Optimal control of infinite dimensional systems

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The state equation

Let $\mathcal{H}$ be a reflexive Banach space. We consider the bilinear equation

$$\dot{y}(t) + \mathcal{A}y(t) = \mathcal{F}(t) + u(t)(\mathcal{B}_1 + \mathcal{B}_2 y(t)); \quad y(0) = y_0;$$

with $y_0 \in \mathcal{H}$; $\mathcal{F} \in L^1(0, T; \mathcal{H})$; $\mathcal{B}_1 \in \mathcal{H}$; $\mathcal{B}_2 \in L(\mathcal{H})$; $u \in L^1(0; T)$.

$\mathcal{A}$ is an unbounded operator on $\mathcal{H}$ that is the infinitesimal generator of the strongly continuous semigroup

$$T(t) := e^{-tA}.$$

C^0 or strongly continuous semigroups

Family $T(t)$, for $t \geq 0$, of bounded linear operators such that $T(0) = I$ and

$$T(s+t) = T(s)T(t), \quad s, t \geq 0$$

$$x = \lim_{t \downarrow 0} T(t)x, \quad \text{for all } x \in \mathcal{H}.$$

Then there exists $M \geq 1$, $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$

Infinitesimal generator of a C^0 semigroup

(Unbounded) linear operator $\mathcal{A}$ in $\mathcal{H}$ such that

$$\mathcal{A}x = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}$$

with domain the set of x such that the above limit exists.

Characterization of C^0 semigroups

Resolvent

If $\lambda I + \mathcal{A}$ is invertible with a bounded inverse, we say that λ belongs to the resolvent set $\rho(\mathcal{A})$ and denote by

$$R_\lambda(\mathcal{A}) := (\lambda I + \mathcal{A})^{-1}$$

the resolvent.

Theorem

A linear operator $\mathcal{A}$ is the infinitesimal generator of a C^0 semigroup $T(t)$ such that $\|T(t)\| \leq Me^{\omega t}$, iff $\mathcal{A}$ is closed with dense domain, and for all $\lambda > \omega$, $\lambda \in \rho(\mathcal{A})$ and

$$\|R_\lambda(\mathcal{A})^n\| \leq M/(\lambda - \omega)^n, \quad n = 1, 2, \ldots$$

If $M = 1$, $\omega = 0$ we have a contraction semigroup: $\|T(t)\| \leq 1$.
Ref: A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer, 1983 (with convention $-\mathcal{A}$ instead of $\mathcal{A}$).

Differential equations

For the semigroup $T(t) = e^{-t\mathcal{A}}$ we have: If $\mathcal{A} \in L(\mathcal{H})$ then

$$e^{-t\mathcal{A}} = I - t\mathcal{A} + \frac{1}{2}t^2\mathcal{A}^2 + \cdots$$

For $f \in L^1(0,T;\mathcal{H})$ consider the differential equation over $(0,T)$:

$$\dot{y} + \mathcal{A}y = f; \quad y(0) = y_0.$$

The mild, or semigroup solution is by the definition

$$y(t) = e^{-t\mathcal{A}}y_0 + \int_0^t e^{-(t-s)\mathcal{A}} f(s)$$

Nonlinear differential equations

For $F : \mathcal{H} \to \mathcal{H}$ we define the mild solution of

$$\dot{y}(t) + \mathcal{A}y(t) = F(y(t)) + f(t);\ t \in (0,T); \quad y(0) = y_0$$

by

$$y(t) = e^{-t\mathcal{A}}y_0 + \int_0^t e^{-(t-s)\mathcal{A}}(F(y(s)) + f(s))\mathrm{d}s$$

whenever this is fixed-point equation, it is well-defined (as is e.g. if F is Lipschitz).

Dual semigroup

If $\mathcal{A}$ (unbounded) linear operator in $\mathcal{H}$ with domain $D(\mathcal{A})$: its dual $\mathcal{A}^*$ is the linear operator over $\mathcal{H}^*$ with domain

$$\{x^* \in \mathcal{H}^*;\ \exists y^* \in \mathcal{H}^*;\ \langle x^*, \mathcal{A}x\rangle = \langle y^*, x\rangle, \quad \text{for all } x \in D(\mathcal{A})\ \}.$$

If $\lambda \in \rho(\mathcal{A})$ then $R_\lambda(\mathcal{A})^* = R_\lambda(\mathcal{A}^*)$.

Theorem

Let $\mathcal{A}$ be the infinitesimal generator of a C^0 semigroup $e^{-t\mathcal{A}}$. Then

the semigroup $(e^{-t\mathcal{A}})^*$ over $\mathcal{H}^*$ is C^0 and its generator is $\mathcal{A}^*$.

