\hat{V}_\infty$ are continuous in x^* then for each $k \in \mathbb{N}_0$ and each sufficiently large N, the closed loop cost satisfies

$$\hat{J}_L^{cl}(k, x, \mu_N) + \hat{V}_\infty(k + L, x_{\mu_N}(L, x)) \leq \hat{V}_\infty(k, x) + L\delta(N) \tag{10}$$

for some function $\delta \in \mathcal{L}$.

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Application: Energy efficient building operation

- PDE model[1] of airflow and temperatures inside a building:

$$\frac{\partial \mathbf{y_1}}{\partial t} + \mathbf{y_1} \cdot \nabla \mathbf{y_1} = -\frac{1}{\rho} \nabla y_2 + \nu \Delta \mathbf{y_1} - \mathbf{g} \alpha (y_3 - \tilde{y}_3) \tag{11}$$

$$\nabla \cdot \mathbf{y_1} = 0 \tag{12}$$

$$\frac{\partial y_3}{\partial t} + \mathbf{y_1} \cdot \nabla y_3 = \kappa \Delta y_3 \tag{13}$$

+ time-varying boundary conditions.

where
$\mathbf{y_1} : \Omega \times [0, \infty) \to \mathbb{R}^d$ is air velocity
$y_2 : \Omega \times [0, \infty) \to \mathbb{R}$ is pressure
$y_3 : \Omega \times [0, \infty) \to \mathbb{R}$ is temperature

and parameters $\rho, \nu, \mathbf{g}, \alpha, \tilde{y}_3, \kappa$.

[1] David J Tritton. *Physical fluid dynamics*. Springer Science & Business Media, 2012.

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$$\text{where} \quad \begin{aligned} \mathbf{y_1} &: \Omega \times [0,\infty) \to \mathbb{R}^d \text{ is air velocity} \\ y_2 &: \Omega \times [0,\infty) \to \mathbb{R} \quad \text{is pressure} \\ y_3 &: \Omega \times [0,\infty) \to \mathbb{R} \quad \text{is temperature} \end{aligned}$$

  and parameters $\rho, \nu, \mathbf{g}, \alpha, \tilde{y}_3, \kappa$.

- Joint work with Thomas Meurer, Julian Andrej (University of Kiel), and Stefan Volkwein, Luca Mechelli (University of Konstanz)

[1] David J Tritton. *Physical fluid dynamics*. Springer Science & Business Media, 2012.

Outline

Summary, Remarks and open questions

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- Time-varying system dynamics leads to time-varying optimal operating behaviour.
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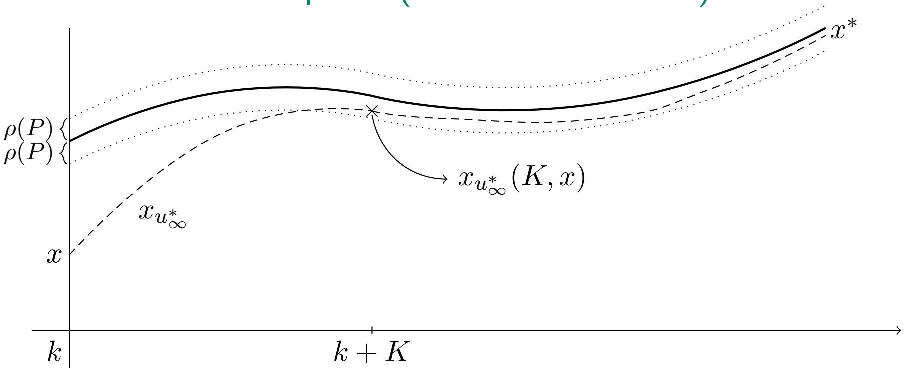
Open questions:

- Convergence of the closed-loop to x^*?
- How to verify dissipativity?

References

- Joël Blot and Naïla Hayek. *Infinite-horizon optimal control in the discrete-time framework*. Springer, 2014.
- Lars Grüne. "Approximation Properties of Receding Horizon Optimal Control". In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 118.1 (2016), pp. 3–37. ISSN: 1869-7135. DOI: 10.1365/s13291-016-0134-5. URL: http://dx.doi.org/10.1365/s13291-016-0134-5.
- Lars Grüne and Marleen Stieler. "Asymptotic stability and transient optimality of economic MPC without terminal conditions". In: *SIAM Journal on Control and Optimization* 52.3 (2014), pp. 1935–1957.
- Matthias Müller and Lars Grüne. "Economic model predictive control without terminal constraints for optimal periodic behavior". In: *Automatica* (2016).
- David J Tritton. *Physical fluid dynamics*. Springer Science & Business Media, 2012.

