Stability of Interconnected Dynamical Systems

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Motivation
    for infinite networks+

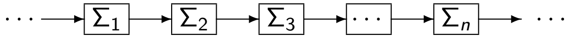
Stability of small Σ
    Only two interconnected systems

ISS-Lyapunov functions for small Σ
    Construction for 2 and $n \in \mathbb{N}$ interconneced systems

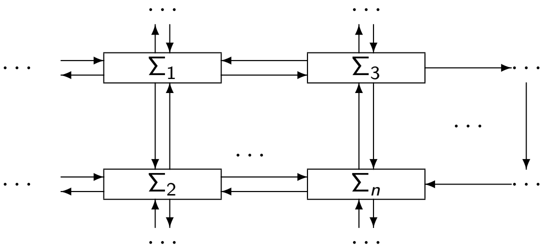
Back to ∞

Systems of systems

vehicle platooning



airplane flight formation



Flocks of birds, schools of fish, . . .

Large networks $n \approx \infty$ are modeled as spatially invariant systems

Problem statement

Given: $\Sigma_i,\ i \in \mathbb{N}$ with state $x_i \in \mathbb{R}^{n_i},\quad n_i \in \mathbb{N}$
Neighbours sending inputs to Σ_i are numbered by indices $I_i \subset \mathbb{N}$
Neighbours state $\bar{x}_i \in \mathbb{R}^{N_i}$ is composed of vectors $x_j \in \mathbb{R}^{n_j},\ j \in I_i$
ordered by the index j and $N_i := \sum_{j \in I_i} n_j$.

$$\Sigma_i : \quad \dot{x}_i = f_i(x_i, \bar{x}_i, u_i),$$

$u_i \in L_\infty^{loc}([0,\infty); \mathbb{R}^{m_i})$, $f_i : \mathbb{R}^{n_i+N_i+m_1} \to \mathbb{R}^{n_i}$ is s.t. $\exists!$ solutions.
Assume that for each Σ_i $\exists$ radially unbounded V s.t.

$$V_i(x_i) \geq \max_{k \in I_i}\{\gamma_{ik}(V_k(x_k)), \gamma_i(|u_i|)\} \ \Rightarrow\ \nabla V_i(x_i) \cdot f_i(x_i, \bar{x}_i, u_i)) \leq -\alpha_i(|x_i|)$$

here $\gamma_{ik} \in \mathcal{K}_\infty$ and α is pos. def.

Question: is the whole interconnection $\Sigma : \quad \dot{x} = f(x, u)$ stable?

Promlem statement

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}, \; u := \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}, \; f(x,u) := \begin{cases} f_1(x_1, \bar{x}_1, u_1) \\ f_2(x_2, \bar{x}_2, u_2) \\ \quad\vdots \end{cases}$$

$$\Sigma : \quad \dot{x} = f(x,u), \qquad f : \ell_\infty \times \ell_\infty \to \ell_\infty$$

Definition

$V : \ell_\infty \to [0,\infty)$ is called an ISS Lyapunov function for Σ if $\exists$ $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$ and pos. def α such that $\forall x, u \in \ell_\infty$ holds

$$\alpha_1(|x|_\infty) \leq V(x) \leq \alpha_2(|x|_\infty)$$

$$V(x) \geq \gamma(|u|_\infty) \quad \Rightarrow \quad \sum_{i=1}^{\infty} \frac{\partial V}{\partial x_i} f_i(x_i, \bar{x}_i, u_i) \leq -\alpha(|u|_\infty)$$

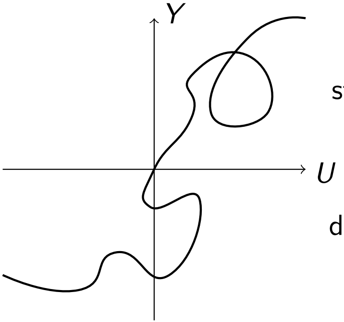
Aim: apply small-gain approach and construct V on the base of given V_i