Adjoint equation

Consider the direct and adjoint differential equation, where $a \in L(\mathcal{H})$, $f \in L^1(0, T; \mathcal{H})$, $g \in L^1(0, T; \mathcal{H})$:

$$\dot{y}(t) + \mathcal{A}y(t) = ay(t) + f(t);\ t \in (0, T); \quad y(0) = y_0.$$

$$-\dot{p}(t) + \mathcal{A}^*p(t) = a^*p(t) + g(t);\ t \in (0, T); \quad p(T) = p_T.$$

The semigroup solutions in $C(0, T; \mathcal{H})$ and $C(0, T; \mathcal{H}^*)$ are

$$y(t) = e^{-t\mathcal{A}}y_0 + \int_0^t e^{-(t-s)\mathcal{A}}(ay(s) + f(s))\mathrm{d}s$$

$$p(t) = e^{-(t-T)\mathcal{A}^*}p_T + \int_t^T e^{-(t-s)\mathcal{A}^*}(a^*p(s) + g(s))\mathrm{d}s$$

Integration by parts (IBP)

$$\dot{y}(t) + \mathcal{A}y(t) = ay(t) + f(t);\ t \in (0,T); \quad y(0) = y_0,$$
$$-\dot{p}(t) + \mathcal{A}^*p(t) = a^*p(t) + g(t);\ t \in (0,T); \quad p(T) = p_T.$$

We have that

$$\langle p(T), y(T)\rangle + \int_0^T \langle g(t), y(t)\rangle \mathrm{d}t = \langle p(0), y(0)\rangle + \int_0^T \langle p(t), f(t)\rangle \mathrm{d}t.$$

Application to optimal control:

y solution of linearized state equation

p costate

LHS = directional derivative of cost
RHS = expression of reduced gradient

The optimal control problem

Bilinear state equation

$$\dot{y} + \mathcal{A}y = \mathcal{F} + u(\mathcal{B}_1 + \mathcal{B}_2 y); \quad y(0) = y_0.$$

Cost function for symmetric $Q, Q_T \in L(\mathcal{H}, \mathcal{H}^*)$ inducing quadratic forms, and $\alpha_1 \in \mathbb{R}$, $\alpha_2 \geq 0$:

$$J(u,y) := \int_0^T \langle Q\eta(t), \eta(t) \rangle_{\mathcal{H}^*,\mathcal{H}} \mathrm{d}t + \langle Q_T \eta(T), \eta(T) \rangle_{\mathcal{H}^*,\mathcal{H}}$$
$$+ \alpha_1 \int_0^T u(t)\mathrm{d}t + \alpha_2 \int_0^T u(t)^2 \mathrm{d}t$$

with

$$\eta(t) := y(t) - y_d(t), \quad y_d \in L^\infty(0,T;\mathcal{H}).$$

The optimal control problem

Control space (scalar)

$$\mathcal{U} := L^1(0,T).$$

Control constraints

$$u_m \leq u(t) \leq u_M.$$

With reduced cost $F(u) := J(u, y[u])$ the optimal control problem is given as

$$\text{Min } F(u); \quad u \in \mathcal{U}_{ad}. \tag{P}$$

$\hat{u} \in L^1(0,T)$ is a local weak minimum if $\hat{u}$ is minimum over

$$\{u \in \mathcal{U}_{\mathsf{ad}} \ : \ \|u - \hat{u}\|_\infty < \varepsilon\}, \quad \varepsilon > 0.$$

Existence

The compactness hypothesis is:

$$\begin{cases} \text{Let } y(y_0, f) \text{ be the solution of} \\ \dot{y} + \mathcal{A}y = \mathcal{F}; \quad y(0) = y_0. \\ \text{For given } y_0 \in \mathcal{H}, \text{ the mapping } \mathcal{F} \mapsto \mathcal{B}_2 y[y_0, f], \\ \text{is compact from } L^2(0,T;\mathcal{H}) \text{ to } L^2(0,T;\mathcal{H}). \end{cases}$$

Theorem

Let the compactness hypothesis hold. Then the optimal control problem has a nonempty set of minima.

Literature

Control-affine ODEs, second order sufficient conditions for singular arcs:

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Control-affine PDEs, second order sufficient conditions for bang-bang case:

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Optimal control in the semigroup setting:

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Linearized state equation

Linearized equation at $(\hat{u}, \hat{y})$: for $v \in \mathcal{U}$

$$\dot{z}(t) + \mathcal{A}z(t) = \hat{u}(t)\mathcal{B}_2 z(t) + v(t)(\mathcal{B}_1 + \mathcal{B}_2 \hat{y}(t)); \quad z(0) = 0;$$

Notation: $z[v]$ is the associated solution.