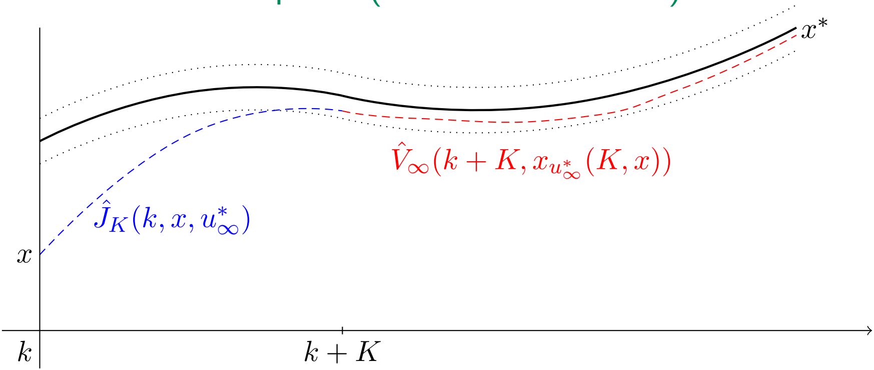
Idea of proof (for infinite horizon)



Pick $P \in \mathbb{N}$ such that $\rho(P) \leq \varepsilon$ (possible because of turnpike property).
$\rightsquigarrow$ For $K \notin \mathcal{Q}(k, x, P, \infty)$:

$$|(x_{u^*_\infty}(K,x), u^*_\infty(K))|_{(x^*(k+K), u^*(k+K))} \leq \rho(P) \leq \varepsilon$$

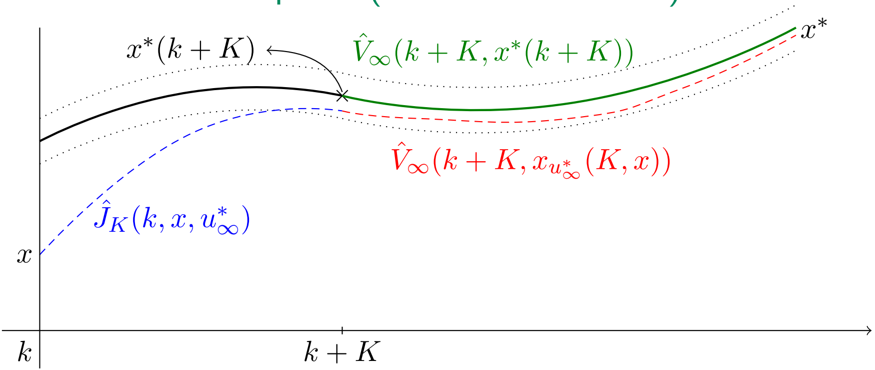
Idea of proof (for infinite horizon)



Dynamic programming principle:

$$\hat{V}_\infty(k,x) = \hat{J}_K(k,x,u_\infty^*) + \hat{V}_\infty(k+K, x_{u_\infty^*}(K,x))$$

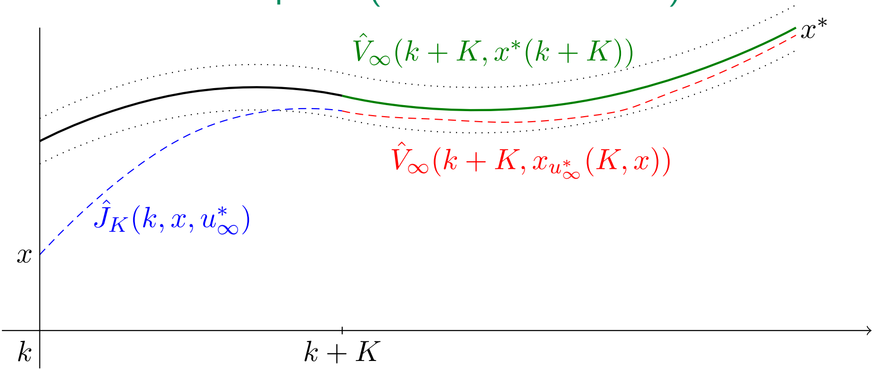
Idea of proof (for infinite horizon)



$$\hat{V}_\infty(k+K, x^*(k+K)) = 0.$$

$$\Rightarrow \hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x)) - \hat{V}_\infty(k+K, x^*(k+K))}_{=:R_1(k,x,K)}$$

