Stabilisierung kontroll-affiner Systeme
durch Lie-Klammer Approximationen mit
Anwendungen zur abstandsbasierten Formationssteuerung

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Motivation: distanced-based formation control

Model: \( N \) kinematic points in \( \mathbb{R}^n \):

\[
\dot{p}_i = \sum_{k=1}^{n} u_{i,k} b_{i,k}, \quad i = 1, \ldots, N,
\]

where

- \( p_i = (p_{i,1}, \ldots, p_{i,n}) \in \mathbb{R}^n \) position of agent \( i \),
- \( b_{i,1}, \ldots, b_{i,n} \in \mathbb{R}^n \) orthonormal basis of velocity direction,
- \( u_{i,1}, \ldots, u_{i,n} \) input channels for velocities.
Motivation: distanced-based formation control

**Measurements:** described by an undirected graph $G = (V, E)$, where

- $V = \{1, \ldots, N\}$ set of nodes,
- $E = \{\{i, j\} \subseteq V \mid |\{i, j\}| = 2\}$ set of edges,
  
we write $ij := \{i, j\}$.

If $ij \in E$, then agent $i$ can measure *distance* (≠relative position) to agent $j$ and vice versa.
**Motivation:** distanced-based formation control

**Objective:** steer agents to state $p = (p_1, \ldots, p_N) \in \mathbb{R}^{nN}$ such that

$$\forall ij \in E : \|p_j - p_i\| = d_{ij},$$

where $d_{ij} > 0$ *desired distance* between agents $i, j$.

**Constraints:**
- autonomous agents $\Rightarrow$ decentralized, distributed control law,
- each agent can only use its own distance measurements,
- no shared clock, no storage of data.
Motivation: distanced-based formation control

Standard approach: gradient-based control law.

Local potential function for agent $i$:

$$\psi_i(p) := \frac{1}{4} \sum_{j \in V : ij \in E} (\|p_j - p_i\|^2 - d_{ij}^2)^2$$

Control law for agent $i$:

$$u_i = (u_{i,1}, \ldots, u_{i,n}) = -\nabla p_i \psi_i(p) = \frac{1}{2} \sum_{j \in V : ij \in E} (p_j - p_i)(\|p_j - p_i\|^2 - d_{ij}^2)$$
Motivation: distanced-based formation control

Standard approach: gradient-based control law.

Local potential function for agent $i$:

$$
\psi_i(p) := \frac{1}{4} \sum_{j \in V: ij \in E} (\|p_j - p_i\|^2 - d_{ij}^2)^2
$$

Control law for agent $i$:

$$
u_i = (u_{i,1}, \ldots, u_{i,n}) = -\nabla_{p_i} \psi_i(p) = \frac{1}{2} \sum_{j \in V: ij \in E} (p_j - p_i)(\|p_j - p_i\|^2 - d_{ij}^2)
$$

Then, set of target formations

$$
F := \{ p = (p_1, \ldots, p_n) \in \mathbb{R}^{nN} \mid \forall ij \in E: \|p_j - p_i\| = d_{ij} \}
$$

- is locally asymptotically stable (Krick et al., 2009; Oh, Ahn, 2014)
- is locally exponentially stable if frameworks $(G, p)$ with $p \in F$ are infinitesimally rigid (Mou et al, 2016, Sun et al., 2016)
Motivation: distanced-based formation control

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Then, set of *target formations*

$$F := \{ p = (p_1, \ldots, p_n) \in \mathbb{R}^{nN} \mid \forall ij \in E : \|p_j - p_i\| = d_{ij} \}$$

- is locally asymptotically stable (Krick et al., 2009; Oh, Ahn, 2014)
- is locally exponentially stable if frameworks $(G, p)$ in $F$ are *infinitesimally rigid* (Mou et al, 2016, Sun et al., 2016)

**Obstacle:** implementation requires measurements of relative positions *actively controlled variable* $\neq$ *sensed variable*
Motivation: distanced-based formation control

**Approach:** approximate the gradient by Lie brackets.

**Toy model:** $N = 1$ agent in $n = 1$ dimension, i.e., $\dot{p} = u \in \mathbb{R}$, with potential $\psi(p) := \frac{1}{2}|p - 1|^2$.

