INSTITUTE OF MATHEMATICS
AND SCIENTIFIC COMPUTING

Stabilisierung von unendlich-dimensionalen nichtlinearen Systemen

Tobias Breiten
Karl Kunisch, Laurent Pfeiffer

12. Elgersburg Workshop

26.02.-01.03.2018

Problem setup

We consider abstract Cauchy problems

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\text{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

- $\mathcal{Y}$ is a Hilbert space
- $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset \mathcal{Y} \to \mathcal{Y}$ generates a C_0-semigroup $\left(e^{\mathcal{A}t}\right)_{t\geq 0}$ on $\mathcal{Y}$:
  - $e^{\mathcal{A}t} \in \mathcal{L}(\mathcal{Y})$ for all $t \geq 0$,
  - $e^{\mathcal{A}0} = I$, $e^{\mathcal{A}(t+s)} = e^{\mathcal{A}t}e^{\mathcal{A}s}$ for all $t, s \geq 0$,
  - $\lim_{t\downarrow 0} \|e^{\mathcal{A}t}y - y\| = 0$,
  - $\mathcal{D}(\mathcal{A}) = \left\{ y \in \mathcal{Y}\colon \lim_{t\downarrow 0} \frac{e^{\mathcal{A}t}y - y}{t} \text{ exists} \right\}$, $\mathcal{A}y = \lim_{t\downarrow 0} \frac{e^{\mathcal{A}t}y - y}{t}$.

Problem setup

We consider abstract Cauchy problems

$$\dot{y}(t) = \mathcal{A}y(t) + \mathcal{F}(y(t)) \qquad + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y},$$
$$y_{\mathrm{obs}}(t) = \mathcal{C}y(t),$$

- $\mathcal{Y}$ is a Hilbert space
- $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset \mathcal{Y} \to \mathcal{Y}$ generates a C_0-semigroup $\left(e^{\mathcal{A}t}\right)_{t\geq 0}$ on $\mathcal{Y}$:
- nonlinear operator with $\mathcal{F}(0) = 0$, linear feedback $u(t) = k(y(t))$

Problem setup

We consider abstract Cauchy problems

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) \qquad + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

- $\mathcal{Y}$ is a Hilbert space
- $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset \mathcal{Y} \to \mathcal{Y}$ generates a C_0-semigroup $\left(e^{\mathcal{A}t}\right)_{t\geq 0}$ on $\mathcal{Y}$:
- nonlinear operator with $\mathcal{F}(0) = 0$, linear feedback $u(t) = k(y(t))$
- bilinear control, polynomial feedback $u(t) = k(y(t))$

Problem setup

We consider abstract Cauchy problems

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) \qquad + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\text{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

- $\mathcal{Y}$ is a Hilbert space
- $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset \mathcal{Y} \to \mathcal{Y}$ generates a C_0-semigroup $\left(e^{\mathcal{A}t}\right)_{t\geq 0}$ on $\mathcal{Y}$:
- nonlinear operator with $\mathcal{F}(0) = 0$, linear feedback $u(t) = k(y(t))$
- bilinear control, polynomial feedback $u(t) = k(y(t))$

Goal: find u such that $\lim_{t\to\infty} \|y(t)\|_? = 0$ if $\|y_0\|_? < \rho$

Nonlinear control systems

Consider a nonlinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{F}(y(t)) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

where, in addition to before

- semigroup $e^{\mathcal{A}t}$ is analytic,
- λ is s.t. $\widehat{\mathcal{A}} := \lambda I - \mathcal{A}$ satisfies $\langle y, \widehat{\mathcal{A}}y \rangle_{\mathcal{Y}} \geq \|y\|^2_{\mathcal{Y}}$ for $y \in \mathcal{D}(\mathcal{A})$,
- $\mathcal{B} \in \mathcal{L}(\mathcal{U}, [\mathcal{D}(\mathcal{A}^*)]')$ and $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for fixed $0 \leq \gamma < 1$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta < \min\left\{1 - \gamma, \frac{1}{2}\right\}$,
- $\mathcal{U}, \mathcal{Z}$ Hilbert spaces.

Nonlinear control systems

Consider a nonlinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{F}(y(t)) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\text{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

where, in addition to before

- semigroup $e^{\mathcal{A}t}$ is analytic,
- λ is s.t. $\widehat{\mathcal{A}} := \lambda I - \mathcal{A}$ satisfies $\langle y, \widehat{\mathcal{A}}y \rangle_{\mathcal{Y}} \geq \|y\|_{\mathcal{Y}}^2$ for $y \in \mathcal{D}(\mathcal{A})$,
- $\mathcal{B} \in \mathcal{L}(\mathcal{U}, [\mathcal{D}(\mathcal{A}^*)]')$ and $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for fixed $0 \leq \gamma < 1$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta < \min\left\{1-\gamma, \frac{1}{2}\right\}$,
- $\mathcal{U}, \mathcal{Z}$ Hilbert spaces.

Nonlinear control systems

Consider a nonlinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{F}(y(t)) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

where, in addition to before

- semigroup $e^{\mathcal{A}t}$ is analytic,
- λ is s.t. $\widehat{\mathcal{A}} := \lambda I - \mathcal{A}$ satisfies $\langle y, \widehat{\mathcal{A}}y\rangle_{\mathcal{Y}} \geq \|y\|^2_{\mathcal{Y}}$ for $y \in \mathcal{D}(\mathcal{A})$,
- $\mathcal{B} \in \mathcal{L}(\mathcal{U}, [\mathcal{D}(\mathcal{A}^*)]')$ and $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for fixed $0 \leq \gamma < 1$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta < \min\left\{1-\gamma, \frac{1}{2}\right\}$,
- $\mathcal{U}, \mathcal{Z}$ Hilbert spaces.

Nonlinear control systems

Consider a nonlinear control system

$$\dot{y}(t) = \mathcal{A}y(t) + \mathcal{F}(y(t)) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y},$$
$$y_{\mathrm{obs}}(t) = \mathcal{C}y(t),$$

where, in addition to before

- semigroup $e^{\mathcal{A}t}$ is analytic,
- λ is s.t. $\widehat{\mathcal{A}} := \lambda I - \mathcal{A}$ satisfies $\langle y, \widehat{\mathcal{A}}y\rangle_{\mathcal{Y}} \geq \|y\|^2_{\mathcal{Y}}$ for $y \in \mathcal{D}(\mathcal{A})$,
- $\mathcal{B} \in \mathcal{L}(\mathcal{U}, [\mathcal{D}(\mathcal{A}^*)]')$ and $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for fixed $0 \leq \gamma < 1$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta < \min\left\{1-\gamma, \frac{1}{2}\right\}$,
- $\mathcal{U}, \mathcal{Z}$ Hilbert spaces.

Nonlinear control systems

Consider a nonlinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{F}(y(t)) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

where, in addition to before

- semigroup $e^{\mathcal{A}t}$ is analytic,
- λ is s.t. $\widehat{\mathcal{A}} := \lambda I - \mathcal{A}$ satisfies $\langle y, \widehat{\mathcal{A}}y \rangle_{\mathcal{Y}} \geq \|y\|^2_{\mathcal{Y}}$ for $y \in \mathcal{D}(\mathcal{A})$,
- $\mathcal{B} \in \mathcal{L}(\mathcal{U}, [\mathcal{D}(\mathcal{A}^*)]')$ and $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for fixed $0 \leq \gamma < 1$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta < \min\left\{1 - \gamma, \frac{1}{2}\right\}$,
- $\mathcal{U}, \mathcal{Z}$ Hilbert spaces.

A general Riccati based approach

1. Consider LQ-problem for linearized system

$$\min_{u\in L^2([0,\infty);\,\mathcal{U})} \frac{1}{2}\left(\int_0^\infty \|\mathcal{C}y(t)\|^2 + \|u(t)\|^2 \,\mathrm{d}t\right)$$

$$\text{s.t. } \dot{y} = \mathcal{A}y + \mathcal{B}u,\ y(0) = y_0,\quad y_{\mathrm{obs}} = \mathcal{C}y$$

If $(\mathcal{A},\mathcal{B})$ stabilizable, $(\mathcal{A},\mathcal{C})$ detectable, obtain $u_{\mathrm{opt}} = -\mathcal{B}^*\mathcal{P}y$ via

$$\langle \mathcal{A}^*\mathcal{P}y_1, y_2\rangle_{\mathcal{Y}} + \langle \mathcal{P}\mathcal{A}y_1, y_2\rangle_{\mathcal{Y}} - \langle \mathcal{B}^*\mathcal{P}y_1, \mathcal{B}^*\mathcal{P}y_2\rangle_{\mathcal{U}} + \langle \mathcal{C}y_1, \mathcal{C}y_2\rangle_{\mathcal{Z}} = 0$$

2. Study local well-posedness of $\dot{y} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})y + \mathcal{F}(y)$ by
   - showing that $\mathcal{F}$ is locally Lipschitz continuous
   - analyzing solutions to $\dot{y}_z = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})y_z + \mathcal{F}(z)$
   - applying a fixed point argument for the nonlinear system

cp. [Badra, Barbu, Benner, Heiland, Lasiecka, Raymond, Triggiani, ...]