Recalling stability of feedback systems

"I did know once, only I've sort of forgotten." (Winnie-the-Pooh)

input $u \longrightarrow \Sigma \longrightarrow y$ output

$u \in U, \quad y \in Y, \quad U, Y$ Banach spaces



graph of Σ is $G_\Sigma := \{(u,y) |\ u \in U\}$

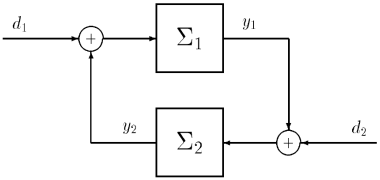
stability $:\Leftrightarrow\ \exists \gamma \in \mathcal{K} : ||y||_Y \leq \gamma(||u||_U)$

inverse graph is $G^I_\Sigma := \{(y,u) |\ u \in U\}$

distance to G_Σ: $d(x, G_\Sigma) := \inf_{z \in G_\Sigma} ||x - z||$

for simplicity: $U = Y$.

Stability of a well-defined interconnection Σ



$$\Sigma : \quad \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Graph separation theorem:

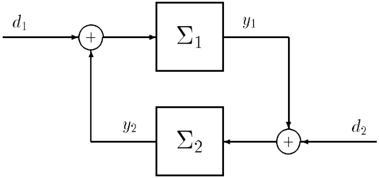
$$\Sigma \quad \text{is stable} \quad \Leftrightarrow \quad \exists \alpha \in \mathcal{K}_\infty : \quad x \in G^I_{\Sigma_2} \Rightarrow ||x|| \leq \alpha(d(x, G_{\Sigma_1}))$$

If a point on the inverse graph of Σ_2 is close to G_{Σ_1}, it must be small. If $x \in G^I_{\Sigma_2}$ is large, then $d(x, G_{\Sigma_1})$ must be large.

Remark: 1) we do not require that Σ_i is stable   2) Note: $\Leftrightarrow$

Proof of the graph separation theorem (Safonov 1977)

Th: Σ is stable $\Leftrightarrow \exists \alpha \in \mathcal{K}_\infty$: $x \in G^I_{\Sigma_2} \quad \Rightarrow \quad ||x|| \leq \alpha(d(x, G_{\Sigma_1}))$.



$$\Sigma : \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Let $\quad x := (y_2, y_1 + d_2) \in G^I_{\Sigma_2} \quad$ and $\quad z := (y_2 + d_1, y_1) \in G_{\Sigma_1}$

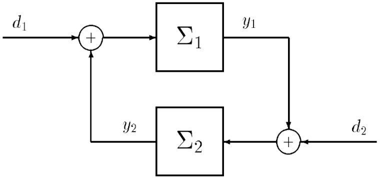
Note that: $x \in G^I_{\Sigma_2}, \quad z \in G_{\Sigma_1} \quad \Rightarrow \quad \begin{cases} (x - z) = (-d_1, d_2), \\ ||x - z|| = ||(d_1, d_2)|| \end{cases}$

"$\Rightarrow$" $\exists x \in G^I_{\Sigma_2}$ with large $||x||$ but small $d(x, G_{\Sigma_1})$, i.e., $\exists z \in G_{\Sigma_1}$ with small $||x - z|| \quad \Rightarrow \quad (d_1, d_2)$ is small. This contradicts stability.

"$\Leftarrow$" for large x $\nexists z$ close to x $\Rightarrow$ only large input can yield large x $\Rightarrow$ stability of Σ.

Remarks on graph separation theorem

Th: Σ is stable $\Leftrightarrow \exists \alpha \in \mathcal{K}_\infty$: $x \in G^I_{\Sigma_2} \quad \Rightarrow \quad ||x|| \leq \alpha(d(x, G_{\Sigma_1}))$.

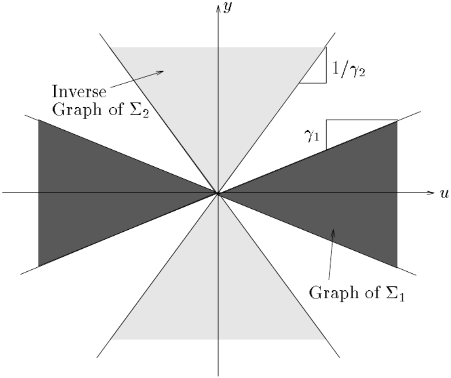


- Note: if and only if condition for stability
- Safonov used this theorem for robustness margins estimations
- Ideas of the theorem go back to conic relations of Zames 1966
- Replace $+$ with max in Σ, then stability $\Leftrightarrow G_{\Sigma_1} \cap G_{\Sigma_2} = 0$
- How this theorem can be used?