Define $u_1, u_2, h_1, h_2 : \mathbb{R} \to \mathbb{R}$ by

$$u_1(t) := \cos t, \quad u_2(t) := \sin t,$$

$$h_1(y) := \begin{cases} 
  y \sin(\log y) & y > 0 \\
  0 & y \leq 0 
\end{cases}, \quad h_2(y) := \begin{cases} 
  y \cos(\log y) & y > 0 \\
  0 & y \leq 0 
\end{cases}.$$  

**Control law:** $u := u_1(t) h_1(\psi(p)) + u_2(t) h_2(\psi(p))$

![Graph](image-url)
Motivation: distanced-based formation control

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Define $u_1, u_2, h_1, h_2 : \mathbb{R} \to \mathbb{R}$ by

\[
\begin{align*}
    u_1(t) &:= \cos t, & \quad u_2(t) &:= \sin t, \\
    h_1(y) &:= \begin{cases} y \sin(\log y) & y > 0 \\ 0 & y \leq 0 \end{cases}, & \quad h_2(y) &:= \begin{cases} y \cos(\log y) & y > 0 \\ 0 & y \leq 0 \end{cases}.
\end{align*}
\]

**Control law:** $u := u_1(t) h_1(\psi(p)) + u_2(t) h_2(\psi(p))$

“Lie bracket” of $f_1 := (h_1 \circ \psi)$ and $f_2 := (h_2 \circ \psi)$:

\[
[f_1, f_2](p) := f_2'(p) f_1(p) - f_1'(p) f_2(p) = -\psi(p) \psi'(p) > 0 \text{ for } p \neq 1
\]
Approximation of Lie Brackets

Closed-loop system: \( \dot{p} = u_1(t) f_1(p) + u_2(t) f_2(p) = \sum_i u_i(t) f_i(p) \).

Integral equation, averaging procedure:

\[
p(t) = p(t_0) + \sum_i \int_{t_0}^{t} u_i(s) f_i(p(s)) \, ds
\]
Approximation of Lie Brackets

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Integral equation, averaging procedure:

\[
p(t) = p(t_0) + \sum_i \int_{t_0}^{t} u_i(s) f_i(p(s)) \, ds
\]

\[
= p(t_0) + \sum_i \left[ U_i(s) f_i(p(s)) \right]_{t_0}^{t} - \sum_{i,j} \int_{t_0}^{t} U_i(s) u_j(s) f'_i(p(s)) f_j(p(s)) \, ds
\]

\[
= p(t_0) + \sum_i \left[ U_i(s) f_i(p(s)) \right]_{t_0}^{t} - \int_{t_0}^{t} \frac{1}{2} [f_1, f_2](p(s)) \, ds
\]

\[
- \sum_{i,j} \int_{t_0}^{t} \overline{U} V_{j,i}(s) f'_i(p(s)) f_j(p(s)) \, ds
\]
Approximation of Lie Brackets

Closed-loop system: 
\[ \dot{p} = u_1(t) f_1(p) + u_2(t) f_2(p) = \sum_i u_i(t) f_i(p). \]

Integral equation, averaging procedure:
\[ p(t) = p(t_0) + \sum_i \int_{t_0}^{t} u_i(s) f_i(p(s)) \, ds \]
\[ = p(t_0) + \sum_i \left[ U_i(s) f_i(p(s)) \right]_{t_0}^{t} - \sum_{i,j} \int_{t_0}^{t} U_i(s) u_j(s) f_i'(p(s)) f_j(p(s)) \, ds \]
\[ = p(t_0) + \sum_i \left[ U_i(s) f_i(p(s)) \right]_{t_0}^{t} - \int_{t_0}^{t} \frac{1}{2} [f_1, f_2](p(s)) \, ds \]
\[ - \sum_{i,j} \int_{t_0}^{t} \tilde{U} V_{j,i}(s) f_i'(p(s)) f_j(p(s)) \, ds \]
\[ = p(t_0) + \left[ D_1(s, p(s)) \right]_{t_0}^{t} + \int_{t_0}^{t} \left( \frac{1}{2} [f_1, f_2](p(s)) + D_2(s, p(s)) \right) \, ds \]

with certain time-varying remainder vector fields \( D_1, D_2 \).
Approximation of Lie Brackets