A general Riccati based approach

1. Consider LQ-problem for linearized system

$$\min_{u\in L^2([0,\infty);\,\mathcal{U})} \quad \frac{1}{2}\left(\int_0^\infty \|\mathcal{C}y(t)\|^2 + \|u(t)\|^2 \,\mathrm{d}t\right)$$

$$\text{s.t. } \dot{y} = \mathcal{A}y + \mathcal{B}u,\; y(0) = y_0, \quad y_{\mathrm{obs}} = \mathcal{C}y$$

If $(\mathcal{A},\mathcal{B})$ stabilizable, $(\mathcal{A},\mathcal{C})$ detectable, obtain $u_{\mathrm{opt}} = -\mathcal{B}^*\mathcal{P}y$ via

$$\langle \mathcal{A}^*\mathcal{P}y_1, y_2\rangle_{\mathcal{Y}} + \langle \mathcal{P}\mathcal{A}y_1, y_2\rangle_{\mathcal{Y}} - \langle \mathcal{B}^*\mathcal{P}y_1, \mathcal{B}^*\mathcal{P}y_2\rangle_{\mathcal{U}} + \langle \mathcal{C}y_1, \mathcal{C}y_2\rangle_{\mathcal{Z}} = 0$$

2. Study local well-posedness of $\dot{y} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})y + \mathcal{F}(y)$ by
   - showing that $\mathcal{F}$ is locally Lipschitz continuous
   - analyzing solutions to $\dot{y}_z = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})y_z + \mathcal{F}(z)$
   - applying a fixed point argument for the nonlinear system

cp. [Badra, Barbu, Benner, Heiland, Lasiecka, Raymond, Triggiani, ...]

Existing Riccati theory [Lasiecka/Triggiani'00]

Under previous assumptions the LQ-problem

$$\min_{u \in L^2([0,\infty);\,\mathcal{U})} \; \frac{1}{2}\left(\int_0^\infty \|\mathcal{C}y(t)\|^2 + \|u(t)\|^2 \,\mathrm{d}t\right)$$

$$\text{s.t. } \dot{y} = \mathcal{A}y + \mathcal{B}u,\; y(0) = y_0,\; y_{\mathrm{obs}} = \mathcal{C}y$$

has a unique optimal solution $u_{\mathrm{opt}} = -\mathcal{B}^*\mathcal{P}y \in L^2(0,\infty;\mathcal{U})$.

In particular, it holds that

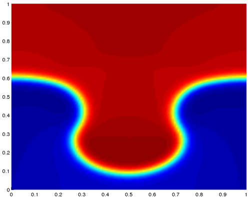
- $(\widehat{\mathcal{A}}^*)^\gamma \mathcal{P} \widehat{\mathcal{A}}^{-\delta} \in \mathcal{L}(\mathcal{Y})$,
- $\mathcal{B}^*\mathcal{P} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^\delta), \mathcal{U})$,
- $\mathcal{A}_\mathcal{P} := \mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}$ generates an exponentially stable, analytic semigroup $e^{\mathcal{A}_\mathcal{P} t}$ on $\mathcal{D}(\widehat{\mathcal{A}}^\delta)$

The monodomain equations

Consider a coupled PDE-ODE system

$$\begin{aligned}
\partial_t v &= \Delta v + av + bw + cv^2 + dv^3 + f_v, && \text{in } \Omega \times (0,\infty),\\
\partial_t w &= \gamma v - \delta w + f_w, && \text{in } \Omega \times (0,\infty),\\
\partial_\nu v &= mu + g, && \text{on } \Gamma \times (0,\infty),\\
v_{\mathrm{obs}} &= \tilde{m} v_{|\Gamma}, &&\\
v(x,0) = v_0(x) &\text{ and } w(x,0) = w_0(x), && \text{in } \Omega,
\end{aligned}$$

- $a, b, c, d \in \mathbb{R}, \gamma, \delta > 0$
- equations simplify bidomain equations,
- model for electric activity of human heart,
- spiral/reentry waves can occur
  $\rightsquigarrow$ fibrillation processes of the heart,
- $m, \tilde{m}$ allow to localize control/observation.



Stabilizing around a stationary solution

Let $(\bar{v}, \bar{w}) \in H^3(\Omega) \times L^\infty(\Omega)$ denote a stationary solution

$$
\begin{aligned}
0 &= \Delta\bar{v} + a\bar{v} + b\bar{w} + c\bar{v}^2 + d\bar{v}^3 + f_v, && \text{in } \Omega \times (0,\infty),\\
0 &= \gamma\bar{v} - \delta\bar{w} + f_w, && \text{in } \Omega \times (0,\infty),\\
\partial_\nu \bar{v} &= g, && \text{on } \Gamma \times (0,\infty),
\end{aligned}
$$

and consider the stabilization problem for $(y, z) = (v - \bar{v}, w - \bar{w})$:

$$
\begin{aligned}
\partial_t y &= \Delta y + \alpha(x) y + bz + dy^3 + \varphi(x) y^2, && \text{in } \Omega \times (0,\infty),\\
\partial_t z &= \gamma y - \delta z, && \text{in } \Omega \times (0,\infty),\\
\partial_\nu y &= mu, && \text{on } \Gamma \times (0,\infty),\\
y_{\mathrm{obs}} &= \tilde{m} y_{|\Gamma}, &&\\
y(x,0) &= v_0(x) - \bar{v}(x) \text{ and } z(x,0) = w_0(x) - \bar{w}(x), && \text{in } \Omega.
\end{aligned}
$$

Stabilizing around a stationary solution

Let $(\bar{v}, \bar{w}) \in H^3(\Omega) \times L^\infty(\Omega)$ denote a stationary solution

$$\begin{aligned} 0 &= \Delta \bar{v} + a\bar{v} + b\bar{w} + c\bar{v}^2 + d\bar{v}^3 + f_v, && \text{in } \Omega \times (0,\infty), \\ 0 &= \gamma \bar{v} - \delta \bar{w} + f_w, && \text{in } \Omega \times (0,\infty), \\ \partial_\nu \bar{v} &= g, && \text{on } \Gamma \times (0,\infty), \end{aligned}$$

and consider the stabilization problem for $(y, z) = (v - \bar{v}, w - \bar{w})$:

$$\begin{aligned} \partial_t y &= \Delta y + \alpha(x) y + bz + dy^3 + \varphi(x) y^2, && \text{in } \Omega \times (0,\infty), \\ \partial_t z &= \gamma y - \delta z, && \text{in } \Omega \times (0,\infty), \\ \partial_\nu y &= mu, && \text{on } \Gamma \times (0,\infty), \\ y_{\text{obs}} &= \tilde{m} y_{|\Gamma}, && \\ y(x,0) &= v_0(x) - \bar{v}(x) \text{ and } z(x,0) = w_0(x) - \bar{w}(x), && \text{in } \Omega. \end{aligned}$$

The abstract linearized system

We can write the linearization as abstract Cauchy problem

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y} + \mathcal{B}u, \quad \mathbf{y}(0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix},$$
$$\mathbf{y}_{\mathrm{obs}} = \mathcal{C}\mathbf{y},$$

where

- $\mathcal{A}\mathbf{y} = \begin{pmatrix} \Delta y + \alpha(x)y + bz \\ \gamma y - \delta z \end{pmatrix}, \; \mathcal{D}(\mathcal{A}) = H^2_\nu(\Omega) \times L^2(\Omega),$
- $\mathcal{B} = \widehat{\mathcal{A}}\mathcal{N}_{\widehat{\mathcal{A}}}$ with Neumann map $\mathcal{N}_{\widehat{\mathcal{A}}}$ defined by $\mathcal{N}_{\widehat{\mathcal{A}}}u = \mathbf{y}$ iff

$$\begin{aligned} \lambda y - \Delta y - \alpha(x)y - bz &= 0, && \text{in } \Omega, \\ \lambda z - \gamma y + \delta z &= 0, && \text{in } \Omega, \\ \partial_\nu y &= mu, && \text{on } \Gamma, \end{aligned}$$

- $\mathcal{C}\mathbf{y} = \tilde{m}y_{|\Gamma}$ the Dirichlet trace operator.

The abstract linearized system

We can write the linearization as abstract Cauchy problem

$$\begin{aligned} \dot{\mathbf{y}} &= \mathcal{A}\mathbf{y} + \mathcal{B}u, \quad \mathbf{y}(0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \\ \mathbf{y}_{\text{obs}} &= \mathcal{C}\mathbf{y}, \end{aligned}$$

where

- $\mathcal{A}\mathbf{y} = \begin{pmatrix} \Delta y + \alpha(x)y + bz \\ \gamma y - \delta z \end{pmatrix}$, $\mathcal{D}(\mathcal{A}) = H^2_\nu(\Omega) \times L^2(\Omega)$,
- $\mathcal{B} = \widehat{\mathcal{A}}\mathcal{N}_{\widehat{\mathcal{A}}}$ with Neumann map $\mathcal{N}_{\widehat{\mathcal{A}}}$ defined by $\mathcal{N}_{\widehat{\mathcal{A}}}u = \mathbf{y}$ iff

$$\begin{aligned} \lambda y - \Delta y - \alpha(x)y - bz &= 0, && \text{in } \Omega, \\ \lambda z - \gamma y + \delta z &= 0, && \text{in } \Omega, \\ \partial_\nu y &= mu, && \text{on } \Gamma, \end{aligned}$$

- $\mathcal{C}\mathbf{y} = \tilde{m}y_{|\Gamma}$ the Dirichlet trace operator.

The abstract linearized system

We can write the linearization as abstract Cauchy problem

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y} + \mathcal{B}u, \quad \mathbf{y}(0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix},$$
$$\mathbf{y}_{\mathrm{obs}} = \mathcal{C}\mathbf{y},$$

where

- $\mathcal{A}\mathbf{y} = \begin{pmatrix} \Delta y + \alpha(x)y + bz \\ \gamma y - \delta z \end{pmatrix}, \ \mathcal{D}(\mathcal{A}) = H^2_\nu(\Omega) \times L^2(\Omega),$
- $\mathcal{B} = \widehat{\mathcal{A}}\mathcal{N}_{\widehat{\mathcal{A}}}$ with Neumann map $\mathcal{N}_{\widehat{\mathcal{A}}}$ defined by $\mathcal{N}_{\widehat{\mathcal{A}}}u = \mathbf{y}$ iff

$$\begin{aligned} \lambda y - \Delta y - \alpha(x)y - bz &= 0, && \text{in } \Omega, \\ \lambda z - \gamma y + \delta z &= 0, && \text{in } \Omega, \\ \partial_\nu y &= mu, && \text{on } \Gamma, \end{aligned}$$

- $\mathcal{C}\mathbf{y} = \tilde{m}y_{|\Gamma}$ the Dirichlet trace operator.