Theorem

The mapping $u \mapsto y[u]$ *from* $\mathcal{U}$ *to* $\mathcal{Y}$ *is of class* C^∞ *and we have that*

$$Dy[u]v = z[v]; \quad \forall v \in \mathcal{U}.$$

First-order necessary conditions

Let p be the solution of costate equation

$$-\dot{p} + \mathcal{A}^* p = Q(y - y_d) + u\mathcal{B}_2^* p; \quad p(T) = Q_T(y(T) - y_{dT}(T)).$$

Theorem

Let

$$\Lambda(t) := \alpha_1 + \alpha_2 u + \langle p(t), \mathcal{B}_1 + \mathcal{B}_2 y(t) \rangle.$$

Then $u \mapsto F(u)$ is of class C^∞ and we have

$$DF(u)v = \int_0^T v(t)\Lambda(t)\mathrm{d}t.$$

The contact sets are

$$I_m(u) \quad := \{t \in (0,T) : u(t) = u_m\}; \quad I_M(u) \quad := \{t \in (0,T) : u(t) = u_M\}.$$

Theorem (First-order necessary optimality conditions)

If $\hat{u}$ is a weak local minimum, then there holds

$$\{t;\ \Lambda(t) > 0\} \subset I_m(\hat{u}), \quad \{t;\ \Lambda(t) < 0\} \subset I_M(\hat{u}).$$

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Quadratic form

For given control $\hat{u}$, corresponding state $\hat{y} = y[\hat{u}]$, and costate $\hat{p} = p[\hat{u}]$ we define the quadratic form

$$\mathcal{Q}(z, v) := \int_0^T \big(a_1 \|z(t)\|^2 + v(t)\langle \hat{p}(t), \mathcal{B}_2 z(t)\rangle\big)\,\mathrm{d}t + a_2\|z(T)\|^2.$$

Lemma

For all $u \in \mathcal{U}$ we have

$$F(u) = F(\hat{u}) + DF(\hat{u})v + \tfrac{1}{2}\mathcal{Q}(z[v]; v) + \mathcal{O}(\|v\|^3).$$

Second-order necessary conditions

Given a feasible control u, the critical cone is defined as

$$C(u) := \left\{ \begin{array}{l} v \in L^1(0,T) \mid \Lambda(t)v(t) = 0 \text{ a.e. on } [0,T], \\ v(t) \geq 0 \text{ a.e. on } I_m(u),\ v(t) \leq 0 \text{ a.e. on } I_M(u) \end{array} \right\}.$$

Theorem (Second-order necessary conditions)

Let $\hat{u}$ be a weak local minimum, then

$$\mathcal{Q}(z[v], v) \geq 0 \quad \textit{for all } v \in C(\hat{u}).$$

Singular case $\alpha_2 = 0$

Goh transformation

For a linear equation

$$\dot{y} + \mathcal{A}y = ay + b^0 v; \quad y(0) = 0;$$

with $a \in L^1(0;T;L(\mathcal{H}))$; $b^0 \in C([0;T];\mathcal{H})$ the Goh transform is defined as

$$w(t) := \int_0^t v(s)\mathrm{d}s; \quad \xi := y - wb^0.$$

We set $b^1 := (a - \mathcal{A})b^0 - \dot{b}^0$.

Lemma

ξ *is a mild solution of*

$$\dot{\xi} + \mathcal{A}\xi = a\xi + wb^1; \quad \xi(0) = 0.$$

Goh transformation of the linearized state equation

With $\mathcal{B} := \mathcal{B}_1 + \mathcal{B}_2 \hat{y}$ we recall the linearized state equation

$$\dot{z}(t) + \mathcal{A}z(t) = \hat{u}(t)\mathcal{B}_2 z(t) + v(t)\mathcal{B}; \quad z(0) = 0.$$

Lemma

Setting

$$\xi := z - w\mathcal{B}; \quad w := \int_0^t v(s)\mathrm{d}s$$

ξ *is a mild solution of*

$$\dot{\xi} + \mathcal{A}\xi = \hat{u}\mathcal{B}_2\xi + wb^1;$$

where $b^1 := -\mathcal{B}_2 f - [\mathcal{A}; \mathcal{B}_2]\hat{y} - \mathcal{A}\mathcal{B}_1.$