Idea of proof (for infinite horizon)



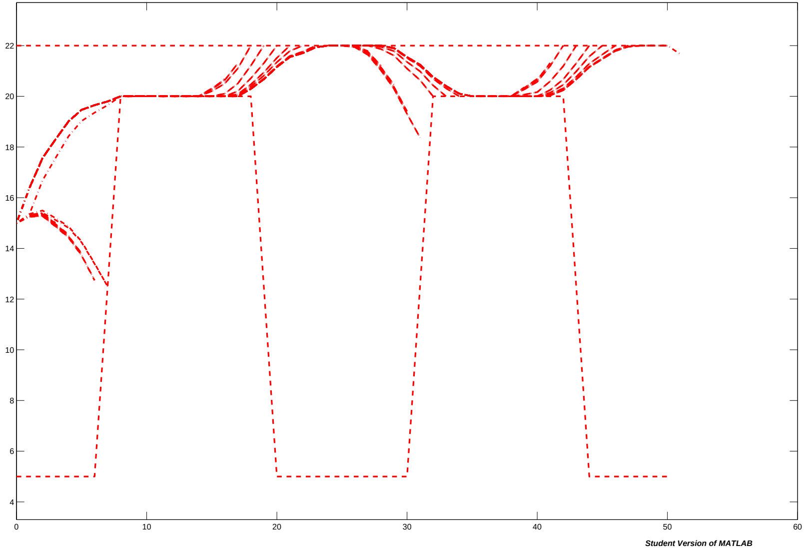
Continuity property:

$$|\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x)) - \hat{V}_\infty(k+K, x^*(k+K))| \leq \omega_V(\rho(P))$$

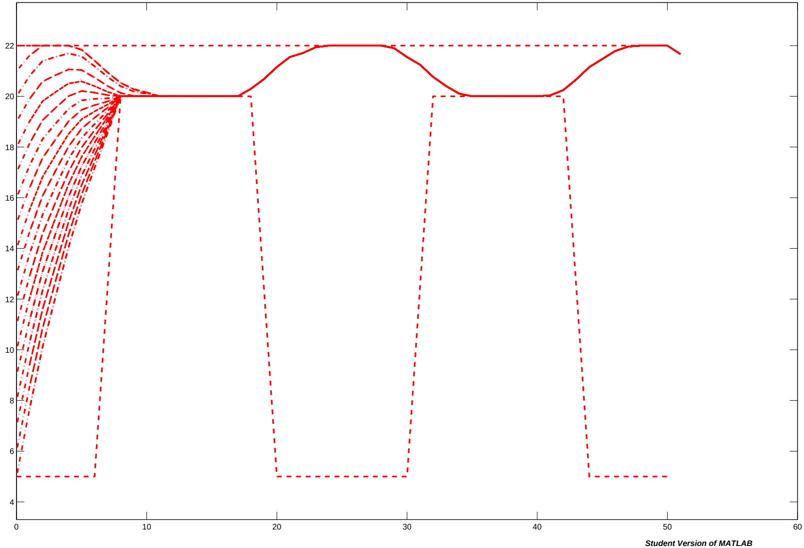
$$\Rightarrow \hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + R_1(k,x,K)$$

with $|R_1(k,x,K)| \leq \omega_V(\sigma(P))$.

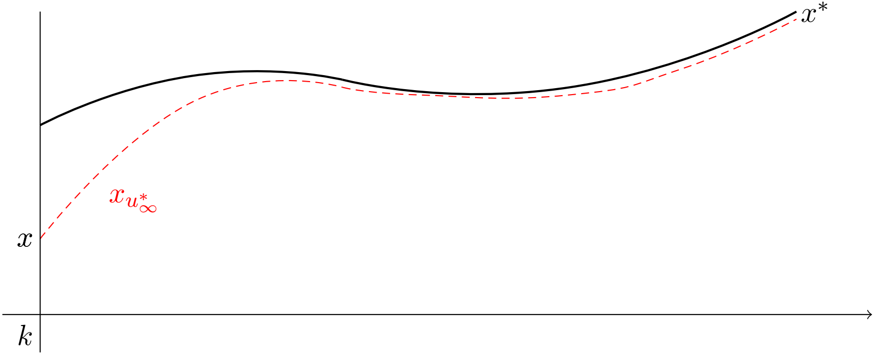
Example: turnpike property for different horizon length N