Properties of the linearized system

We have the following results:

- $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates an analytic semigroup on $\mathcal{Y} = L^2(\Omega) \times L^2(\Omega)$,
- the point spectrum $\sigma_p(\mathcal{A})$ is given by

$$\sigma_p(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda = -\frac{\delta - \mu}{2} \pm \frac{1}{2}\sqrt{(\delta + \mu)^2 + 4\gamma b} \right\},$$

  where μ denote the eigenvalues of the operator $\Delta + \alpha(x)$,
- $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for $\gamma > \frac{1}{4}$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta > \frac{1}{4}$

Properties of the linearized system

We have the following results:

- $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates an analytic semigroup on $\mathcal{Y} = L^2(\Omega) \times L^2(\Omega)$,
- the point spectrum $\sigma_p(\mathcal{A})$ is given by

$$\sigma_p(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda = -\frac{\delta - \mu}{2} \pm \frac{1}{2}\sqrt{(\delta + \mu)^2 + 4\gamma b} \right\},$$

  where μ denote the eigenvalues of the operator $\Delta + \alpha(x)$,

- $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for $\gamma > \frac{1}{4}$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta > \frac{1}{4}$

Properties of the linearized system

We have the following results:

- $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates an analytic semigroup on $\mathcal{Y} = L^2(\Omega) \times L^2(\Omega)$,
- the point spectrum $\sigma_p(\mathcal{A})$ is given by

$$\sigma_p(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda = -\frac{\delta - \mu}{2} \pm \frac{1}{2}\sqrt{(\delta + \mu)^2 + 4\gamma b} \right\},$$

  where μ denote the eigenvalues of the operator $\Delta + \alpha(x)$,
- $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for $\gamma > \frac{1}{4}$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta > \frac{1}{4}$

Properties of the linearized system

We have the following results:

- $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates an analytic semigroup on $\mathcal{Y} = L^2(\Omega) \times L^2(\Omega)$,
- the point spectrum $\sigma_p(\mathcal{A})$ is given by

$$\sigma_p(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda = -\frac{\delta - \mu}{2} \pm \frac{1}{2}\sqrt{(\delta + \mu)^2 + 4\gamma b} \right\},$$

  where μ denote the eigenvalues of the operator $\Delta + \alpha(x)$,

- $\widehat{\mathcal{A}}^{-\gamma}\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for $\gamma > \frac{1}{4}$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\delta}), \mathcal{Z})$ for $\delta > \frac{1}{4}$

Null controllability and β-stabilizability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

∞-dim Hautus test [Weiss/Rebarber'00]

If $(\mathcal{A}, \mathcal{B})$ is stabilizable, then there exist $\kappa > 0$ and $m > 0$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -\kappa$

$$\|(sI - \mathcal{A}^*)y\| + \|\mathcal{B}^*y\| \geq m\|y\| \quad \forall y \in \mathcal{D}(\mathcal{A}^*).$$

For $\beta \geq \delta$ Hautus test can be shown to fail for $(\mathcal{A} + \beta I, \mathcal{B})$:

consider $\lambda \in \sigma_p(\mathcal{A}^* + \beta I)$ with eigenfunction $\mathbf{y} = (y, \frac{b}{\delta - \beta + \lambda} y)$

Consequence: $(\mathcal{A}, \mathcal{B})$ not β-stabilizable, not null controllable

Null controllability and β-stabilizability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

∞-dim Hautus test [Weiss/Rebarber'00]

If $(\mathcal{A}, \mathcal{B})$ is stabilizable, then there exist $\kappa > 0$ and $m > 0$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -\kappa$

$$\|(sI - \mathcal{A}^*)y\| + \|\mathcal{B}^* y\| \geq m\|y\| \quad \forall y \in \mathcal{D}(\mathcal{A}^*).$$

For $\beta \geq \delta$ Hautus test can be shown to fail for $(\mathcal{A} + \beta I, \mathcal{B})$:

consider $\lambda \in \sigma_p(\mathcal{A}^* + \beta I)$ with eigenfunction $\mathbf{y} = (y, \frac{b}{\delta - \beta + \lambda} y)$

Consequence: $(\mathcal{A}, \mathcal{B})$ not β-stabilizable, not null controllable

Null controllability and β-stabilizability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

∞-dim Hautus test [Weiss/Rebarber'00]

If $(\mathcal{A}, \mathcal{B})$ is stabilizable, then there exist $\kappa > 0$ and $m > 0$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -\kappa$

$$\|(sI - \mathcal{A}^*)y\| + \|\mathcal{B}^* y\| \geq m\|y\| \quad \forall y \in \mathcal{D}(\mathcal{A}^*).$$

For $\beta \geq \delta$ Hautus test can be shown to fail for $(\mathcal{A} + \beta I, \mathcal{B})$:

consider $\lambda \in \sigma_p(\mathcal{A}^* + \beta I)$ with eigenfunction $\mathbf{y} = (y, \frac{b}{\delta - \beta + \lambda} y)$

Consequence: $(\mathcal{A}, \mathcal{B})$ not β-stabilizable, not null controllable

Null controllability and β-stabilizability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

∞-dim Hautus test [Weiss/Rebarber'00]

If $(\mathcal{A}, \mathcal{B})$ is stabilizable, then there exist $\kappa > 0$ and $m > 0$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -\kappa$

$$\|(sI - \mathcal{A}^*)y\| + \|\mathcal{B}^* y\| \geq m\|y\| \quad \forall y \in \mathcal{D}(\mathcal{A}^*).$$

For $\beta \geq \delta$ Hautus test can be shown to fail for $(\mathcal{A} + \beta I, \mathcal{B})$:

consider $\lambda \in \sigma_p(\mathcal{A}^* + \beta I)$ with eigenfunction $\mathbf{y} = (y, \frac{b}{\delta - \beta + \lambda} y)$

Consequence: $(\mathcal{A}, \mathcal{B})$ not β-stabilizable, not null controllable

Stabilizability and detectability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

For $\beta < \delta$ stabilizability of $(\mathcal{A} + \beta I, \mathcal{B})$ can be shown as follows:

- consider auxiliary LQ-problem

$$\min_{u \in L^2([0,\infty);\,\mathcal{U})} \frac{1}{2}\left(\int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \,\mathrm{d}t\right)$$

$$\text{s.t.} \begin{cases} \partial_t y = \Delta y + \alpha(x) y + \kappa y, & \text{in } \Omega \times (0,\infty), \\ \partial_\nu y = mu, & \text{on } \Gamma \times (0,\infty), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

  with $\kappa = \beta + \gamma \frac{|b|}{\delta - \beta}$,

- optimal feedback gain k_{aux} stabilizes original problem
  $\rightsquigarrow$ use, e.g., spectral argument (+analyticity of the semigroup)
- similar argument for detectability of $(\mathcal{A}, \mathcal{C})$

Stabilizability and detectability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

For $\beta < \delta$ stabilizability of $(\mathcal{A} + \beta I, \mathcal{B})$ can be shown as follows:

- consider auxiliary LQ-problem

$$\min_{u \in L^2([0,\infty);\, \mathcal{U})} \frac{1}{2} \left(\int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \, \mathrm{d}t \right)$$

$$\text{s.t.} \begin{cases} \partial_t y = \Delta y + \alpha(x) y + \kappa y, & \text{in } \Omega \times (0, \infty), \\ \partial_\nu y = mu, & \text{on } \Gamma \times (0, \infty), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

  with $\kappa = \beta + \gamma \frac{|b|}{\delta - \beta}$,

- optimal feedback gain k_{aux} stabilizes original problem
  $\rightsquigarrow$ use, e.g., spectral argument (+analyticity of the semigroup)

- similar argument for detectability of $(\mathcal{A}, \mathcal{C})$

Stabilizability and detectability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

For $\beta < \delta$ stabilizability of $(\mathcal{A} + \beta I, \mathcal{B})$ can be shown as follows:

- consider auxiliary LQ-problem

$$\min_{u \in L^2([0,\infty);\, \mathcal{U})} \frac{1}{2} \left(\int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \, \mathrm{d}t \right)$$

$$\text{s.t.} \begin{cases} \partial_t y = \Delta y + \alpha(x) y + \kappa y, & \text{in } \Omega \times (0, \infty), \\ \partial_\nu y = mu, & \text{on } \Gamma \times (0, \infty), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

  with $\kappa = \beta + \gamma \frac{|b|}{\delta - \beta}$,

- optimal feedback gain k_{aux} stabilizes original problem
  $\rightsquigarrow$ use, e.g., spectral argument (+analyticity of the semigroup)

- similar argument for detectability of $(\mathcal{A}, \mathcal{C})$

Stabilizability and detectability

Recall: we need stabilizability/detectability of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$

For $\beta < \delta$ stabilizability of $(\mathcal{A} + \beta I, \mathcal{B})$ can be shown as follows:

- consider auxiliary LQ-problem

$$\min_{u \in L^2([0,\infty);\, \mathcal{U})} \frac{1}{2} \left(\int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \, \mathrm{d}t \right)$$

$$\text{s.t.} \begin{cases} \partial_t y = \Delta y + \alpha(x) y + \kappa y, & \text{in } \Omega \times (0,\infty), \\ \partial_\nu y = mu, & \text{on } \Gamma \times (0,\infty), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

  with $\kappa = \beta + \gamma \frac{|b|}{\delta - \beta}$,

- optimal feedback gain k_{aux} stabilizes original problem
  $\rightsquigarrow$ use, e.g., spectral argument (+analyticity of the semigroup)
- similar argument for detectability of $(\mathcal{A}, \mathcal{C})$

Analyzing the nonlinearity

Recall: nonlinearity for the monodomain equations given by

$$\mathcal{F}(\mathbf{y}) := \begin{pmatrix} \varphi(x)y^2 + dy^3 \\ 0 \end{pmatrix} =: \begin{pmatrix} F(y) \\ 0 \end{pmatrix}$$

On $Q_\infty = \Omega \times (0,\infty)$ consider

$$H^{2,1}(Q_\infty) := L^2(0,\infty; H^2(\Omega)) \cap H^1(0,\infty; L^2(\Omega)).$$

For $y_1, y_2 \in H^{2,1}(Q_\infty)$ it holds that

$$\|F(y_1) - F(y_2)\|_{L^2(Q_\infty)} \\ \leq M\left(\|y_1\|_{H^{2,1}} + \|y_2\|_{H^{2,1}} + \|y_1\|^2_{H^{2,1}} + \|y_2\|^2_{H^{2,1}}\right) \|y_1 - y_2\|_{H^{2,1}(Q_\infty)}$$

Consequence: F and, thus, $\mathcal{F}$ are locally Lipschitz continuous

Analyzing the nonlinearity

Recall: nonlinearity for the monodomain equations given by

$$\mathcal{F}(\mathbf{y}) := \begin{pmatrix} \varphi(x)y^2 + dy^3 \\ 0 \end{pmatrix} =: \begin{pmatrix} F(y) \\ 0 \end{pmatrix}$$

On $Q_\infty = \Omega \times (0,\infty)$ consider

$$H^{2,1}(Q_\infty) := L^2(0,\infty; H^2(\Omega)) \cap H^1(0,\infty; L^2(\Omega)).$$

For $y_1, y_2 \in H^{2,1}(Q_\infty)$ it holds that

$$\|F(y_1) - F(y_2)\|_{L^2(Q_\infty)} \le M \left(\|y_1\|_{H^{2,1}} + \|y_2\|_{H^{2,1}} + \|y_1\|^2_{H^{2,1}} + \|y_2\|^2_{H^{2,1}}\right) \|y_1 - y_2\|_{H^{2,1}(Q_\infty)}$$

Consequence: F and, thus, $\mathcal{F}$ are locally Lipschitz continuous

Analyzing the nonlinearity

Recall: nonlinearity for the monodomain equations given by

$$\mathcal{F}(\mathbf{y}) := \begin{pmatrix} \varphi(x)y^2 + dy^3 \\ 0 \end{pmatrix} =: \begin{pmatrix} F(y) \\ 0 \end{pmatrix}$$

On $Q_\infty = \Omega \times (0, \infty)$ consider

$$H^{2,1}(Q_\infty) := L^2(0, \infty; H^2(\Omega)) \cap H^1(0, \infty; L^2(\Omega)).$$

For $y_1, y_2 \in H^{2,1}(Q_\infty)$ it holds that

$$\begin{aligned} &\|F(y_1) - F(y_2)\|_{L^2(Q_\infty)} \\ &\quad \leq M \left(\|y_1\|_{H^{2,1}} + \|y_2\|_{H^{2,1}} + \|y_1\|^2_{H^{2,1}} + \|y_2\|^2_{H^{2,1}} \right) \|y_1 - y_2\|_{H^{2,1}(Q_\infty)} \end{aligned}$$

Consequence: F and, thus, $\mathcal{F}$ are locally Lipschitz continuous

Analyzing the nonlinearity

Recall: nonlinearity for the monodomain equations given by

$$\mathcal{F}(\mathbf{y}) := \begin{pmatrix} \varphi(x)y^2 + dy^3 \\ 0 \end{pmatrix} =: \begin{pmatrix} F(y) \\ 0 \end{pmatrix}$$

On $Q_\infty = \Omega \times (0, \infty)$ consider

$$H^{2,1}(Q_\infty) := L^2(0, \infty; H^2(\Omega)) \cap H^1(0, \infty; L^2(\Omega)).$$

For $y_1, y_2 \in H^{2,1}(Q_\infty)$ it holds that

$$\begin{aligned} &\|F(y_1) - F(y_2)\|_{L^2(Q_\infty)} \\ &\quad \leq M \left(\|y_1\|_{H^{2,1}} + \|y_2\|_{H^{2,1}} + \|y_1\|^2_{H^{2,1}} + \|y_2\|^2_{H^{2,1}} \right) \|y_1 - y_2\|_{H^{2,1}(Q_\infty)} \end{aligned}$$

Consequence: F and, thus, $\mathcal{F}$ are locally Lipschitz continuous

Some regularity results

Let us turn to the (nonhomogeneous) linearized closed-loop system

$$\dot{\mathbf{y}} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y} + \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \qquad (*)$$

$$\mathbf{y}_{\mathrm{obs}} = \mathcal{C}\mathbf{y}$$

If $f \in L^2(Q_\infty)$ and $\mathbf{y}_0 \in H^1(\Omega) \times L^2(\Omega)$, then $(*)$ has a unique solution

$$\mathbf{y} \in \left(H^{2,1}(Q_\infty) \cap C_b([0,\infty); H^1(\Omega))\right) \times H^2(0,\infty; L^2(\Omega))$$

Regularity of analytic semigroups [Bensoussan et al.'93]

If $\mathcal{A}$ generates an exp. stable analytic sg. on $\mathcal{Y}$ then for all $0 \leq \theta \leq 1$,

$$h \colon y \mapsto (\dot{y} - \mathcal{A}y, y(0))$$

$$L^2(0,\infty; [\mathcal{D}(\mathcal{A}), \mathcal{Y}]_\theta) \cap H^1(0,\infty; [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta})$$

$$\to L^2(0,\infty; [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta}) \times [[\mathcal{D}(\mathcal{A}), \mathcal{Y}]_\theta, [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta}]_{\frac{1}{2}}$$

is an isomorphism.

Some regularity results

Let us turn to the (nonhomogeneous) linearized closed-loop system

$$\dot{\mathbf{y}} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y} + \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \qquad (*)$$

$$\mathbf{y}_{\mathrm{obs}} = \mathcal{C}\mathbf{y}$$

If $f \in L^2(Q_\infty)$ and $\mathbf{y}_0 \in H^1(\Omega) \times L^2(\Omega)$, then $(*)$ has a unique solution

$$\mathbf{y} \in \left(H^{2,1}(Q_\infty) \cap C_b([0,\infty); H^1(\Omega))\right) \times H^2(0,\infty; L^2(\Omega))$$

Regularity of analytic semigroups [Bensoussan et al.'93]

If $\mathcal{A}$ generates an exp. stable analytic sg. on $\mathcal{Y}$ then for all $0 \leq \theta \leq 1$,

$$h \colon y \mapsto (\dot{y} - \mathcal{A}y, y(0))$$

$$L^2(0,\infty; [\mathcal{D}(\mathcal{A}), \mathcal{Y}]_\theta) \cap H^1(0,\infty; [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta})$$

$$\to L^2(0,\infty; [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta}) \times [[\mathcal{D}(\mathcal{A}), \mathcal{Y}]_\theta, [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta}]_{\frac{1}{2}}$$

is an isomorphism.

Some regularity results

Let us turn to the (nonhomogeneous) linearized closed-loop system

$$\dot{\mathbf{y}} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y} + \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \qquad (*)$$

$$\mathbf{y}_{\mathrm{obs}} = \mathcal{C}\mathbf{y}$$

If $f \in L^2(Q_\infty)$ and $\mathbf{y}_0 \in H^1(\Omega) \times L^2(\Omega)$, then $(*)$ has a unique solution

$$\mathbf{y} \in \left(H^{2,1}(Q_\infty) \cap C_b([0,\infty); H^1(\Omega))\right) \times H^2(0,\infty; L^2(\Omega))$$

Regularity of analytic semigroups [BENSOUSSAN ET AL.'93]

If $\mathcal{A}$ generates an exp. stable analytic sg. on $\mathcal{Y}$ then for all $0 \leq \theta \leq 1$,

$$h\colon y \mapsto (\dot{y} - \mathcal{A}y, y(0))$$

$$L^2(0,\infty; [\mathcal{D}(\mathcal{A}), \mathcal{Y}]_\theta) \cap H^1(0,\infty; [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta})$$

$$\to L^2(0,\infty; [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta}) \times [[\mathcal{D}(\mathcal{A}), \mathcal{Y}]_\theta, [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]'_{1-\theta}]_{\frac{1}{2}}$$

is an isomorphism.

The final result

Local well-posedness of the closed-loop system

Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$. Then there exist $\mu_1, \mu_2 > 0$ such that if $\|\mathbf{y}_0\|_{H^1(\Omega)\times L^2(\Omega)} \leq \mu_1$, then

$$\dot{\mathbf{y}} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y} + \mathcal{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

admits a unique solution in the set

$$D_\mu = \Big\{\mathbf{y} \in \Big(H^{2,1}(Q_\infty) \cap C_b([0,\infty); H^1(\Omega))\Big) \times H^2(0,\infty; L^2(\Omega)), \\ \|\mathbf{y}\|_{H^{2,1}(Q_\infty)\times H^2(0,\infty;L^2(\Omega))} \leq \mu_2 \Big\}.$$

In particular: $\lim_{t\to\infty} \|\mathbf{y}\|_{H^1(\Omega)} = 0$

Proof idea: show that $\mathcal{M}: \mathbf{z} \to \mathbf{y_z}$ defined by

$$\dot{\mathbf{y}}_\mathbf{z} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y}_\mathbf{z} + \mathcal{F}(\mathbf{z}), \; \mathbf{y}_\mathbf{z}(0) = \mathbf{y}_0,$$

is a contraction in D_μ.