Hypotheses for the Goh transform

Hypotheses:
We assume the existence of subspace $E \subset \mathcal{H}$ with continuous inclusion having a restriction property[1], and such that $\mathrm{dom}(\mathcal{A}) \subset E$; and

$$\begin{cases} \text{(i)} & \mathcal{B}_1 \in \mathrm{dom}(\mathcal{A}), \\ \text{(ii)} & \mathcal{B}_2 \,\mathrm{dom}(\mathcal{A}) \subset \mathrm{dom}(\mathcal{A}), \quad \mathcal{B}_2^* \,\mathrm{dom}(\mathcal{A}^*) \subset \mathrm{dom}(\mathcal{A}^*), \\ \text{(iii)} & \text{for } k = 1,2 : \left[\mathcal{A}, \mathcal{B}_2^k\right] \text{ has a continuous extension to } E, \\ \text{(iv)} & f \in L^\infty(0,T;\mathcal{H}); \quad [\mathcal{A}, \mathcal{B}_2^k]^* \hat{p} \in L^\infty(0,T;\mathcal{H}^*),\ k = 1,2, \\ \text{(v)} & \hat{y} \in L^2(0,T;E_1); \ \ [[\mathcal{A}, \mathcal{B}_2^1], \mathcal{B}_2]\hat{y} \in L^\infty(0,T;\mathcal{H}). \end{cases}$$

[1] Properties of the restriction of the semigroup to the subspace.

Goh transformation of the quadratic form

Applying the Goh transformation we obtain a new quadratic form

$$\widehat{\mathcal{Q}}(\xi, w, h) := \mathcal{Q}_T(\xi, h) + \mathcal{Q}_a(\xi, w) + \mathcal{Q}_b(w),$$

with

$$\mathcal{Q}_b(w) := \int_0^T w^2(t) R(t) \mathrm{d}t,$$

$$\mathcal{Q}_T(\xi, h) := a_2 \left\|\xi(T) + h\mathcal{B}(T)\right\|_{\mathcal{H}}^2 + h^2(\hat{p}_T, \mathcal{B}_2\mathcal{B}_1 + \mathcal{B}_2^2\hat{y}_T)_{\mathcal{H}} + h(\hat{p}_T, \mathcal{B}_2\xi_T)_{\mathcal{H}},$$

$$\mathcal{Q}_a(\xi, w) := \int_0^T \Big(a_1\|\xi\|_{\mathcal{H}}^2 + 2a_1 w(\xi, \mathcal{B})_{\mathcal{H}} + 2a_1 w(\hat{y} - y_d, \mathcal{B}_2\xi)_{\mathcal{H}} - 2w([\mathcal{A}, \mathcal{B}_2]^*\hat{p}, \xi)_{\mathcal{H}} \Big) \mathrm{d}t,$$

with $R \in L^\infty(0, T; \mathcal{H})$ given by

$$R(t) := a_1(\|\mathcal{B}\|_{\mathcal{H}}^2 + (\hat{y} - y_d, \mathcal{B}_2\mathcal{B})_{\mathcal{H}}) + (\hat{p}, r(t))_{\mathcal{H}},$$
$$r(t) := \mathcal{B}_2^2 f(t) - \mathcal{A}\mathcal{B}_2\mathcal{B}_1 + 2\mathcal{B}_2\mathcal{A}\mathcal{B}_1 - \big[[\mathcal{A}, \mathcal{B}_2], \mathcal{B}_2\big]\hat{y}.$$

Second order condition in transformed variables

Theorem

For $v \in L^1(0;T)$ and $w \in AC(0;T)$ given by Goh transformation, there holds

$$\mathcal{Q}(z[v];v) = \widehat{\mathcal{Q}}(\xi[w], w, w(T)).$$

Let $PC_2(\hat{u})$ be the closure in the $L^2 \times \mathbb{R}$–topology of

$$\{(w;h) \in W^{1,\infty}(0;T) \times \mathbb{R} \,:\, \dot{w} \in C(\hat{u});\; w(0) = 0; w(T) = h\}.$$

Theorem (Second order necessary condition)

We have that

$$\widehat{\mathcal{Q}}(\xi[w], w, h) \geq 0 \text{ for all } (w;h) \in PC_2(\hat{u}).$$

Legendre Clebsch condition

Singular arc: $(t_1;t_2) \subset [0;T]$ such that for all $\theta > 0$;

$$\exists \varepsilon > 0 : \quad \hat{u}(t) \in [u_m + \varepsilon; u_M - \varepsilon]; \text{ for a.a. } t \in (t_1 + \vartheta; t_2 - \vartheta).$$

Corollary

If $\hat{u}$ is a local weak minimum then

$$R(t) \geq 0 \text{ for a.a. } t \in (t_1;t_2); \text{ for all } (t_1,t_2) \text{ singular arc.}$$

Further we define

Lower bound arc: $(t_1;t_2)$ such that $\hat{u}(t) = u_m$ for a.a. $t \in (t_1;t_2)$,

Upper bound arc: $(t_1;t_2)$ such that $\hat{u}(t) = u_M$ for a.a. $t \in (t_1;t_2)$.

