Milestones in mathematical system theory

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Systems and Control before 1960

• In the 1950s and 1960s, control was considered an electrical engineering subject. Dynamical systems were input/output and frequency-domain based. Transfer functions were believed to be the way to characterize a system.

• For electrical engineers systems were filters passing some frequencies more easily that others.

• Mathematical language of control at the time: transfer functions, applied almost uniquely to continuous-time single-input/single-output systems. A differential equation

$$\mathbf{p}(d/dt)\mathbf{y} = \mathbf{q}(d/dt)\mathbf{u}, \quad \text{with} \;\; \mathbf{p},\; \mathbf{q} \;\; \text{real polynomials,}$$

was immediately transformed to a transfer function.

• Practical and useful procedures were developed, for instance:

lead/lag compensation
gain and phase margins
Bode plots
Nyquist diagrams,
Nichols charts
root-locus graphs

• Despite the restrictive nature of transfer functions, there were two imprtant success stories: Passive network synthesis and Wiener filtering.

Passive network synthesis

In his 1931 dissertation, O. Brune proved the remarkable result that a transfer function can be realized as the impedance of a circuit containing an interconnection of (positive) resistors, inductors, capacitors, and transformers if and only if it is rational and positive real.

This result was later strengthened in a half-a-page paper by Bott and Duffin who showed that in the scalar case transformers are not needed. However, the realizations were far from minimal.

Later, the multivariable case was also covered, but transformerless synthesis remains an open problem in the multivariable case even today. Today because of solid state technology passivity is no longer an issue.

• O. Brune, ”Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency,” Journal of mathematical physics, vol. 10, pp. 191 - 236, 1931.

• R. Bott and R.J. Duffin, ”Impedance synthesis without use of transformers”, Journal of Applied Physics, p. 816 (1949)

• More recently: R.E. Kalman, Oberwolfach (2008) and M. Smith have been concerned with passive synthesis.

Wiener filtering

• The problem is to estimate a signal from observations corrupted by noise. Wiener formulated this problem in the setting of stationary stochastic processes.

• It is easy to derive this filter if for the present estimate we are permitted to use the observations for all time.

• But the construction of the filter which uses only the past observations (causal filter) was much more difficult. This problem was solved by Wiener in 1942, but the report was made public after the war (1949).

• The discrete-time equivalent of Wiener's work was derived independently by Andrey Kolmogorov and published in 1941. Hence the theory is often called the Wiener-Kolmogorov filtering theory.

• N. Wiener, "The interpolation, extrapolation and smoothing of stationary time series", Report of the Services 19, Research Project DIC-6037 MIT, February 1942.

• A.N. Kolmogorov, "Stationary sequences in Hilbert space", (In Russian) Bull. Moscow Univ. 1941 vol.2 no.6 1-40. English translation in Kailath T. (ed.) Linear least squares estimation Dowden, Hutchinson & Ross 1977

Around 1960, the basic model for studying dynamics shifted from

$$\mathbf{p}(d/dt)\mathbf{y} = \mathbf{q}(d/dt)\mathbf{u} \qquad \text{to} \qquad \boxed{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, t)}$$

Consequently one could cover:

- Initial conditions
- Multivariable systems
- Nonlinearities
- Time-variation
- Finite state machines and Automata
- Systems described by PDE's (to some extent).
- Input/state/output systems had more modeling power and were far richer mathematically.

The main idea is to explicitly display the system's memory in terms of the state.

The move to state space models constituted a true paradigm shift.

The credit for this paradigm shift must go to scientists from the Soviet Union.

When Pontryagin and the researchers around him started thinking about control, they chose $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, as the model for articulating optimality.

The result was the **maximum principle**. (The principle states, that the control Hamiltonian must take an extreme value over permissible controls.)

The number of examples that gave nontrivial results, as time-optimal control of the harmonic oscillator, was extensive. The results were surprising and mathematically deep.

Optimal control was picked up immediately in the US.

• L.S. Pontryagin, V.G. Boltyanski, R.V. Gamkrelidze, E.F. Mishchenko, ”The mathematical theory of optimal processes”, Wiley-Interscience, (translated from the Russian) 1962.

Later Boltyanski complained about Pontryagin’s behavior in this regard. See:

• V.G. Boltyanski, The maximum principle - How it came to be? Report 526, Schwerpunktprogramm der DFG, Anwendungsbezogene Optimierung und Steurung, 1994.

R.E. Kalman applied the state space model to the same filtering problem discussed by Wiener, with surprising results.

- The solution was recursive.
- The fact that the estimates could use only the past of the observations posed no difficulties.
- The filter gains were derived from a solution of a differential equation, which was later called the Riccati equation.
- The infinite-time theory involved the algebraic Riccati equation (ARE), which is a quadratic matrix equation.
- The ARE has multiple solutions, and obtaining the correct one involved the newly introduced notions of controllability and observability.