The final result

Local well-posedness of the closed-loop system

Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$. Then there exist $\mu_1, \mu_2 > 0$ such that if $\|\mathbf{y}_0\|_{H^1(\Omega)\times L^2(\Omega)} \leq \mu_1$, then

$$\dot{\mathbf{y}} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y} + \mathcal{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

admits a unique solution in the set

$$D_\mu = \Big\{\mathbf{y} \in \Big(H^{2,1}(Q_\infty) \cap C_b([0,\infty); H^1(\Omega))\Big) \times H^2(0,\infty; L^2(\Omega)), \\ \|\mathbf{y}\|_{H^{2,1}(Q_\infty)\times H^2(0,\infty;L^2(\Omega))} \leq \mu_2\Big\}.$$

In particular: $\lim_{t\to\infty} \|\mathbf{y}\|_{H^1(\Omega)} = 0$

Proof idea: show that $\mathcal{M}: \mathbf{z} \to \mathbf{y}_\mathbf{z}$ defined by

$$\dot{\mathbf{y}}_\mathbf{z} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P})\mathbf{y}_\mathbf{z} + \mathcal{F}(\mathbf{z}), \ \mathbf{y}_\mathbf{z}(0) = \mathbf{y}_0,$$

is a contraction in D_μ.

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, 1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2}, 1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, \frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, 1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2}, 1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, \frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0,1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2},1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0,\frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, 1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2}, 1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, \frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, 1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2}, 1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0, \frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0,1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2},1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0,\frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Remarks and open questions

One can obtain similar results in the case that

- $\Omega \subset \mathbb{R}^2$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0,1)$
- $\Omega \subset \mathbb{R}^3$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (\frac{1}{2},1)$
- $\Omega \subset \mathbb{R}^2$, Dirichlet control and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ with $\varepsilon \in (0,\frac{1}{2})$

It is not clear (to me) if

- stabilization by Dirichlet control in $\Omega \subset \mathbb{R}^3$ is possible
- stabilization around time-varying trajectory is possible
  $\rightsquigarrow$ major problem: existence of a stabilizing control

Bilinear control systems

Consider a bilinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

with

- Hilbert spaces $V \subset \mathcal{Y} \subset V'$, dense and compact embedding $V \hookrightarrow \mathcal{Y}$
- $-\mathcal{A}$ associated to a V-$\mathcal{Y}$-coercive bilinear form $a \colon V \times V \to \mathbb{R}$

  $$\exists \delta > 0 \text{ and } \lambda \in \mathbb{R} \colon a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_{\mathcal{Y}}^2, \quad \forall v \in V$$

- $\mathcal{N} \in \mathcal{L}(V,\mathcal{Y}) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}),V)$ and $\mathcal{N}^* \in \mathcal{L}(V,\mathcal{Y})$
- $\mathcal{B} \in \mathcal{Y}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, $(\mathcal{A},\mathcal{B})$ stabilizable, $(\mathcal{A},\mathcal{C})$ detectable
- $[\mathcal{D}(\mathcal{A}),\mathcal{Y}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}^*),\mathcal{Y}]_{\frac{1}{2}} = V$

Bilinear control systems

Consider a bilinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

with

- Hilbert spaces $V \subset \mathcal{Y} \subset V'$, dense and compact embedding $V \hookrightarrow \mathcal{Y}$
- $-\mathcal{A}$ associated to a V-$\mathcal{Y}$-coercive bilinear form $a\colon V \times V \to \mathbb{R}$

$$\exists \delta > 0 \text{ and } \lambda \in \mathbb{R}\colon a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_{\mathcal{Y}}^2, \quad \forall v \in V$$

- $\mathcal{N} \in \mathcal{L}(V, \mathcal{Y}) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}), V)$ and $\mathcal{N}^* \in \mathcal{L}(V, \mathcal{Y})$
- $\mathcal{B} \in \mathcal{Y}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$, $(\mathcal{A}, \mathcal{B})$ stabilizable, $(\mathcal{A}, \mathcal{C})$ detectable
- $[\mathcal{D}(\mathcal{A}), \mathcal{Y}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]_{\frac{1}{2}} = V$

Bilinear control systems

Consider a bilinear control system

$$\dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y},$$
$$y_{\mathrm{obs}}(t) = \mathcal{C}y(t),$$

with

- Hilbert spaces $V \subset \mathcal{Y} \subset V'$, dense and compact embedding $V \hookrightarrow \mathcal{Y}$
- $-\mathcal{A}$ associated to a V-$\mathcal{Y}$-coercive bilinear form $a\colon V \times V \to \mathbb{R}$

$$\exists \delta > 0 \text{ and } \lambda \in \mathbb{R}\colon a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_{\mathcal{Y}}^2, \quad \forall v \in V$$

- $\mathcal{N} \in \mathcal{L}(V,\mathcal{Y}) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}),V)$ and $\mathcal{N}^* \in \mathcal{L}(V,\mathcal{Y})$
- $\mathcal{B} \in \mathcal{Y}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, $(\mathcal{A},\mathcal{B})$ stabilizable, $(\mathcal{A},\mathcal{C})$ detectable
- $[\mathcal{D}(\mathcal{A}),\mathcal{Y}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}^*),\mathcal{Y}]_{\frac{1}{2}} = V$

Bilinear control systems

Consider a bilinear control system

$$\dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y},$$
$$y_{\text{obs}}(t) = \mathcal{C}y(t),$$

with

- Hilbert spaces $V \subset \mathcal{Y} \subset V'$, dense and compact embedding $V \hookrightarrow \mathcal{Y}$
- $-\mathcal{A}$ associated to a V-$\mathcal{Y}$-coercive bilinear form $a\colon V \times V \to \mathbb{R}$

$$\exists \delta > 0 \text{ and } \lambda \in \mathbb{R}\colon a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_{\mathcal{Y}}^2, \quad \forall v \in V$$

- $\mathcal{N} \in \mathcal{L}(V,\mathcal{Y}) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}), V)$ and $\mathcal{N}^* \in \mathcal{L}(V,\mathcal{Y})$
- $\mathcal{B} \in \mathcal{Y}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, $(\mathcal{A},\mathcal{B})$ stabilizable, $(\mathcal{A},\mathcal{C})$ detectable
- $[\mathcal{D}(\mathcal{A}),\mathcal{Y}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}^*),\mathcal{Y}]_{\frac{1}{2}} = V$

Bilinear control systems

Consider a bilinear control system

$$\dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y},$$
$$y_{\mathrm{obs}}(t) = \mathcal{C}y(t),$$

with

- Hilbert spaces $V \subset \mathcal{Y} \subset V'$, dense and compact embedding $V \hookrightarrow \mathcal{Y}$
- $-\mathcal{A}$ associated to a V-$\mathcal{Y}$-coercive bilinear form $a\colon V \times V \to \mathbb{R}$

$$\exists \delta > 0 \text{ and } \lambda \in \mathbb{R}\colon a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_{\mathcal{Y}}^2, \quad \forall v \in V$$

- $\mathcal{N} \in \mathcal{L}(V, \mathcal{Y}) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}), V)$ and $\mathcal{N}^* \in \mathcal{L}(V, \mathcal{Y})$
- $\mathcal{B} \in \mathcal{Y}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$, $(\mathcal{A}, \mathcal{B})$ stabilizable, $(\mathcal{A}, \mathcal{C})$ detectable
- $[\mathcal{D}(\mathcal{A}), \mathcal{Y}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}^*), \mathcal{Y}]_{\frac{1}{2}} = V$

Bilinear control systems

Consider a bilinear control system

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y(0) = y_0 \in \mathcal{Y}, \\ y_{\mathrm{obs}}(t) &= \mathcal{C}y(t), \end{aligned}$$

with

- Hilbert spaces $V \subset \mathcal{Y} \subset V'$, dense and compact embedding $V \hookrightarrow \mathcal{Y}$
- $-\mathcal{A}$ associated to a V-$\mathcal{Y}$-coercive bilinear form $a\colon V \times V \to \mathbb{R}$

$$\exists \delta > 0 \text{ and } \lambda \in \mathbb{R}\colon a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_{\mathcal{Y}}^2, \quad \forall v \in V$$

- $\mathcal{N} \in \mathcal{L}(V,\mathcal{Y}) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}), V)$ and $\mathcal{N}^* \in \mathcal{L}(V,\mathcal{Y})$
- $\mathcal{B} \in \mathcal{Y}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, $(\mathcal{A},\mathcal{B})$ stabilizable, $(\mathcal{A},\mathcal{C})$ detectable
- $[\mathcal{D}(\mathcal{A}),\mathcal{Y}]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}^*),\mathcal{Y}]_{\frac{1}{2}} = V$

An infinite horizon problem

In the following, we focus on

$$\inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0) := \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2 \,\mathrm{d}t,$$
$$\text{s.t.} \begin{cases} \dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), & \text{for } t > 0, \\ y(0) = y_0. \end{cases} \tag{P}$$

We call y a solution of the system if for each $T > 0$

$$y \in W(0,T) = \left\{ y \in L^2(0,T;V) \,|\, \dot{y} \in L^2(0,T;V') \right\}.$$