Classical small-gain theorem

Let Σ_1 and Σ_2 be finite gain stable, i.e., $\exists\, \gamma_1, \gamma_2 > 0$ s.t.

$$||y_1|| \leq \gamma_1 ||u_1|| \quad \text{and} \quad ||y_2|| \leq \gamma_2 ||u_2||$$

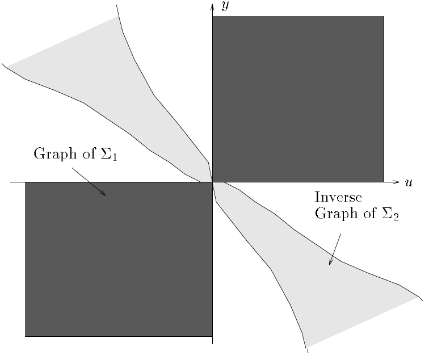


Theorem:
$\gamma_1\gamma_2 < 1 \Rightarrow \Sigma$ is stable

Remark: This and the next 3 figures are taken from "Input-output Stability" by Teel, Georgiou, Praly and Sontag in "The Control Handbook", CRC Press 1996

Classical passivity theorem

Σ is passive if $\forall\,(u,y)\in G_\Sigma$ holds $\langle u,y\rangle := \int_0^\infty u^T(t)y(t)\,dt \geq 0$
Σ is strictly passive if $\exists\,\varepsilon>0$ with $\langle u,y\rangle \geq \varepsilon(||u||_2^2+||y||_2^2)$
Σ is input strictly passive: $\langle u,y\rangle \geq \varepsilon(||u||_2^2)$
Σ is output strictly passive: $\langle u,y\rangle \geq \varepsilon(||y||_2^2)$



Theorem:
If Σ_1 is passive and Σ_2 scaled by -1 is strictly passive
$\Rightarrow$ Σ is stable wrt $||\cdot||_2$

Classical passivity theorem

Σ is passive if $\forall (u,y) \in G_\Sigma$ holds $\langle u,y \rangle := \int_0^\infty u^T(t) y(t)\, dt \geq 0$
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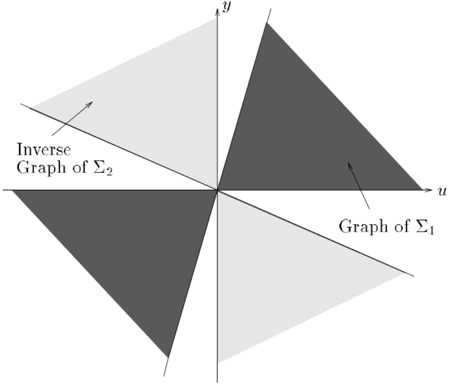
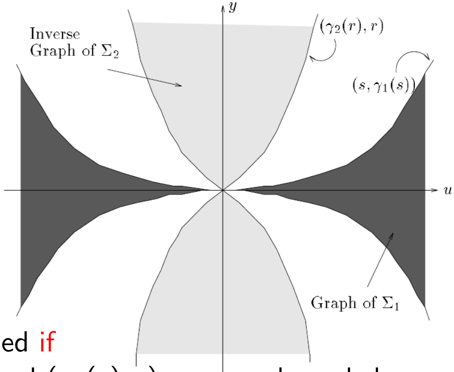


Theorem:
Σ_1 strictly input (output) passive
Σ_2 scaled by -1 strictly input (output) passive
$\Rightarrow \Sigma$ is stable wrt $||\cdot||_2$

Remark: NL: $\langle u,y \rangle \geq ||u||_2 \rho(||u||_2) + ||y||_2 \rho(||y||_2), \quad \rho \in \mathcal{K}_\infty$