Closed-loop system: \( \dot{p} = u_1(t) f_1(p) + u_2(t) f_2(p) \)

Integral equation, averaging procedure:

\[
p(t) = p(t_0) + [D_1(s, p(s))]_{t_0}^t + \int_{t_0}^t \left( \frac{1}{2}[f_1, f_2](p(s)) + D_2(s, p(s)) \right) ds
\]
Approximation of Lie Brackets

Closed-loop system: \( \dot{p} = u_1(t) f_1(p) + u_2(t) f_2(p) \)

Integral equation, averaging procedure:

\[
p(t) = p(t_0) + \left[ D_1(s, p(s)) \right]_{t_0}^{t} + \int_{t_0}^{t} \left( \frac{1}{2} [f_1, f_2](p(s)) + D_2(s, p(s)) \right) \, ds
\]

A detailed analysis shows: \( D_1, D_2 \) vanish sufficiently fast compared to \([f_1, f_2]\) when \( \psi(p) \to 0 \). Thus, for \( \psi(p) \) close to 0:

\[
p(t) \approx p(t_0) + \int_{t_0}^{t} \frac{1}{2} [f_1, f_2](p(s)) \, ds = p(t_0) - \frac{1}{2} \int_{t_0}^{t} \psi(p(s)) \psi'(p(s)) \, ds
\]

\( \Rightarrow \) trajectories follow approximately \([f_1, f_2]\).
Approximation of Lie Brackets

**Toy model:** system $\dot{p} = u_1(t) f_1(p) + u_2(t) f_2(p)$ on $\mathbb{R}$.


$$\dot{p} = f_0(p) + \sum_{i=1}^{m} u_i^\omega(t) f_i(p),$$

with

- a drift vector field $f_0: \mathbb{R}^n \to \mathbb{R}^n$,
- suff. smooth control vector fields $f_1, \ldots, f_m: \mathbb{R}^n \to \mathbb{R}^n$,
- time-varying functions $u_1^\omega, \ldots, u_m^\omega: \mathbb{R} \to \mathbb{R}$ of the form

$$u_i^\omega(t) = \Lambda_i(t) \omega^{e_i} \sin(\omega \Omega_i t + \Phi_i),$$

where $\omega > 0$ is a frequency parameter.
Approximation of Lie Brackets

Control-affine system

\[
\dot{p} = f_0(p) + \sum_{i=1}^{m} u_i^\omega(t) f_i(p),
\]

Integral equation, averaging procedure:

\[
p(t) = p(t_0) + \left[ D_1^\omega(s, p(s)) \right]_{t_0}^{t} \\
+ \int_{t_0}^{t} \left( f_0(p(s)) + \sum_{0 < |l| \leq r} \frac{v_l(s)}{|l|} [f_l](p(s)) + D_2^\omega(s, p(s)) \right) \, ds,
\]

where

- \( D_1^\omega, D_2^\omega \) certain time-varying remainder vector fields,
- \( l = (i_1, \ldots, i_k) \) multi-index of length \(|l| = k|\),
- \( v_l : \mathbb{R} \to \mathbb{R} \) time-varying function (depending on \( u_i^\omega \)),
- \([f_l] := [f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{i_k}] \cdots]] \) iterated Lie bracket.

One can show that \( D_1^\omega, D_2^\omega \to 0 \) as \( \omega \to \infty \) on compact sets.
Approximation of Lie Brackets

$\omega$-dependent system

$$\Sigma^\omega : \dot{p} = f_0(p) + \sum_{i=1}^{m} u_i^\omega(t) f_i(p)$$

Limit system

$$\Sigma^\infty : \dot{p} = f_0(p) + \sum_{0 < |l| \leq r} \frac{v_l(t)}{|l|} [f_l](p)$$

Theorem (Kurzweil, Jarník (1988); Sussmann, Liu (1991))