Notation: $W_\infty := W(0,\infty)$ and $W_\infty^0 = \{y \in W_\infty \,|\, y(0) = 0\}$

Remark: if $y \in W_\infty \Rightarrow \lim_{t\to\infty} \|y(t)\|_{\mathcal{Y}} = 0$

An infinite horizon problem

In the following, we focus on

$$\inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0) := \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2 \,\mathrm{d}t,$$

$$\text{s.t.} \begin{cases} \dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), & \text{for } t > 0, \\ y(0) = y_0. \end{cases} \tag{P}$$

We call y a solution of the system if for each $T > 0$

$$y \in W(0,T) = \left\{ y \in L^2(0,T;V) \,|\, \dot{y} \in L^2(0,T;V') \right\}.$$

Notation: $W_\infty := W(0,\infty)$ and $W_\infty^0 = \{y \in W_\infty \,|\, y(0) = 0\}$

Remark: if $y \in W_\infty \Rightarrow \lim_{t\to\infty} \|y(t)\|_{\mathcal{Y}} = 0$

An infinite horizon problem

In the following, we focus on

$$\inf_{u\in L^2(0,\infty)} \mathcal{J}(u,y_0) := \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|^2\,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2\,\mathrm{d}t,$$

$$\text{s.t.} \begin{cases} \dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), & \text{for } t > 0, \\ y(0) = y_0. \end{cases} \tag{P}$$

We call y a solution of the system if for each $T > 0$

$$y \in W(0,T) = \left\{ y \in L^2(0,T;V) \,|\, \dot{y} \in L^2(0,T;V') \right\}.$$

Notation: $W_\infty := W(0,\infty)$ and $W_\infty^0 = \{y \in W_\infty \,|\, y(0) = 0\}$

Remark: if $y \in W_\infty \Rightarrow \lim_{t\to\infty} \|y(t)\|_{\mathcal{Y}} = 0$

An infinite horizon problem

In the following, we focus on

$$\inf_{u\in L^2(0,\infty)} \mathcal{J}(u,y_0) := \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2 \,\mathrm{d}t,$$

$$\text{s.t.} \begin{cases} \dot{y}(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), & \text{for } t > 0, \\ y(0) = y_0. \end{cases} \tag{P}$$

We call y a solution of the system if for each $T > 0$

$$y \in W(0,T) = \left\{ y \in L^2(0,T;V) \,|\, \dot{y} \in L^2(0,T;V') \right\}.$$

Notation: $W_\infty := W(0,\infty)$ and $W_\infty^0 = \{y \in W_\infty \,|\, y(0) = 0\}$

Remark: if $y \in W_\infty \Rightarrow \lim_{t\to\infty} \|y(t)\|_{\mathcal{Y}} = 0$

A local detectability result

Local detectability

There exists $\delta, M > 0$ such that if

- $\|u\|_{L^2(0,\infty)} \leq \delta$
- the solution y to

$$\dot{y} = \mathcal{A}y + \mathcal{N}yu + \mathcal{B}u + f, \quad y(0) = y_0,$$

satisfies $\mathcal{C}y \in L^2(0,\infty;\mathcal{Z})$

then

$$\|y\|_{W_\infty} \leq M \left(\|y_0\|_{\mathcal{Y}} + \|u\|_{L^2(0,\infty)} + \|f\|_{L^2(0,\infty;V')} + \|\mathcal{C}y\|_{L^2(0,\infty;\mathcal{Z})} \right).$$

Consequence: if $\|u\|_{L^2(0,\infty)} \leq \delta$ and $\mathcal{J}(u, y_0) < \infty$ then $y \in W_\infty$

A local detectability result

Local detectability

There exists $\delta, M > 0$ such that if

- $\|u\|_{L^2(0,\infty)} \leq \delta$
- the solution y to

$$\dot{y} = \mathcal{A}y + \mathcal{N}yu + \mathcal{B}u + f, \quad y(0) = y_0,$$

  satisfies $\mathcal{C}y \in L^2(0,\infty;\mathcal{Z})$

then

$$\|y\|_{W_\infty} \leq M\left(\|y_0\|_{\mathcal{Y}} + \|u\|_{L^2(0,\infty)} + \|f\|_{L^2(0,\infty;V')} + \|\mathcal{C}y\|_{L^2(0,\infty;\mathcal{Z})}\right).$$

Consequence: if $\|u\|_{L^2(0,\infty)} \leq \delta$ and $\mathcal{J}(u, y_0) < \infty$ then $y \in W_\infty$

Feasible and optimal controls

Existence of a feasible control

There exists $\delta > 0$ such that if $\|y_0\|_{\mathcal{Y}} < \delta$, then (P) is feasible.

Existence of optimal controls

Let $y_0 \in \mathcal{Y}$ and assume that (P) is feasible. Then, there exists an optimal control $\bar{u}$ with $\bar{y} \in W_\infty$ and a costate $p \in W_\infty$ such that for a.e. $t \geq 0$,

$$-\dot{p} = \mathcal{A}^* p + \bar{u}\mathcal{N}^* p + \mathcal{C}^*\mathcal{C}\bar{y}$$
$$\alpha\bar{u} + \langle \mathcal{N}\bar{y} + \mathcal{B}, p\rangle_{\mathcal{Y}} = 0.$$

Proof idea: sensitivity analysis based on the mapping

$$\Phi\colon (y,u,p) \mapsto \begin{pmatrix} y(0) \\ \dot{y} - (\mathcal{A}y + \mathcal{N}yu + \mathcal{B}u) \\ -\dot{p} - \mathcal{A}^*p - \mathcal{N}^*pu - \mathcal{C}^*\mathcal{C}y \\ \alpha u + \langle \mathcal{N}y + B, p\rangle_{\mathcal{Y}} \end{pmatrix}$$

Feasible and optimal controls

Existence of a feasible control

There exists $\delta > 0$ such that if $\|y_0\|_{\mathcal{Y}} < \delta$, then (P) is feasible.

Existence of optimal controls

Let $y_0 \in \mathcal{Y}$ and assume that (P) is feasible. Then, there exists an optimal control $\bar{u}$ with $\bar{y} \in W_\infty$ and a costate $p \in W_\infty$ such that for a.e. $t \geq 0$,

$$-\dot{p} = \mathcal{A}^* p + \bar{u}\mathcal{N}^* p + \mathcal{C}^*\mathcal{C}\bar{y}$$
$$\alpha\bar{u} + \langle \mathcal{N}\bar{y} + \mathcal{B}, p\rangle_{\mathcal{Y}} = 0.$$

Proof idea: sensitivity analysis based on the mapping

$$\Phi\colon (y, u, p) \mapsto \begin{pmatrix} y(0) \\ \dot{y} - (\mathcal{A}y + \mathcal{N}yu + \mathcal{B}u) \\ -\dot{p} - \mathcal{A}^*p - \mathcal{N}^*pu - \mathcal{C}^*\mathcal{C}y \\ \alpha u + \langle \mathcal{N}y + B, p\rangle_{\mathcal{Y}} \end{pmatrix}$$

Feasible and optimal controls

Existence of a feasible control

There exists $\delta > 0$ such that if $\|y_0\|_{\mathcal{Y}} < \delta$, then (P) is feasible.

Existence of optimal controls

Let $y_0 \in \mathcal{Y}$ and assume that (P) is feasible. Then, there exists an optimal control $\bar{u}$ with $\bar{y} \in W_\infty$ and a costate $p \in W_\infty$ such that for a.e. $t \geq 0$,

$$-\dot{p} = \mathcal{A}^* p + \bar{u}\mathcal{N}^* p + \mathcal{C}^*\mathcal{C}\bar{y}$$
$$\alpha\bar{u} + \langle \mathcal{N}\bar{y} + \mathcal{B}, p\rangle_{\mathcal{Y}} = 0.$$

Proof idea: sensitivity analysis based on the mapping

$$\Phi\colon (y,u,p) \mapsto \begin{pmatrix} y(0) \\ \dot{y} - (\mathcal{A}y + \mathcal{N}yu + \mathcal{B}u) \\ -\dot{p} - \mathcal{A}^* p - \mathcal{N}^* pu - \mathcal{C}^*\mathcal{C}y \\ \alpha u + \langle \mathcal{N}y + B, p\rangle_{\mathcal{Y}} \end{pmatrix}$$

The value function

For $y_0 \in \mathcal{Y}$ with $\mathcal{V}$ we denote the value function associated with (P)

$$\mathcal{V}(y_0) = \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0).$$

Regularity of $\mathcal{V}$

There exists $\delta > 0$ such that $\mathcal{V}$ is infinitely differentiable on $B_{\mathcal{Y}}(\delta)$.

An optimal feedback law

There exists $\delta > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$ the optimal control $\bar{u}$ is given by

$$\bar{u}(t) = -\frac{1}{\alpha} D\mathcal{V}(\bar{y}(t))(\mathcal{N}\bar{y}(t) + \mathcal{B}).$$

The Hamilton-Jacobi-Bellman equation

There exists $\delta > 0$ s.t. for all $y \in B_V(\delta) \cap \mathcal{D}(\mathcal{A})$ the HJB equation holds:

$$D\mathcal{V}(y)\mathcal{A}y + \frac{1}{2}\|\mathcal{C}y\|_{\mathcal{Z}}^2 - \frac{1}{2\alpha}(D\mathcal{V}(y)(\mathcal{N}y + \mathcal{B}))^2 = 0.$$

The value function

For $y_0 \in \mathcal{Y}$ with $\mathcal{V}$ we denote the value function associated with (P)

$$\mathcal{V}(y_0) = \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0).$$

Regularity of $\mathcal{V}$

There exists $\delta > 0$ such that $\mathcal{V}$ is infinitely differentiable on $B_{\mathcal{Y}}(\delta)$.