Nonlinear small-gain theorem



Consider Σ_i with
stability gain function $\gamma_i \in \mathcal{K}_\infty$:

$$||y_i|| \leq \gamma_i(||u_i||), \quad i = 1, 2.$$

Graph separation condition is satisfied if
dist. between the curves $(s, \gamma_1(s))$ and $(\gamma_2(r), r)$ grows unbounded
$\Leftrightarrow$
$\exists\, \rho \in \mathcal{K}_\infty$ s.t. the curves $(s, \gamma_1(s) + \rho(s))$ and $(\gamma_2(r) + \rho(r), r)$
have only one common point 0
$\Leftrightarrow$
$(\gamma_1 + \rho) \circ (\gamma_2 + \rho)(r) < r, \quad r > 0$ (in max-case $\gamma_1 \circ \gamma_2(r) < r$)

Derivation of the nonlinear gain γ

Consider a stable system $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$

Let $V : \mathbb{R}^n :\to \mathbb{R}_+$ be smooth and $\exists\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ s.t.

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

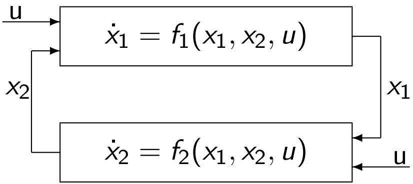
$$\dot{V}(x) := \nabla V \cdot f(x,u) \leq -\alpha_3(|x|) + \alpha_4(|u|)$$

$h(0,0) = 0$ and continuity $\Rightarrow$ $\exists \phi_x, \phi_u \in \mathcal{K}_\infty$ s.t.
$|h(x,u)| \leq \phi_x(|x|) + \phi_u(|u|)$

Then we can take

$$\gamma := \phi_x \circ \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \circ \alpha_4 + \phi_u$$

see Sontag 1989



$$V_1(x_1) \geq \max\left\{gamma_{12}(V_2(x_2)),\ \gamma_1(|u|)\right\} \ \Rightarrow\ \nabla V_1(x_1) f_1(x_1, x_2, u) \leq -\alpha_1($$

$$V_2(x_2) \geq \max\left\{\gamma_{21}(V_1(x_1)),\ \gamma_2(|u|)\right\} \ \Rightarrow\ \nabla V_2(x_2) f_2(x_1, x_2, u) \leq -\alpha_2(V_2)$$

Theorem (Jiang, Mareels, Wang 1996)

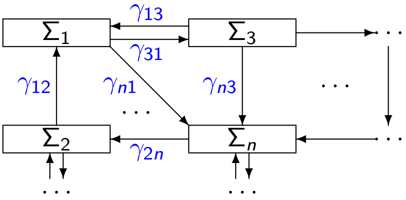
$$\begin{matrix}\forall r > 0 \\ \gamma_{12} \circ \gamma_{21}(r) < r\end{matrix} \Rightarrow \quad \dot{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\bullet} = \begin{pmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{pmatrix} =: f(x, u)\ \mathit{ISS}$$

with ISS-Lyapunov function $V(x_1, x_2) = \max\{\sigma(V_1(x_1)), V_2(x_2)\}$
where σ *is any* $\mathcal{K}_\infty$*- function with* $\gamma_{21} \leq \sigma \leq \gamma_{12}^{-1}$

Stability of n coupled systems

Consider

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u) \end{aligned}$$



with

$$|x_i(t)| \leq \max\left\{\beta_i(|x_i(0)|, t),\ \max_{j=1}^{n} \gamma_{ij}(||x_j||_\infty),\ \eta_i(||u||_\infty)\right\}$$

and $\quad \gamma_{ij} \equiv 0 \quad$ or $\quad \gamma_{ij} \in \mathcal{K}_\infty, \quad$ and $\quad \gamma_{ii} := 0.$