For every initial condition $(t_0, p_0)$, every $T > 0$, and every $\varepsilon > 0$, there exists $\omega_0 > 0$ such that for every $\omega \geq \omega_0$, and every $t \in [t_0, t_0 + T]$: \[ \|x^\infty(t) - x^\omega(t)\| \leq \varepsilon. \]

Application: motion planning for nonholonomic systems.
Approximation of Lie Brackets

$\omega$-dependent system \[ \Sigma^\omega : \dot{p} = f_0(p) + \sum_{i=1}^{m} u_i^\omega(t) f_i(p) \]

Limit system \[ \Sigma^\infty : \dot{p} = f_0(p) + \sum_{0 < |l| \leq r} \frac{v_l(t)}{|l|}[f_l](p) \]

**Theorem (Dürr et al. (2013))**

Case $r = 2$ and $u_i^\omega(t) = \Lambda_i(t) \sqrt{\omega} \sin(\omega \Omega_i t + \Phi_i)$, $\Lambda_i$ bounded.

For every compact set $K \subseteq \mathbb{R}^n$, every $T > 0$, and every $\varepsilon > 0$, there exists $\omega_0 > 0$ such that for every $\omega \geq \omega_0$, every $(t_0, p_0) \in \mathbb{R} \times K$, and every $t \in [t_0, t_0 + T]$:

\[ \|x^\infty(t) - x^\omega(t)\| \leq \varepsilon \]

as long as $x^\infty$ stays in $K$.

Application: stabilization of control-affine systems.
\[ \Sigma^\omega : \dot{p} = f_0(p) + \sum_{i=1}^{m} u^\omega_i(t) f_i(p) \]

Limit system

\[ \Sigma^\infty : \dot{p} = f_0(p) + \sum_{0 < |l| \leq r} \frac{v_l(t)}{|l|} [f_l](p) \]

**Theorem (Dürr et al. (2013))**

Case \( r = 2 \) and \( u^\omega_i(t) = \Lambda_i(t) \sqrt{\omega} \sin(\omega \Omega_i t + \Phi_i) \), \( \Lambda_i \) bounded.

For a compact \( K \subseteq \mathbb{R}^n \), the following implication holds:

\[ K \text{ is LUAS for } \Sigma^\infty \quad \Rightarrow \quad K \text{ is } \omega\text{-PLUAS for } \Sigma^\omega \]

LUAS = \textit{locally uniformly asymptotically stable.}

\( \omega\text{-PLUAS} = \textit{practically LUAS} \) if \( \omega \) sufficiently large.
Theorem (S. (2017))

General $r \in \mathbb{N}$, and $K \subseteq \mathbb{R}^n$ compact.

Suppose that $u^\omega_i$ satisfy certain averaging conditions as $\omega \to 0$.

(a) $K$ is LUAS for $\Sigma^\infty$ $\implies$ $K$ is $\omega$-PLUAS for $\Sigma^\omega$

(b) If vector fields satisfy certain boundedness conditions, then:

$$K \text{ is LUES for } \Sigma^\infty \implies K \text{ is } \omega\text{-LUES for } \Sigma^\omega$$

$LUES=$locally uniformly exponentially stable.

$\omega\text{-LUES=}LUES$ if $\omega$ sufficiently large.
Application to extremum seeking control.

**Objective:** find a control law $u$ that steers the system to an extremum of $\psi$ (w.l.o.g. minimum of $\psi$)

**Restrictions:**
- no information about the current system state $x$,
- no explicit knowledge about the output function $x \mapsto \psi(x)$,
- only real-time measurements of $y = \psi(x)$ are available
Application to extremum seeking control.

$u \in \mathbb{R}^m$ nonlinear system $\Sigma$

$x \in \mathbb{R}^n$ output function $y = \psi(x)$

$y \in \mathbb{R}$

$u = u(t, y)$

Idea by Dürr et al. (2013): approximate descent directions of $\psi$ by Lie brackets (feed in sinusoidal perturbations).