Example: turnpike property for different initial value x_0

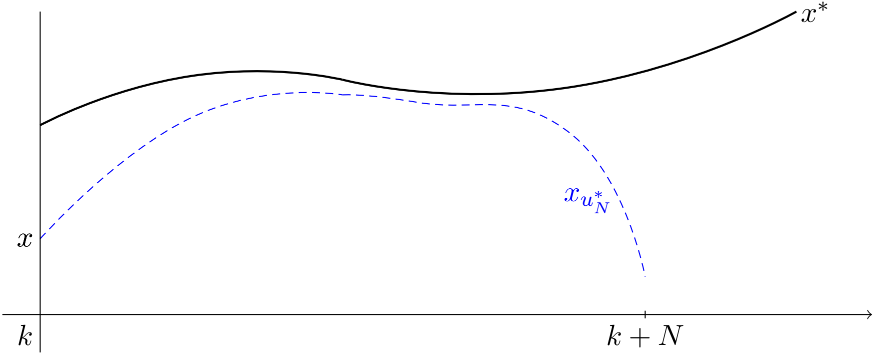


Idea of proof for Lemma 2



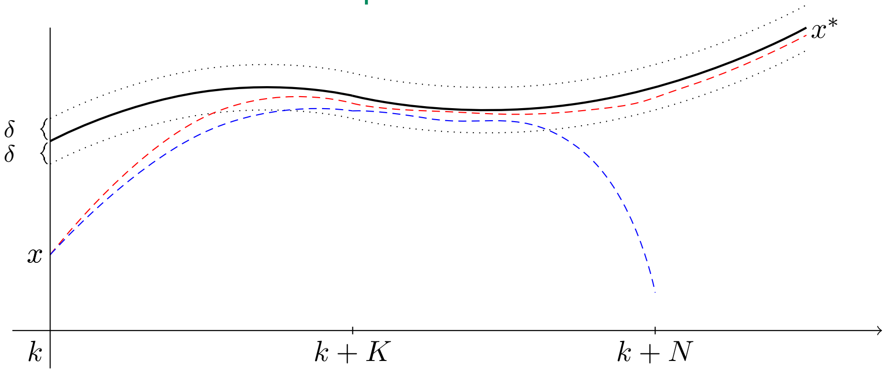
Open-loop on infinite horizon: $x_{u_\infty^*}$

Idea of proof for Lemma 2



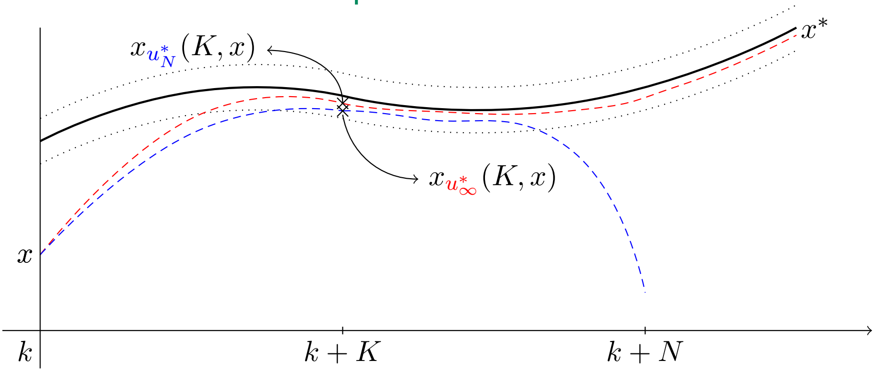
Open-loop on finite horizon: $x_{u_N^*}$

Idea of proof for Lemma 2



Choose P large enough, s.t. $\delta := \max\{\sigma(P), \rho(P)\} < \varepsilon$. Pick $K \in \{0, \ldots, N\} \setminus (\mathcal{Q}(k, x, P, N) \cup \mathcal{Q}(k, x, P, \infty))$.