The gain matrix

$$\Gamma := (\gamma_{ij}) = \begin{bmatrix} 0 & \gamma_{12} & \cdots & \cdots & \gamma_{1n} \\ \gamma_{21} & 0 & \gamma_{23} & \cdots & \gamma_{2n} \\ \vdots & & & & \vdots \\ \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-2} & 0 & \gamma_{n-1,n} \\ \gamma_{n1} & \cdots & \cdots & \gamma_{n,n-1} & 0 \end{bmatrix}$$

$$\Gamma_{\max} : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \qquad \Gamma(s) = \begin{bmatrix} \max_{j=1}^n \gamma_{1j}(s_j) \\ \vdots \\ \max_{j=1}^n \gamma_{nj}(s_j) \end{bmatrix}$$

In case of gain summation

$$\Gamma_{\sum} : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \qquad \Gamma(s) = \begin{bmatrix} \sum_{j=1}^n \gamma_{1j}(s_j) \\ \vdots \\ \sum_{j=1}^n \gamma_{nj}(s_j) \end{bmatrix}$$

Small-gain theorem

$$\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \quad \Gamma(s) = \begin{bmatrix} \max_{j=1}^n \gamma_{1j}(s_j) \\ \vdots \\ \max_{j=1}^n \gamma_{nj}(s_j) \end{bmatrix}$$

Theorem (S.D., B. Rüffer, F. Wirth 2007)

$$\Gamma(s) \not\geq s \quad \forall\ s \in \mathbb{R}^n_+,\ s \neq 0 \quad \Rightarrow \quad \dot{x} = f(x,u) \qquad ISS$$

Notation: $x = (x_1^T, \ldots, x_n^T)^T$ and $f = (f_1^T, \ldots, f_n^T)^T$

Remarks: 1) For $x, y \in \mathbb{R}^n$ holds $x \not\geq y \quad \Leftrightarrow \quad \exists i$ with $x_i < y_i$.
2) Equivalence to cycle condition: all cycles are contractions.
3) In case of linear gains: $\Gamma(s) \not\geq s \quad \Leftrightarrow \quad \rho(\Gamma) < 1$

Associate discrete time system

Let Γ be a nonlinear operator as above. Consider

$$s_{k+1} := \Gamma(s_k), \qquad s_0 \in \mathbb{R}^n_+, \quad k = 0, 1, 2, \dots, \qquad (*)$$

Theorem

$\Gamma(s) \not\geq s$ for all $s \in \mathbb{R}^n_+ \setminus \{0\} \quad \Leftrightarrow \quad$ (*) is GAS

Remarks:

- The system (*) can be used as a comparison system
- Note the dimension reduction compared with original system

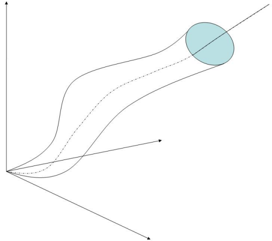
Topological property

The above observations help to prove:

Theorem (S.D., B. Rüffer, F. Wirth 2010)

$$\Gamma(s) \not\geq s \quad \forall\, s \in \mathbb{R}^n_+ \setminus \{0\} \quad \Rightarrow \quad \exists\, \sigma_1, \ldots, \sigma_n \in \mathcal{K}_\infty :$$

$$\forall\, t > 0 : \quad \Gamma(\sigma(t)) < \sigma(t), \quad \sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))^T$$



$\sigma : [0, \infty) \to \mathbb{R}^n_+$
is called Ω-path

Geometric interpretation

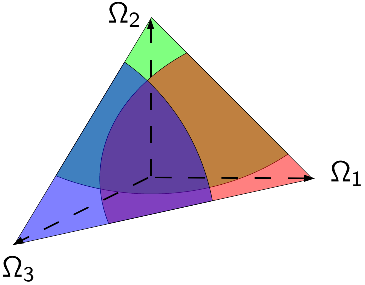
Cones for inverse graphs

$$\Omega_i = \left\{ x \in \mathbb{R}^n : \ s_i > \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

$$\tilde{\Omega}_i = \left\{ x \in \mathbb{R}^N : \ |x_i| > \sum_{j \neq i} \gamma_{ij}(|x_j|) \right\}.$$

$\Gamma(s) \not\geq s \ \forall s \neq 0, s \geq 0$ is equivalent to

- $\bigcup_{i=1}^{n} \tilde{\Omega}_i = \mathbb{R}^N \setminus \{0\}$ and
- $\bigcap_{i=1}^{n} \tilde{\Omega}_i \neq \emptyset$.