Lower and upper bound arcs: bang arcs

Expansion in transformed variables

$\hat{u}$ is a Pontryagin minimum if it is minimum over $\{u \in \mathcal{U}_{\text{ad}} \ : \ \|u - \hat{u}\|_1 < \varepsilon\}$ for some $\varepsilon > 0$:

A bounded sequence $(v_k) \subset L^\infty(0;T)$ converges to 0 in the Pontryagin sense if $\|v\|_1 \to 0$.

Lemma

Let (v_k) converge to 0 in the Pontryagin sense. Then

$$J(\hat{u} + v_k) = J(\hat{u}) + \int_0^T \Lambda(t) v_k(t) \mathrm{d}t + \tfrac{1}{2} \widehat{\mathcal{Q}}(\xi[w_k]; w_k; w_k(T)) + o(\|w_k\|_2^2 + w_k(T)^2)$$

where $(\xi[w_k]; w_k)$ is obtained by the Goh transformation.

Hypotheses

(i) finite structure: there are finitely many boundary and singular maximal arcs and the closure of their union is $[0;T]$;

(ii) strict complementarity: Λ has nonzero values over the interior of bang arcs

(iii) letting TBB denote the set of bang-bang junctions, we assume

$$R(t) > 0; \quad t \in \mathrm{TBB}\,.$$

Set

$$\widehat{PC}_2(\hat{u}) := \left\{ \begin{array}{l} (w;h) \in L^2(0;T) \times \mathbb{R};\ w \text{ is constant over boundary arcs,} \\ w = 0 \text{ over initial bang arc,} \\ w = h \text{ over terminal bang arc.} \end{array} \right\}.$$

Lemma (Characterization of the critical cone)

$$PC_2(\hat{u}) = \left\{ (w,h) \in \widehat{PC}_2(\hat{u}) \ :\ w \textit{ is continuous at bang-bang junctions.} \right\}.$$

ABDLK12 M. S. Aronna, J. F. Bonnans, A. V. Dmitruk, and P. A. Lotito, Quadratic order conditions for bang-singular extremals, Numer. Algebra Control Optim. 2 (2012), no. 3, 511–546.

Second order sufficient condition

Uniform positivity condition:

$$\exists \alpha > 0 : \widehat{\mathcal{Q}}(\xi[w], w, h) \geq \alpha(\|w\|_2^2 + h^2); \text{ for all } (w; h) \in \widehat{PC_2}(\hat{u}).$$

Weak quadratic growth condition: $\exists \beta > 0$ such that for any $u \in \mathcal{U}_{\mathrm{ad}}$; setting $v := u - \hat{u}$ and $w(t) := \int_0^t v(s)\mathrm{d}s$:

$$F(u) \geq J(\hat{u}) + \beta(\|w\|_2^2 + w(T)^2), \quad \text{if } \|v\|_1 \text{ is small enough.}$$

Theorem (Sufficient condition (Aronna, Bonnans, K. (2017))

Let $\hat{u} \in \mathcal{U}_{ad}$ satisfy the first order conditions. If the uniform positivity condition holds then $\hat{u}$ is a Pontryagin minimum for which the weak quadratic growth condition is satisfied.

Content

Control of heat equation

Setting: $\Omega \subset \mathbb{R}^3$ open, bounded, smooth boundary.

We set $V := H_0^1(\Omega)$ and $H := L^2(\Omega)$.

Heat equation: $b \in H_0^1(\Omega) \cap W^{2,\infty}(\Omega)$, $y_0 \in H_0^1(\Omega)$, $y = y(x,t)$

$$\begin{cases} \partial_t y - \Delta y = u(t)b(x)y & \text{in } Q := \Omega \times [0,T], \\ y = 0 \text{ on } \partial\Omega \times [0,T]; & y(\cdot,0) = y_0. \end{cases}$$

Control of heat equation

Cost function:

$$F(u) = \frac{1}{2}\int_0^T \int_\Omega (y(x,t) - y_d(x,t))^2 \mathrm{d}x\mathrm{d}t, \quad y_d \in C(0,T;H).$$