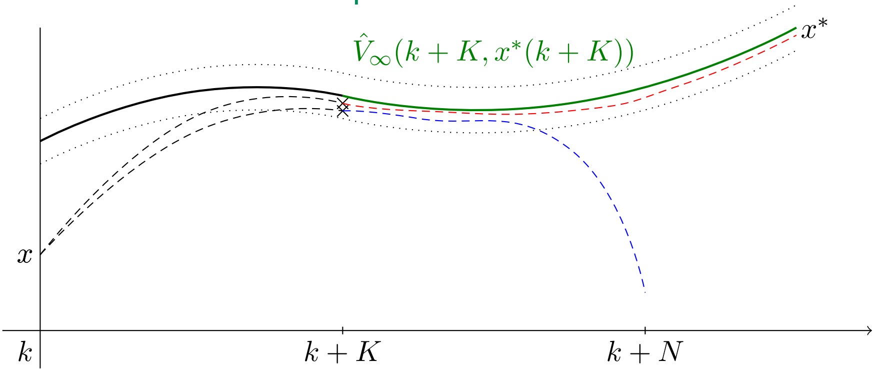
Idea of proof for Lemma 2



Optimality of u_∞^*:

$$\hat{J}_K(k,x,u_\infty^*) + \hat{V}_\infty(k+K, x_{u_\infty^*}(K,x)) \leq \hat{J}_K(k,x,u_N^*) + \hat{V}_\infty(k+K, x_{u_N^*}(K,x))$$

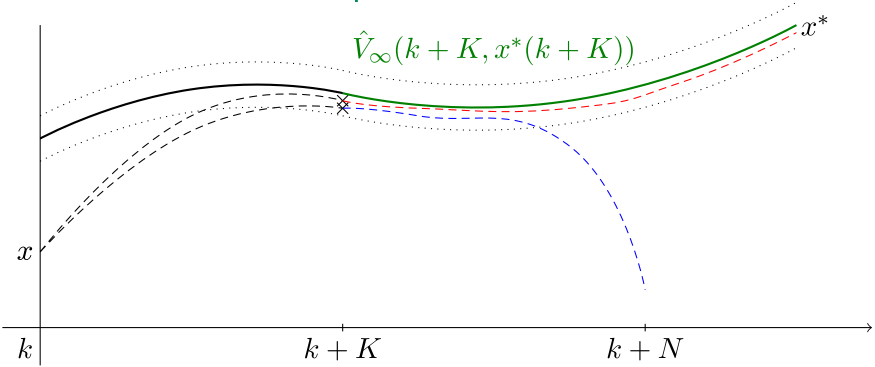
Idea of proof for Lemma 2



Apply Lemma 1:

$$\hat{J}_K(k,x,u^*_\infty) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x))}_{R_1(k,x,K)+\hat{V}_\infty(k+K,x^*(k+K))} \leq \hat{J}_K(k,x,u^*_N) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_N}(K,x))}_{\tilde{R}_2(k,x,K,N)+\hat{V}_\infty(k+K,x^*(k+K))}$$

Idea of proof for Lemma 2



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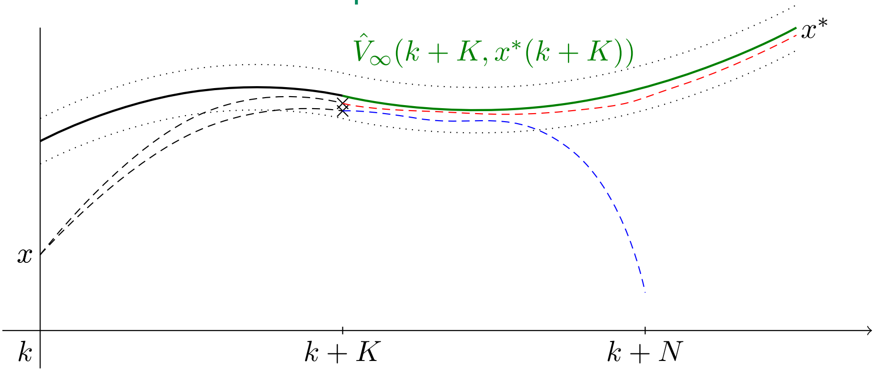
$$\Rightarrow \hat{J}_K(k,x,u^*_\infty) \leq \hat{J}_K(k,x,u^*_N) - R_1(k,x,K) + \tilde{R}_2(k,x,K,N)$$