On $\tilde{\Omega}_i$ there exist ISS Lyapunov functions V_i with $\dot{V}_i(x) = \nabla V_i(x) \cdot f_i(x) < 0$ if $x \in \Omega_i$.

The ISS-Lyapunov function for the network is $V(x) = \max_i \{\sigma_i^{-1}(V_i(x_i))\}$.

ISS-Lyapunov function for networks

Let V_i be an ISS-Lyapunov function for the i-th system:

$$\psi_{i1}(|x_i|) \le V_i(x_i) \le \psi_{i2}(|x_i|), \quad x_i \in \mathbb{R}^{N_i}$$

$$V_i(x_i) \ge \max_{j=1,\dots,n}\{\gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\} \quad \Rightarrow \quad \dot{V}_i(x) \le -\alpha_i(V_i(x_i))$$

Define $\quad \Gamma = (\gamma_{ij})_{i,j=1,\dots,n}, \qquad \Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+.$

Theorem (S.D., B. Rüffer, F. Wirth (2010))

$$\Gamma(s) \not\geq s \qquad \forall\, s \in \mathbb{R}^n_+ \setminus \{0\} \quad \Rightarrow \quad V(x) = \max_i\{\sigma_i^{-1}(V_i(x_i))\}$$

is ISS-Lyapunov function for $\dot{x} = f(x,u)$.

Small-gain theory for other types of systems

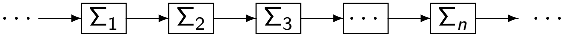
These small gain results were recently extended to other classes of systems:

- discrete time systems
- switched systems
- impulsive systems
- hybrid systems
- infinite dimensional systems

Additional poprties or conditions are sometimes needed (dwell-time, uniformity, Zeno solutions).

Example 1

Cascade of finite number of ISS Σ_i is ISS, but for $u, x_i \in \mathbb{R}, \; i \in \mathbb{N}$



$$\begin{cases} \Sigma_1 : & \dot{x}_1 = -x_1 + u \\ \Sigma_2 : & \dot{x}_2 = -x_2 + 2x_1 \\ \dots & \\ \Sigma_k : & \dot{x}_{k+1} = -x_{k+1} + 2x_k \\ \dots & \end{cases}$$

each Σ_i is ISS, however the cascade is not ISS:
take $u = 1$ and $x_i(0) = 1, \; i \in \mathbb{N}$ then
$\forall\, t \geq 0 \quad \dot{x}_2 > 0, \dot{x}_3 > 0, \dots \quad \Rightarrow \quad |x|_{\ell_\infty}$ grows at any time,i.e.

the solution grows unbounded to the constant point given by
$x_1 = 1, \; x_2 = 2, \dots, x_k = 2^{k-1}, \dots$ and in particular
$\lim_{t\to\infty} |x(t)|_\infty = \infty$ contradicting ISS

Example 2: we make the gains smaller

In the next example all gains are identities:

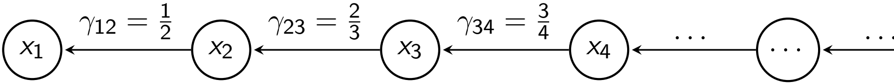
$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + x_1 \\ \dots \\ \dot{x}_{k+1} = -x_{k+1} + x_k \\ \dots \end{cases}$$

Taking zero input $u = 0$ and initial state $x_i(0) = 1,\ i \in \mathbb{N}$ the solution is given by $x_i(t) = e^{-t}\left(\sum_{k=0}^{i} \frac{t^k}{k!}\right),\ i \in \mathbb{N}$ for which $\lim_{t\to\infty} ||x||_\infty = 1 \neq 0$ contradicting the ISS property.