Level sets of $\psi$

Trajectory of $\Sigma$ under extremum seeking control

Minimum of $\psi$
Application to extremum seeking control

Applicable to control-affine systems

\[ \dot{p} = g_0(p) + \sum_{k=1}^{m} u_k g_k(p), \quad y = \psi(p). \]

Dürr et al. (2013):

- Define \( u_{2k-1}^\omega(t) := \sqrt{\omega \Omega_k} \cos(\omega \Omega_k t), \quad u_{2k}^\omega(t) := \sqrt{\omega \Omega_k} \sin(\omega \Omega_k t). \)
- Define \( h_1(y) := y \) and \( h_2(y) := 1 \) for every \( y \in \mathbb{R}. \)

Control law: \( u_k = u_{2k-1}^\omega(t) h_1(\psi(p)) + u_{2k}^\omega(t) h_2(\psi(p)). \)
Application to extremum seeking control

Applicable to control-affine systems

\[ \dot{p} = g_0(p) + \sum_{k=1}^{m} u_k g_k(p), \quad y = \psi(p). \]

Dürr et al. (2013):

- Define \( u_{2k-1}(t) := \sqrt{\omega \Omega_k} \cos(\omega \Omega_k t) \), \( u_{2k}(t) := \sqrt{\omega \Omega_k} \sin(\omega \Omega_k t) \).
- Define \( h_1(y) := y \) and \( h_2(y) := 1 \) for every \( y \in \mathbb{R} \).

Control law: \( u_k = u_{2k-1}(t) h_1(\psi(p)) + u_{2k}(t) h_2(\psi(p)). \)

New vector fields \( f_{2k-1} := (h_1 \circ \psi)g_k \) and \( f_{2k} := (h_2 \circ \psi)g_k \).

\( \omega \)-dependent system \( \Sigma^\omega : \dot{p} = g_0(p) + \sum_{i=1}^{2m} u_i(t) f_i(p), \)

Limit system \( \Sigma^\infty : \dot{p} = g_0(p) + \frac{1}{2} \sum_{k=1}^{m} [f_{2k-1}, f_{2k}](p), \)

with descent directions \( [f_{2k-1}, f_{2k}](p) = -(\mathcal{L}_{g_k} \psi)(p)g_k(p). \)
Application to extremum seeking control

\[ h_1(y) := y, \quad h_2(y) := 1. \]

\( \omega \)-dependent system

\[ \Sigma^\omega : \dot{p} = g_0(p) + \sum_{k=1}^m u_k g_k(p) \]

under control law

\[ u_k = u_{2k-1}^\omega(t) h_1(\psi(p)) + u_{2k}^\omega(t) h_2(\psi(p)), \]

Limit system

\[ \Sigma^\infty : \dot{p} = g_0(p) - \frac{1}{2} \sum_{k=1}^m (\mathcal{L}_{g_k \psi})(p) g_k(p), \]

**Theorem (Dürr et al. (2013))**

Suppose that \( K \subseteq \mathbb{R}^n \) is a compact set of minima of \( \psi \). Then, the following implication holds:

\[ K \text{ is LUAS for } \Sigma^\infty \quad \implies \quad K \text{ is } \omega\text{-PLUAS for } \Sigma^\omega. \]
Application to extremum seeking control

\[ h_1(y) := \sqrt{y} \sin(\log y), \quad h_2(y) := \sqrt{y} \cos(\log y). \]

\( \omega \)-dependent system

\[ \Sigma^\omega: \dot{p} = g_0(p) + \sum_{k=1}^{m} u_k g_k(p) \]

under control law

\[ u_k = u_{2k-1}^\omega(t) h_1(\psi(p)) + u_{2k}^\omega(t) h_2(\psi(p)), \]

Limit system

\[ \Sigma^\infty: \dot{p} = g_0(p) - \frac{1}{2} \sum_{k=1}^{m} (\mathcal{L}_{g_k\psi})(p) g_k(p), \]

**Theorem (S. (2017))**

Suppose that \( \psi \geq 0 \) and that \( K := \psi^{-1}(\{0\}) \) is compact.