Idea of proof for Lemma 2



Similarly: converse equality from Lemma 1 (2):

$$\hat{J}_K(k, x, u_N^*) \leq \hat{J}_K(k, x, u_\infty^*) - R_2(k, x, K, N) + \tilde{R}_1(k, x, K, N)$$

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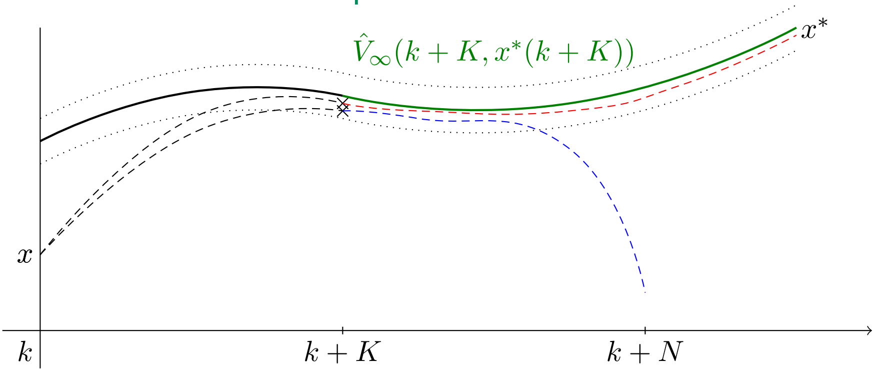

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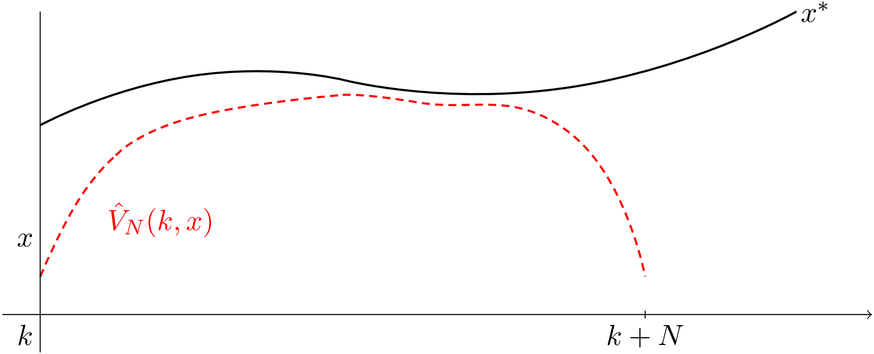
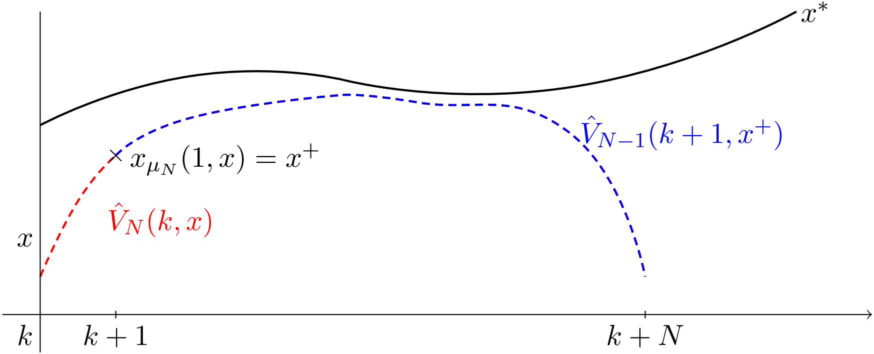
$$\Rightarrow \hat{J}_K(k,x,u_\infty^*) = \hat{J}_K(k,x,u_N^*) + R_3(k,x,K,N)$$

Idea of proof for Theorem 1



- Optimal trajectory starting in x with horizon N. $\rightsquigarrow \hat{V}_N(k,x)$