Example 3: we make the gains smaller again

In the next example all gains are smaller than the identity

$$
\begin{cases}
\dot{x}_1 = -2x_1 + x_2 + u_1 \\
\dot{x}_2 = -\frac{3}{2}x_2 + x_3 + u_2 \\
\dots \\
\dot{x}_k = -\frac{k+1}{k}x_k + x_{k+1} + u_k \\
\dots
\end{cases}
$$



This is an infinite cascade with the interconnection gains

$\gamma_{ij} = 0 \ \Leftrightarrow \ j \neq i + 1$ and $\gamma_{k,k+1} = \frac{k}{k+1} < \text{id}, \ k \in \mathbb{N}$

The matrix A of the system has an unbounded inverse A^{-1}

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{3}{2} & 1 & 0 & 0 & \cdots \\ 0 & 0 & -\frac{4}{3} & 1 & 0 & \cdots \\ 0 & 0 & 0 & -\frac{5}{4} & 1 & \cdots \\ 0 & 0 & 0 & 0 & -\frac{5}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} & -\frac{1}{6} & \cdots \\ 0 & -\frac{2}{3} & -\frac{1}{2} & -\frac{2}{5} & -\frac{1}{3} & \cdots \\ 0 & 0 & -\frac{3}{4} & -\frac{3}{5} & -\frac{1}{2} & \cdots \\ 0 & 0 & 0 & -\frac{4}{5} & -\frac{2}{3} & \cdots \\ 0 & 0 & 0 & 0 & -\frac{5}{6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\Rightarrow \lambda = 0 \in \sigma(A) \quad \Rightarrow$ the system is not 0-GAS, hence it is not ISS.

$$\Gamma = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{2}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{4}{5} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \Rightarrow \quad \forall s \in \ell_\infty^+ \setminus \{0\} \quad \Gamma(s) < s.$$

Γ satisfies the usual SGC, but Σ is not ISS

This example shows: previously known SGC is not enough.
Moreover in spite of $\Gamma(s) < s$ we have $1 \in \sigma(\Gamma)$, because $(\mathrm{id} - \Gamma)^{-1}$

$$= \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 1 & -\frac{2}{3} & 0 & 0 & \cdots \\ 0 & 0 & 1 & -\frac{3}{4} & 0 & \cdots \\ 0 & 0 & 0 & 1 & -\frac{4}{5} & \cdots \\ 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 0 & 1 & \frac{2}{3} & \frac{1}{2} & \frac{2}{5} & \cdots \\ 0 & 0 & 1 & \frac{3}{4} & \frac{3}{5} & \cdots \\ 0 & 0 & 0 & 1 & \frac{4}{5} & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is unbounded. The spectral radius of Γ does not satisfy $r(\Gamma) < 1$, which is different from the finite dimensional case.

$$x_k = \Gamma(x_{k-1}) \text{ is GAS} \quad \not\Leftarrow \quad \Gamma(s) \not\geq s \text{ for all } s \in \mathbb{R}^n_+ \setminus \{0\}$$

Hypothesis

For $i \in \mathbb{N}$ consider Σ_i with gains $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j \in \mathbb{N}$

- The number of neigbours of any Σ_i is uniformly bounded
- Any cycle $(i_1 = i_k)$ is a contraction:
  $\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_{k-1} i_k}(r) < r, \quad r > 0$
- $\exists c \in (0,1),\ M \in \mathbb{N} \quad \forall\, i_1, i_2 \ldots, i_k$ with $k \geq M$
  $\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_{k-1} i_k}(r) < cr, \quad r > 0$

Then

- $Q(x) := sup\{x, \Gamma(x), \Gamma^2(x), \ldots\}$ is a well defined map $Q : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfying $\Gamma(Q(x)) \leq Q(x),\ x \in \mathbb{R}^n_+$
- $\sigma_i(r) := [Q(\bar{1})]_i$ is a $\mathcal{K}_\infty$ function
- $V(x) := \sup_j\{\sigma_j^{-1}\}(V_j(x_j))$ satisfies

$$V(x) \geq \gamma(|u|_\infty) \quad \Rightarrow \quad \sum_{i=1}^{\infty} \frac{\partial V}{\partial x_i} f_i(x_i, \bar{x}_i, u_i) \leq -\alpha(|u|_\infty)$$