(a) \( K \) is LUAS for \( \Sigma^\infty \) \( \implies \) \( K \) is \( \omega \)-PLUAS for \( \Sigma^\omega \)

(b) If the drift \( g_0 \) satisfies certain boundedness conditions, then:

\( K \) is LUES for \( \Sigma^\infty \) \( \implies \) \( K \) is \( \omega \)-LUES for \( \Sigma^\omega \)
Stabilization of control-affine systems

\[ h_1(y) := \sqrt{y} \sin(\log y), \quad h_2(y) := \sqrt{y} \cos(\log y) \] for \( y > 0 \).

Example

Let \( A \in \mathbb{R}^{n \times n}, \, B \in \mathbb{R}^{m \times n} \). Consider the system

\[
\Sigma^\omega : \quad \dot{x} = Ax + Bu, \quad y = \psi(x) := \|x\|^2
\]

with \( u = (u_1, \ldots, u_m)^\top \) under the control law

\[
u_k = \lambda u_{2k-1}(t) h_1(\psi(x)) + \lambda u_{2k}(t) h_2(\psi(x)).\]

with \( \lambda, \omega > 0 \). Assume that rank \( B = n \). Limit system

\[
\Sigma^\infty : \quad \dot{x} = (A - \lambda^2 B^\top B)x
\]

If \( \lambda, \omega > 0 \) are sufficiently large, the \( \{0\} \) is LUES for \( \Sigma^\omega \).
Stabilization of control-affine systems

Remark: choice of functions $h_1, h_2$ in

$$u_k = u_{2k-1}^\omega (t) h_1(\psi(p)) + u_{2k}^\omega (t) h_2(\psi(p))$$

- Dürr et al. (2013): $h_1(y) = y$ and $h_2(y) = 1$
  $\rightarrow \omega$-PLUAS

- Scheinker et al. (2014): $h_1(y) = \sin y$ and $h_2(y) = \cos y$
  $\rightarrow \omega$-PLUAS

- S. (2017): $h_1(y) = \sqrt{y} \sin(\log y)$ and $h_2(y) = \sqrt{y} \cos(\log y)$
  $\rightarrow \omega$-PLUAS and also $\omega$-LUES

- S., Sun (2018): $h_1(y) = y \sin(\log y)$ and $h_2(y) = y \cos(\log y)$
  $\rightarrow$ LUAS (for every $\omega > 0$), but not LUES

Integral equation, averaging procedure:

$$p(t) = p(t_0) + \left[D_1^\omega (s, p(s))\right]_t^{t_0} + \int_{t_0}^{t} \left(f^\infty(p(s)) + D_2^\omega (s, p(s))\right) ds,$$
"Team" of control-affine systems

\[
\dot{p}_i = \sum_{k=1}^{n} u_{i,k} b_{i,k}, \quad y_i = \psi_i(p), \quad i = 1, \ldots, N,
\]

output functions = local potential functions

\[
\psi_i(p) := \frac{1}{4} \sum_{j \in V : ij \in E} \left( \|p_j - p_i\|^2 - d_{ij}^2 \right)^2
\]

can be computed by individual distance measurements.
Application to distanced-based formation control

\[ h_1(y) := y \sin(\log y), \quad h_2(y) := y \cos(\log y) \text{ for } y > 0. \]

\(\omega\)-dependent system \(\Sigma^\omega: \dot{p}_i = \sum_{k=1}^{n} u_{i,k} b_{i,k}, \quad i = 1, \ldots, N\)

under control law \(u_{i,k} = u_{i,k,1}(t) h_1(\psi_i(p)) + u_{i,k,2}(t) h_2(\psi_i(p))\),

Limit system \(\Sigma^\infty: \dot{p}_i = -\frac{1}{2} \psi_i(p) \nabla p_i \psi_i(p), \quad i = 1, \ldots, N.\)

**Theorem (S., Sun(2018))**

*Suppose that the frameworks \((G, p)\) with \(p \in F\) are infinitesimally rigid. Then, for every \(\omega > 0\), the set \(F\) is LUAS for \(\Sigma^\omega\).*

**Remark:** the domain of attraction increases when \(\omega\) increases.
Application to distance-based formation control

\[ x_i - 2 - 1 0 1 2 \]
\[ y_i - 2 - 1 0 1 2 \]
\[ z_i - 2 - 1 0 1 2 \]
Thank you for your attention!