Idea of proof for Theorem 1



- Optimal trajectory starting in x^+ with horizon $N-1$.
  $\rightsquigarrow \hat{V}_{N-1}(k+1, x^+)$
- Stage cost along the MPC closed loop solution:

$$\hat{\ell}(k, x, \mu_N(x)) = \quad \hat{V}_N(k, x) - \hat{V}_{N-1}(k+1, x^+)$$

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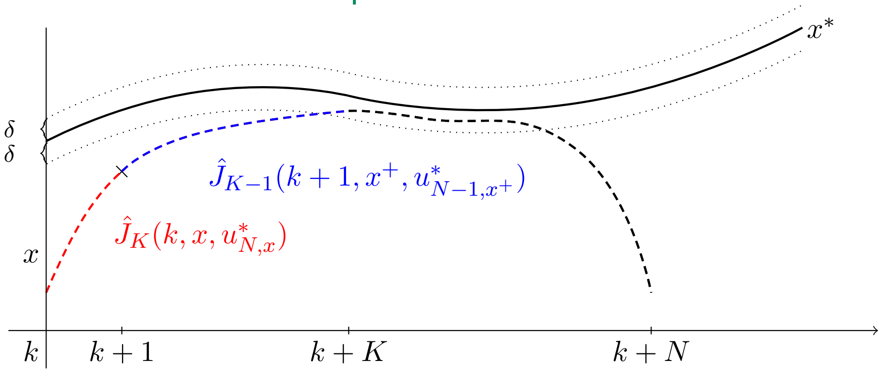

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- Stage cost along the MPC closed loop solution:

$$\begin{aligned}\hat{\ell}(k, x, \mu_N(x)) &= \hat{V}_N(k, x) - \hat{V}_{N-1}(k+1, x^+) \\ &= \hat{J}_K(k, x, u^*_{N,x}) - \hat{J}_{K-1}(k+1, x^+, u^*_{N-1,x^+})\end{aligned}$$

Idea of proof for Theorem 1

- Stage cost along the MPC closed loop solution:

$$\hat{\ell}(k,x,\mu_N(x)) = \hat{J}_K(k,x,u^*_{N,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{N-1,x^+})$$
$$\overset{\text{Lemma 2}}{=} \hat{J}_K(k,x,u^*_{\infty,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{\infty,x^+}) + R_3(\ldots) - R_3(\ldots)$$

Idea of proof for Theorem 1

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$$
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& + \underbrace{R_1(\dots) - R_1(\dots) + R_3(\dots) - R_3(\dots)}_{=:R_4(k,x,K,N)}
\end{aligned}
$$

Idea of proof for Theorem 1

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- One can show that $|R_4(k,x,K,N)| \leq \delta(N)$ for some $\delta \in \mathcal{L}$.

Idea of proof for Theorem 1

- Stage cost along the MPC closed loop solution:

$$\begin{aligned}
\hat{\ell}(k,x,\mu_N(x)) = &\quad \hat{J}_K(k,x,u^*_{N,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{N-1,x^+}) \\
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\end{aligned}$$

- One can show that $|R_4(k,x,K,N)| \leq \delta(N)$ for some $\delta \in \mathcal{L}$.
- Sum $\hat{\ell}(k+j,x,\mu_N(x))$ along the closed loop:

$$\begin{aligned}
\hat{J}^{\text{cl}}_L(k,x,\mu_N) &= \sum_{j=0}^{L-1} \hat{\ell}(k+j,x_{\mu_N}(j,x),\mu_N(x_{\mu_N}(j,x))) \\
&= \sum_{j=0}^{L-1} \hat{V}_\infty(j,x) - \hat{V}_\infty(j+1,x^+) + R_4(k,x,K,N) \\
&\leq \hat{V}_\infty(k,x) - \hat{V}_\infty(k+L,x_{\mu_N}(L,x)) + L\delta(N)
\end{aligned}$$

Boundary conditions

Temperature: $y_3 = u$ on Γ_1

$-\partial_n y_3 = \gamma(y_3 - y_{\text{out}})$ on $\Gamma \setminus \Gamma_1$

Air velocity: $\mathbf{y_1} = 0$ on Γ (No-slip BC)

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Optimal control problem:

$$\begin{aligned} &\min_u \frac{1}{2} \int_0^\infty \int_\Gamma u^2 \, dx \, dt \\ &\text{s.t.} \quad (11), (12), (13), \\ &\qquad \underline{y_3}(t) \leq y_3(t) \leq \overline{y_3}(t) \quad \forall t \in [0, \infty), \\ &\qquad \underline{u}(t) \leq u(t) \leq \overline{u}(t) \quad \forall t \in [0, \infty). \end{aligned} \tag{14}$$