Commutator is a 'first order' operator: Here $\mathcal{A} = -\Delta$ with domain $\mathrm{dom}(\mathcal{A}) := H_0^1(\Omega) \cap H^2(\Omega)$. We have to compute (cancellation of $b\Delta y$)

$$[-\Delta, b]y = (-\Delta b)y + 2\nabla b \cdot \nabla y.$$

Known regularity result: if $y_0 \in H_0^1(\Omega)$ and $\hat{u} \in L^2(0,T)$ then

$$y \in C(0,T;H_0^1(\Omega)) \quad \Rightarrow \quad [-\Delta, b]y \in C(0,T;L^2(\Omega)).$$

ABK17 M.S. Aronna, F. Bonnans, A. K., *Optimal control of a infinite dimensional bilinear systems; Application to the heat and wave equation*, published online in Mathematical Programming, 2016.

Control of the wave equation

Wave equation:

$$\begin{cases} \partial_{tt} y - \Delta y = f_2 + u(b_1 + b_2 y) & \text{in } \Omega \times (0,T), \\ y(\cdot,0) = y_{01}(x), \quad \partial_t y(\cdot,0) = y_{02} & \text{in } \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

We set

$$\mathcal{A}_W := \begin{pmatrix} 0 & -\operatorname{id} \\ -\Delta & 0 \end{pmatrix}, \ \mathcal{B}_1 := \begin{pmatrix} 0 \\ b_1 \end{pmatrix}, \ \mathcal{B}_2 := \begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix}, \quad \mathcal{F} := \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \ y_0 = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$$

and

$$\mathcal{H} := V \times H, \quad \operatorname{dom}(\mathcal{A}_W) := (H^2(\Omega) \cap V) \times V.$$

Commutator is a 'zero' order operator: $[\mathcal{A}, \mathcal{B}_2] = \begin{pmatrix} -b_2 & 0 \\ 0 & b_2 \end{pmatrix}$.

ABK17 M.S. Aronna, F. Bonnans, A. K., *Optimal control of a infinite dimensional bilinear systems; Application to the heat and wave equation*, published online in Mathematical Programming, 2016.

Control of the damped wave equation

Damped wave equation:

$$\begin{cases} \partial_{tt} y - \Delta y = f_2 + u(b_1 + b_2 \dot{y}) & \text{in } \Omega \times (0,T), \\ y(\cdot,0) = y_{01}(x), \quad \partial_t y(\cdot,0) = y_{02} & \text{in } \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

Here we set

$$\mathcal{B}_2 := \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Commutator is a second-order operator: $[\mathcal{A}, \mathcal{B}_2] = \begin{pmatrix} 0 & -b_2 \\ b_2 \Delta & 0 \end{pmatrix}$.

$\longrightarrow$ requires additional regularity.

BF18 F. Bethke and A. Kröner, *Sufficient optimality conditions for bilinear optimal control of the linear damped wave equations*, in preparation.

Application to the Schrödinger equation

Schrödinger equation:

$$\begin{cases} \partial_t y - i\Delta y = -iu(t)b(x)y & \text{in } \Omega \times [0,T], \\ y = 0 \text{ on } \partial\Omega \times [0,T]; \quad y(\cdot,0) = y_0. \end{cases}$$

with $\bar{H} := L^2(\Omega;\mathbb{C})$, $\bar{V} := H_0^1(\Omega;\mathbb{C})$ and

$$y_0 \in \bar{V}, \ \ b_2^k \in W_0^{2,\infty}(\Omega), \ k = 1,2, \quad f \in L^2(0,T;\bar{V}) \cap C(0,T;\bar{\mathcal{H}}).$$

Commutator: Here $\mathcal{A} = -i\Delta$ with domain $\mathrm{dom}(\mathcal{A}) := \bar{V} \cap \bar{H}^2(\Omega)$.
We obtain a first order commutator

$$[-i\Delta, b]y = -i(\Delta b)y + i2\nabla b \cdot \nabla y.$$

ABK2016 M.S. Aronna, J.F. Bonnans, and A. Kröner, *Optimal control of PDEs in a complex space setting; application to the Schrödinger equation*, Research report, INRIA, 2016.