An optimal feedback law

There exists $\delta > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$ the optimal control $\bar{u}$ is given by

$$\bar{u}(t) = -\frac{1}{\alpha} D\mathcal{V}(\bar{y}(t))(\mathcal{N}\bar{y}(t) + \mathcal{B}).$$

The Hamilton-Jacobi-Bellman equation

There exists $\delta > 0$ s.t. for all $y \in B_V(\delta) \cap \mathcal{D}(\mathcal{A})$ the HJB equation holds:

$$D\mathcal{V}(y)\mathcal{A}y + \frac{1}{2}\|\mathcal{C}y\|_{\mathcal{Z}}^2 - \frac{1}{2\alpha}(D\mathcal{V}(y)(\mathcal{N}y + \mathcal{B}))^2 = 0.$$

The value function

For $y_0 \in \mathcal{Y}$ with $\mathcal{V}$ we denote the value function associated with (P)

$$\mathcal{V}(y_0) = \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0).$$

Regularity of $\mathcal{V}$

There exists $\delta > 0$ such that $\mathcal{V}$ is infinitely differentiable on $B_{\mathcal{Y}}(\delta)$.

An optimal feedback law

There exists $\delta > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$ the optimal control $\bar{u}$ is given by

$$\bar{u}(t) = -\frac{1}{\alpha} D\mathcal{V}(\bar{y}(t))(\mathcal{N}\bar{y}(t) + \mathcal{B}).$$

The Hamilton-Jacobi-Bellman equation

There exists $\delta > 0$ s.t. for all $y \in B_V(\delta) \cap \mathcal{D}(\mathcal{A})$ the HJB equation holds:

$$D\mathcal{V}(y)\mathcal{A}y + \frac{1}{2}\|\mathcal{C}y\|_{\mathcal{Z}}^2 - \frac{1}{2\alpha}(D\mathcal{V}(y)(\mathcal{N}y + \mathcal{B}))^2 = 0.$$

The value function

For $y_0 \in \mathcal{Y}$ with $\mathcal{V}$ we denote the value function associated with (P)

$$\mathcal{V}(y_0) = \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0).$$

Regularity of $\mathcal{V}$

There exists $\delta > 0$ such that $\mathcal{V}$ is infinitely differentiable on $B_{\mathcal{Y}}(\delta)$.

An optimal feedback law

There exists $\delta > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$ the optimal control $\bar{u}$ is given by

$$\bar{u}(t) = -\frac{1}{\alpha} D\mathcal{V}(\bar{y}(t))(\mathcal{N}\bar{y}(t) + \mathcal{B}).$$

The Hamilton-Jacobi-Bellman equation

There exists $\delta > 0$ s.t. for all $y \in B_V(\delta) \cap \mathcal{D}(\mathcal{A})$ the HJB equation holds:

$$D\mathcal{V}(y)\mathcal{A}y + \frac{1}{2}\|\mathcal{C}y\|_{\mathcal{Z}}^2 - \frac{1}{2\alpha}(D\mathcal{V}(y)(\mathcal{N}y + \mathcal{B}))^2 = 0.$$

Taylor expansions of $\mathcal{V}$

Idea: Expand $\mathcal{V}$ around 0 as follows

$$\mathcal{V}(y) = \underbrace{\mathcal{V}(0)}_{\in\mathbb{R}} + \underbrace{D\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y},\mathbb{R})}(y) + \frac{1}{2!}\underbrace{D^2\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y}^2,\mathbb{R})}(y,y) + \frac{1}{3!}\underbrace{D^3\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y}^3,\mathbb{R})}(y,y,y) + \dots$$

and approximate optimal feedback law via

$$u_k = -\frac{1}{\alpha}\sum_{j=2}^{k}\frac{1}{(j-1)!}D^j\mathcal{V}(0)(\mathcal{N}y+\mathcal{B},y,\dots,y).$$

Finite-dimensional case:

[Aguilar, Al'brekht, Cebuhar, Costanza, Garrard, Krener, Lukes, ...]

Infinite-dimensional case: [Thevenet/Buchot/Raymond]

Taylor expansions of $\mathcal{V}$

Idea: Expand $\mathcal{V}$ around 0 as follows

$$\mathcal{V}(y) = \underbrace{\mathcal{V}(0)}_{\in\mathbb{R}} + \underbrace{D\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y},\mathbb{R})}(y) + \frac{1}{2!}\underbrace{D^2\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y}^2,\mathbb{R})}(y,y) + \frac{1}{3!}\underbrace{D^3\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y}^3,\mathbb{R})}(y,y,y) + \dots$$

and approximate optimal feedback law via

$$u_k = -\frac{1}{\alpha}\sum_{j=2}^{k}\frac{1}{(j-1)!}D^j\mathcal{V}(0)(\mathcal{N}y+\mathcal{B},y,\dots,y).$$

Finite-dimensional case:

[Aguilar, Al'brekht, Cebuhar, Costanza, Garrard, Krener, Lukes, ...]

Infinite-dimensional case: [Thevenet/Buchot/Raymond]

Taylor expansions of $\mathcal{V}$

Idea: Expand $\mathcal{V}$ around 0 as follows

$$\mathcal{V}(y) = \underbrace{\mathcal{V}(0)}_{\in\mathbb{R}} + \underbrace{D\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y},\mathbb{R})}(y) + \frac{1}{2!}\underbrace{D^2\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y}^2,\mathbb{R})}(y,y) + \frac{1}{3!}\underbrace{D^3\mathcal{V}(0)}_{\in\mathcal{L}(\mathcal{Y}^3,\mathbb{R})}(y,y,y) + \dots$$

and approximate optimal feedback law via

$$u_k = -\frac{1}{\alpha}\sum_{j=2}^{k}\frac{1}{(j-1)!}D^j\mathcal{V}(0)(\mathcal{N}y + \mathcal{B}, y, \dots, y).$$

Finite-dimensional case:

[Aguilar,Al'brekht,Cebuhar,Costanza,Garrard,Krener,Lukes,...]

Infinite-dimensional case: [Thevenet/Buchot/Raymond]

The Riccati equation and $D^2\mathcal{V}(0)$

First nontrivial term in expansion is determined via

$$\langle \mathcal{A}^*\mathcal{P}y_1, y_2\rangle + \langle \mathcal{P}\mathcal{A}y_1, y_2\rangle - \langle \mathcal{B}, \mathcal{P}y_1\rangle \langle \mathcal{B}, \mathcal{P}y_2\rangle + \langle \mathcal{C}y_1, \mathcal{C}y_2\rangle = 0,$$

for $y_1, y_2 \in \mathcal{D}(\mathcal{A})$. Since $\mathcal{P} \in \mathcal{L}(\mathcal{Y})$, we can define a bilinear form $\mathcal{T}_2$ by

$$\mathcal{T}_2 \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad \mathcal{T}_2(y_1, y_2) = \langle y_1, \mathcal{P}y_2\rangle_{\mathcal{Y}}.$$

Remark: $\mathcal{T}_2(\cdot,\cdot)$ corresponds to $D^2\mathcal{V}(0)(\cdot,\cdot)$

Question: Precise structure of $D^j\mathcal{V}(0)$ for $j \geq 3$?

By $\mathcal{A}_{\mathcal{P}} := \mathcal{A} - \frac{1}{\alpha}\mathcal{B}\mathcal{B}^*\mathcal{P}$ denote linearized closed-loop system.

The Riccati equation and $D^2\mathcal{V}(0)$

First nontrivial term in expansion is determined via

$$\langle \mathcal{A}^*\mathcal{P}y_1, y_2\rangle + \langle \mathcal{P}\mathcal{A}y_1, y_2\rangle - \langle \mathcal{B}, \mathcal{P}y_1\rangle \langle \mathcal{B}, \mathcal{P}y_2\rangle + \langle \mathcal{C}y_1, \mathcal{C}y_2\rangle = 0,$$

for $y_1, y_2 \in \mathcal{D}(\mathcal{A})$. Since $\mathcal{P} \in \mathcal{L}(\mathcal{Y})$, we can define a bilinear form $\mathcal{T}_2$ by

$$\mathcal{T}_2 \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad \mathcal{T}_2(y_1, y_2) = \langle y_1, \mathcal{P}y_2\rangle_{\mathcal{Y}}.$$

Remark: $\mathcal{T}_2(\cdot,\cdot)$ corresponds to $D^2\mathcal{V}(0)(\cdot,\cdot)$

Question: Precise structure of $D^j\mathcal{V}(0)$ for $j \geq 3$?

By $\mathcal{A}_{\mathcal{P}} := \mathcal{A} - \frac{1}{\alpha}\mathcal{B}\mathcal{B}^*\mathcal{P}$ denote linearized closed-loop system.

The general structure

For $i, j \in \mathbb{N}$, consider the following set of permutations:

$$S_{i,j} = \{\sigma \in S_{i+j} \,|\, \sigma(1) < \ldots < \sigma(i) \text{ and } \sigma(i+1) < \ldots < \sigma(i+j)\},$$

where S_{i+j} is the set of permutations of $\{1, \ldots, i+j\}$.