Content

Setting

$\Omega \subset \mathbb{R}^3$ bounded and smooth, $\varphi(y) := y^3$,

Semilinear heat equation $Q := \Omega \times [0,T]$, $y = y(x,t)$
$b_i \in H_0^1(\Omega) \cap W^{2,\infty}(\Omega)$, $y_0 \in C(\bar{\Omega}) \cap H_0^1(\Omega)$, $f \in L^2(Q)$:

$$\begin{cases} \dot{y} - \Delta y + \varphi(y) = f(x,t) + \sum_{i=1}^m u_i(t) b_i(x) y & \text{in } Q \\ y = 0 \text{ on } \partial\Omega \times [0,T]; \quad y(\cdot,0) = y_0. \end{cases} \tag{1}$$

For $u \in L^2(0,T)^m$, unique solution $y[u] \in H^{2,1}(Q)$

Well-posedness of the state equation

Lemma

The mapping $L^2(0,T) \times L^\infty(\Omega) \times L^\infty(0,T;L^2(\Omega))$,

$$(u_i, b_i, y) \mapsto u_i b_i y$$

has image in $L^2(Q)$, *is of class* C^∞, *and*

$$\|u_i b_i y\|_{L^2(Q)} \leq \|u_i\|_{L^2(0,T)} \|b_i\|_{L^\infty(\Omega)} \|y\|_{L^\infty(0,T;L^2(\Omega))}.$$

A priori estimates

Multiplying the state equation by $y = y[u]$ and integrating in space:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|^2_{L^2(\Omega)} + \|\nabla y(t)\|^2_{L^2(\Omega)} + \int_\Omega y^4 \mathrm{d}x = \int_\Omega f y \mathrm{d}x + \sum_{i=0}^m u_i(t) \int_\Omega b_i(x) y^2 \mathrm{d}x.$$

So, $\eta(t) := \|y(t)\|^2_{L^2(\Omega)}$ satisfies

$$\tfrac{1}{2}\dot{\eta}(t) \leq \tfrac{1}{2}\|f(\cdot,t)\|^2_{L^2(\Omega)} + \left(\tfrac{1}{2} + \sum_{i=0}^m |u_i(t)|\|b_i\|_\infty\right)\eta(t).$$

Gronwall lemma:

- estimate of y in $L^\infty(0,T;L^2(\Omega))$,
- $\sum_i u_i b_i y$ in $L^2(Q)$,
- y in $Y := H^{2,1}(Q)$:

state equation: $u \mapsto y[u]$ well-posed.

Compactness for the state

see J.L. Lions [p. 14, 1969], Edwards [Thm 8.20.5, 1965]:

$$\begin{cases} \text{For any } \mu \in [1,10), \text{ the following injection is compact:} \\ Y \subset L^{\mu}(0,T;L^{10}(\Omega)), \quad \text{when } n \leq 3. \end{cases}$$

Setting

Cost function, affine w.r.t. the control; $y_d \in L^2(Q)$, $\alpha \in \mathbb{R}^m$:

$$F(u) = \tfrac{1}{2}\int_Q (y[u](x,t) - y_d(x,t))^2 \mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\int_\Omega (y[u](x) - y_{dT}(x))^2 \mathrm{d}x\mathrm{d}t + \sum_{i=1}^m \alpha_i \int_0^T u_i(t)\mathrm{d}t.$$

Control bounds: $u_i^m < u_i^M$

$$U_{ad} := \{u \in L^2(0,T)^m;\ u_i^m \leq u_i(t) \leq u_i^M, \quad i = 1 \text{ to } m, \text{ for a.a. } t.\}$$

State constraints

Same problem with additional state constraints

$$g_j(y(t)) \leq 0, \quad j = 1, \ldots, q$$

where for some $c_j \in H_0^1(\Omega) \cap H^2(\Omega)$:

$$g_j(y(t)) := \int_\Omega c_j(x) y(x,t) \mathrm{d}x$$

Lagrangian

The *Lagrangian* is, choosing the multiplier of the state equation to be $(p, p_0) \in L^2(Q) \times H_0^1(\Omega)$:

$$\mathcal{L}(u, y, p, \mu) = J(u, y) - \langle p_0, y(\cdot, 0) - y_0 \rangle_{H_0^1(\Omega)}$$
$$\int_Q p\big(\Delta y(x,t) - y^3(x,t) + f(x,t) + \sum_{i=0}^m u_i(t) b_i(x) y(x,t) - \dot{y}(x,t)\big) \mathrm{d}x\mathrm{d}t$$
$$+\sum_{j=1}^q \int_0^T g_j(\bar{y}) \mathrm{d}\mu_j(t).$$

Here $\mu \in BV(0,T)^q$ with $\mu(T) = 0$.