Example

$$\begin{aligned} S_{2,2} &= \{\sigma \in S_4 \,|\, \sigma(1) < \sigma(2) \text{ and } \sigma(3) < \sigma(4)\} \\ &= \{(1,2,3,4), (1,3,2,4), (1,4,2,3), \\ &\qquad (2,3,1,4), (2,4,1,3), (3,4,1,2)\} \end{aligned}$$

For given multilinear form $\mathcal{T}$ (of order $i+j$), we define

$$\mathrm{Sym}_{i,j}(\mathcal{T})(z_1, \ldots, z_{i+j}) := \binom{i+j}{i}^{-1} \Big[\sum_{\sigma \in S_{i,j}} \mathcal{T}(z_{\sigma(1)}, \ldots, z_{\sigma(i+j)}) \Big].$$

The general structure

For $j \geq 3$ and $y_1, \ldots, y_j \in \mathcal{D}(\mathcal{A})$ consider

$$\sum_{i=1}^{j} \mathcal{T}_j(y_1, ..., y_{i-1}, \mathcal{A}_{\mathcal{P}} y_i, y_{i+1}, ..., y_j) = \frac{1}{2\alpha} \mathcal{R}_j(y_1, \ldots, y_j) \quad (*),$$

The general structure

For $j \geq 3$ and $y_1, \dots, y_j \in \mathcal{D}(\mathcal{A})$ consider

$$\sum_{i=1}^{j} \mathcal{T}_j(y_1, ..., y_{i-1}, \mathcal{A}_\mathcal{P} y_i, y_{i+1}, ..., y_j) = \frac{1}{2\alpha} \mathcal{R}_j(y_1, \dots, y_j) \quad (*),$$

where $\mathcal{R}_j$ is given by:

$$\mathcal{R}_j = 2j(j-1)\mathrm{Sym}_{1,j-1}\left(\mathcal{F}_1 \otimes \mathcal{G}_{j-1}\right) + \sum_{i=2}^{j-2} \binom{j}{i} \mathrm{Sym}_{i,j-i}\Big((\mathcal{F}_i + i\mathcal{G}_i) \otimes (\mathcal{F}_{j-i} + (j-i)\mathcal{G}_{j-i}) \Big),$$

and:

$$\mathcal{F}_i(y_1, ..., y_i) = \mathcal{T}_{i+1}(\mathcal{B}, y_1, ..., y_i)$$

$$\mathcal{G}_i(y_1, ..., y_i) = \frac{1}{i} \Big[\sum_{\ell=1}^{i} \mathcal{T}_i(y_1, ..., y_{\ell-1}, \mathcal{N} y_\ell, y_{\ell+1}, ..., y_i) \Big].$$

A multilinear operator equation

Well-posedness of $\mathcal{T}_j \equiv D^j\mathcal{V}(0)$

For $j \geq 3$, and $y_1, \ldots, y_j \in \mathcal{Y}$ define the multilinear form

$$\mathcal{T}_j : \mathcal{Y} \times \cdots \times \mathcal{Y} \to \mathbb{R},$$

$$\mathcal{T}_j(y_1, \ldots, y_j) = -\frac{1}{2\alpha} \int_0^\infty \mathcal{R}_j(e^{\mathcal{A}_\mathcal{P} t} y_1, \ldots, e^{\mathcal{A}_\mathcal{P} t} y_j) \,\mathrm{d}t.$$

Then $\mathcal{T}_j$ is the unique solution of $(*)$. Moreover, it holds that

$$|\mathcal{T}_j(y_1, \ldots, y_j)| \leq C \prod_{i=1}^{j} \|y_i\|_\mathcal{Y}.$$

The closed-loop system

Consider now the polynomial feedback law

$$u_k(y) = -\frac{1}{\alpha}\sum_{j=2}^{k}\frac{1}{(j-1)!}\mathcal{T}_j(\mathcal{N}y+\mathcal{B}, y, \ldots, y)$$

and the corresponding (nonlinear) closed-loop system

$$\dot{y} = \mathcal{A}y + (\mathcal{N}y+\mathcal{B})u_k(y), \quad y(0) = y_0. \tag{CL}$$

Local well-posedness

There exist constants $\delta_1, \delta_2 > 0$ such that: if $\|y_0\|_{\mathcal{Y}} \le \delta_1$, then

- (CL) admits a unique solution $y \in W_\infty$ satisfying $\|y\|_{W_\infty} \le \delta_2$.

The closed-loop system

Consider now the polynomial feedback law

$$u_k(y) = -\frac{1}{\alpha}\sum_{j=2}^{k}\frac{1}{(j-1)!}\mathcal{T}_j(\mathcal{N}y+\mathcal{B}, y, \ldots, y)$$

and the corresponding (nonlinear) closed-loop system

$$\dot{y} = \mathcal{A}y + (\mathcal{N}y+\mathcal{B})u_k(y), \quad y(0) = y_0. \tag{CL}$$

Local well-posedness

There exist constants $\delta_1, \delta_2 > 0$ such that: if $\|y_0\|_{\mathcal{Y}} \le \delta_1$, then

- (CL) admits a unique solution $y \in W_\infty$ satisfying $\|y\|_{W_\infty} \le \delta_2$.

A perturbed cost function

It can be shown that u_k is optimal for a perturbed cost function

$$\mathcal{J}_k(u, y_0) := \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|^2\,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u^2(t)\,\mathrm{d}t + R_k(y)$$

with $R_k(y) = \displaystyle\int_0^\infty r_k(y(t))\,\mathrm{d}t$ and r_k is a polynomial remainder term.

Error analysis

There exist $\delta > 0$ and $M > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$,

$$\max\left(\|y_k - \bar{y}\|_{W_\infty}, \|u_k - \bar{u}\|_{L^2(0,\infty)}, \|p_k - \bar{p}\|_{L^2(0,\infty;V)}\right) \leq M\|y_0\|_{\mathcal{Y}}^k,$$

where the costate p_k associated to y_k satisfies

$$-\dot{p}_k = \mathcal{A}^* p_k + \mathcal{N}^* p_k u_k + \mathcal{C}^*\mathcal{C}y_k + DR_k(y_k) \qquad \text{in } (W_\infty^0)^*,$$

Moreover: $\mathcal{J}(y_0, u_k) \leq \mathcal{V}(y_0) + M\|y_0\|_{\mathcal{Y}}^{2k}$.

Idea: $\|(\bar{y}, \bar{u}, \bar{p}) - (y_k, u_k, p_k)\| = \|\Phi^{-1}(y_0, 0, 0, 0) - \Phi^{-1}(y_0, 0, DR_k(y_k), 0)\|$

A perturbed cost function

It can be shown that u_k is optimal for a perturbed cost function

$$\mathcal{J}_k(u, y_0) := \frac{1}{2} \int_0^\infty \|\mathcal{C}y(t)\|^2 \, \mathrm{d}t + \frac{\alpha}{2} \int_0^\infty u^2(t) \, \mathrm{d}t + R_k(y)$$

with $R_k(y) = \int_0^\infty r_k(y(t)) \, \mathrm{d}t$ and r_k is a polynomial remainder term.

Error analysis

There exist $\delta > 0$ and $M > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$,

$$\max \left(\|y_k - \bar{y}\|_{W_\infty}, \|u_k - \bar{u}\|_{L^2(0,\infty)}, \|p_k - \bar{p}\|_{L^2(0,\infty;V)} \right) \leq M \|y_0\|_{\mathcal{Y}}^k,$$

where the costate p_k associated to y_k satisfies

$$-\dot{p}_k = \mathcal{A}^* p_k + \mathcal{N}^* p_k u_k + \mathcal{C}^* \mathcal{C} y_k + DR_k(y_k) \qquad \text{in } (W_\infty^0)^*,$$

Moreover: $\mathcal{J}(y_0, u_k) \leq \mathcal{V}(y_0) + M \|y_0\|_{\mathcal{Y}}^{2k}$.

Idea: $\|(\bar{y}, \bar{u}, \bar{p}) - (y_k, u_k, p_k)\| = \|\Phi^{-1}(y_0, 0, 0, 0) - \Phi^{-1}(y_0, 0, DR_k(y_k), 0)\|$

A perturbed cost function

It can be shown that u_k is optimal for a perturbed cost function

$$\mathcal{J}_k(u, y_0) := \frac{1}{2}\int_0^\infty \|\mathcal{C}y(t)\|^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u^2(t)\,\mathrm{d}t + R_k(y)$$

with $R_k(y) = \int_0^\infty r_k(y(t))\,\mathrm{d}t$ and r_k is a polynomial remainder term.

Error analysis

There exist $\delta > 0$ and $M > 0$ s.t. for all $y_0 \in B_{\mathcal{Y}}(\delta)$,

$$\max\left(\|y_k - \bar{y}\|_{W_\infty}, \|u_k - \bar{u}\|_{L^2(0,\infty)}, \|p_k - \bar{p}\|_{L^2(0,\infty;V)}\right) \le M\|y_0\|_{\mathcal{Y}}^k,$$

where the costate p_k associated to y_k satisfies

$$-\dot{p}_k = \mathcal{A}^* p_k + \mathcal{N}^* p_k u_k + \mathcal{C}^*\mathcal{C} y_k + DR_k(y_k) \qquad \text{in } (W_\infty^0)^*,$$

Moreover: $\mathcal{J}(y_0, u_k) \le \mathcal{V}(y_0) + M\|y_0\|_{\mathcal{Y}}^{2k}$.

Idea: $\|(\bar{y}, \bar{u}, \bar{p}) - (y_k, u_k, p_k)\| = \|\Phi^{-1}(y_0, 0, 0, 0) - \Phi^{-1}(y_0, 0, DR_k(y_k), 0)\|$

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, Z)$ is possible

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Thank you for your attention!

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Thank you for your attention!

Remarks and open questions

- abstract setting is applicable to a (natural) control problem for the Fokker-Planck equation
- (our) numerics rely on model reduction and tensor calculus
- generalizing results to semilinear equations is work in progress

It is not clear (to me) if

- $\mathcal{B} \in V'$ is possible
- $\mathcal{C} \in \mathcal{L}(V, \mathcal{Z})$ is possible

Thank you for your attention!