Costate equation

The costate equation is the condition of stationarity of the Lagrangian w.r.t. the state, that is, for $z \in Y$:

$$\left\{ \begin{array}{l} \int_Q p(\dot{z} + Az)\mathrm{d}x\mathrm{d}t + \langle p_0, z(\cdot,0)\rangle_{H_0^1(\Omega)} = \sum_{j=1}^q \int_0^T g_j'(\bar{y}) z \mathrm{d}x \mathrm{d}\mu_j(t), \\ + \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x. \end{array} \right. \tag{2}$$

With $(\varphi, \psi) \in L^2(Q) \times H_0^1(\Omega)$, associate $z = z[\varphi, \psi] \in Y$, unique solution of

$$\dot{z} + Az = \varphi; \quad z(\cdot, 0) = \psi$$

with $A = A(t)$ the linear, continuous operator $L^2(0,T; H^2(\Omega)) \to L^2(Q)$ defined by

$$Az := -\Delta z + 3\gamma \bar{y}^2 z - \sum_{i=0}^m \bar{u}_i b_i z. \tag{3}$$

The costate equation can be rewritten as $z = z[\varphi, \psi]$

$$\begin{cases} \int_Q p\varphi \mathrm{d}x\mathrm{d}t + \langle p_0, \psi\rangle_{H_0^1(\Omega)} = \sum_{j=1}^q \int_0^T g_j'(\bar{y}) z \mathrm{d}x\mathrm{d}\mu_j(t), \\ + \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x. \end{cases} \tag{4}$$

The r.h.s. is a continuous linear form on $L^2(Q) \times H_0^1(\Omega)$. By the Riesz theorem, the costate equation has a unique solution $(p, p_0) \in L^2(Q) \times H^{-1}(\Omega)$.

Remark If p is smooth enough, we can integrate by parts in time and we have the initial-terminal conditions

$$p(\cdot, T) = \bar{y}(\cdot, T) - y_{dT}(\cdot), \quad p(0) = p_0.$$

Alternative costates

$$p^1 := p + \sum_{j=1}^{q} c_j \mu_j(t); \quad p_0^1 := p_0 + \sum_{j=1}^{q} c_j \mu_j(0). \tag{5}$$

Lemma

(i) *Given* $(p, p_0) \in L^2(Q) \times H^{-1}(\Omega)$ *solution of* (4), *defining* p^1 *as in* (5) *we get that* $p^1 \in Y$ *and satisfies*

$$-\dot{p}^1 + Ap^1 = \bar{y} - y_d + \sum_{j=1}^{q} \mu_j(t) A c_j, \quad p^1(\cdot, T) = \bar{y}(\cdot, T) - y_{dT}.$$

BJ10 J. F. Bonnans and P. Jaisson, Optimal control of a parabolic equation with time-dependent state constraints, SIAM J. Control Optim. 48 (2010), no. 7, 4550–4571.

Let $(\bar{u}, \bar{y})$ be a feasible point of problem (P). Define

$$M_+(0,T) := \{\mu \in BV(0,T)^q;\ \ \mu(T) = 0;\ \ \mathrm{d}\mu \geq 0\}.$$

We say that $\mu \in M_+(0,T)$ is complementary to the state constraint for $\bar{y}$ if

$$\int_0^T g_i(\bar{y}(t))\mathrm{d}\mu_i(t) = 0, \quad i = 1, \ldots, q.$$

Let

$$(\beta, p, \mu) \in \mathbb{R}_+ \times L^\infty(0,T; H_0^1(\Omega)) \times M_+(0,T)$$

Lagrange multiplier

We say that $p \in L^\infty(0,T;H_0^1(\Omega))$ is the costate associated with $(\beta, \bar{u}, \bar{y}, \mu)$, or in short with (β, μ), if it is solution of (2).

Definition

We say that (p, μ) is a Lagrange multiplier if it satisfies the following *first-order optimality conditions:* μ is complementary to the state constraint, p is the costate associated with (β, μ), For $i = 1$ to m, setting

$$y_i(t) := \alpha_i(t) + \int_\Omega b_i(x)\bar{y}(x,t)p(t)\mathrm{d}x$$

one has:

$$\sum_{j=1}^{q} \int_0^T y_i(t)(\bar{u}_i(t) - u_i(t))\mathrm{d}t \leq 0, \quad \text{for every } u \in \mathcal{U}_{\text{ad}}.$$

Existence of Lagrange multipliers

Lemma

Under a standard qualification hypothesis:

With any local solution is associated a nonempty set of triples satisfying the first-order optimality conditions.

ABK18 M.S. Aronna, F. Bonnans, A. K., *Optimal control of the bilinear heat equation subject to state and control constraints, in preparation.*

Summary

- Framework to study sufficient optimality conditions for control-affine optimal control problems
- Applications to PDEs
- Extensions to nonlinear and state constrained problems

Literature

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Thank you for your attention.