Controllability and Observability

In the late 1950s, Kalman introduced the concepts of

- Controllability/reachability, and
- Observability/reconstructibiliy.

Given a siso linear system $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, it was known at that time that the non-singularity of

$$\det \underbrace{\begin{bmatrix} \mathbf{b}, & \mathbf{Ab}, & \cdots, & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix}}_{\mathcal{R}(\mathbf{A},\mathbf{b})} \qquad \text{and} \qquad \det \underbrace{\begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{cA}^{n-1} \end{bmatrix}}_{\mathcal{O}(\mathbf{c},\mathbf{A})}$$

played a role in various problems. What Kalman did, was make **concepts** out of the non-singularity of these quantities (and the full rank of similar matrices for mimo systems).

Therefore it wouldn't be unfair to call them

the Kalman controllability and observability concepts.

These concepts have appropriate generalizations for many other kinds of systems (e.g. non-linear, infinite dimensional, etc.)

- R.E. Kalman, Lectures on controllability and observability, September 1968.

The LQG problem

The solution of the linear-quadratic-gaussian (LQG) problem was considered the main result in control of the 1960s. The algorithms were based on Riccati equations, the solution showed separation between estimation and control; the Kalman filter was a subsystem of the controller.

Consider the continuous-time linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{v}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{w},$$

where $\mathbf{x}$ is the state, $\mathbf{u}$ the control inputs and $\mathbf{y}$ the measured outputs.
The system is affected by additive white Gaussian noise $\mathbf{v}$ and $\mathbf{w}$.

The objective is to find a control input $\mathbf{u}$, such that the following cost function is minimized:

$$J = \mathcal{E}\left[\mathbf{x}^{\mathrm{T}}(T)\mathbf{F}\mathbf{x}(T) + \int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\, dt\right], \text{ where } \mathbf{F},\ \mathbf{Q} \geq 0,\ \ \mathbf{R} > 0,$$

and $\mathbf{E}$ denotes the expected value. The LQG controller is characterized by:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}\left(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}\right), \quad \hat{\mathbf{x}}(0) = \mathcal{E}\left[\mathbf{x}(0)\right], \quad \mathbf{u} = -\mathbf{L}\hat{\mathbf{x}}.$$

At each time t this filter generates estimates $\hat{\mathbf{x}}$ of the state $\mathbf{x}$ using the past measurements and inputs. The **Kalman gain** $\mathbf{K}$ is computed by solving the matrix Riccati differential equation:

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathrm{T}} - \mathbf{P}\mathbf{C}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{C}\mathbf{P} + \mathbf{V}, \;\; \mathbf{P}(0) = \mathcal{E}\left[\mathbf{x}(0)\mathbf{x}^{\mathrm{T}}(0)\right] \;\Rightarrow\; \mathbf{K} = \mathbf{P}\mathbf{C}^{\mathrm{T}}\mathbf{W}^{-1}.$$

The matrix $\mathbf{L}$ is called the feedback gain matrix and is determined through the following matrix Riccati differential equation:

$$-\dot{\mathbf{S}} = \mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S} + \mathbf{Q} \quad \Rightarrow \quad \mathbf{L} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}.$$

Observe the duality between the two matrix Riccati differential equations, the first one running forward in time, the second one running backward in time.

The first matrix Riccati differential equation solves the linear-quadratic estimation problem (LQE), while the second one solves the linear-quadratic regulator problem (LQR). Thus the LQG problem separates into the LQE and LQR problems which can be solved independently. Therefore, the LQG problem is called **separable**.

When $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q}, \mathbf{R}$ and the noise intensity matrices $\mathbf{V}$, $\mathbf{W}$ do not depend on time, and $T \to \infty$, the LQG controller becomes time-invariant. In that case both matrix Riccati differential equations are replaced by algebraic Riccati equations.

Around the same time, the pole placement problem with state and output feedback was attacked, with again controllability and observability as important properties of the underlying system.

Given a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, find a state feedback $\mathbf{u} = -\mathbf{K}\mathbf{x}$ such that the closed loop system

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

has eigenvalues in preassigned locations.

The pole placement problem has a solution $\Leftrightarrow$ the pair $(\mathbf{A}, \mathbf{B})$ is controllable (a result shown by Wohnham in 1969).

• **Remark**. At the time, there was skepticism concerning the paradigm shift and the ensuing mathematization.

Reason: the absence of concrete industrial applications?

Dissipative Systems

According to Jan Willems, the idea of dissipativity is simple. Consider a state space system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}),$$

a real-valued function of the input and output, $\mathbf{s}(\mathbf{u}, \mathbf{y})$, called the supply rate, and a state function $\mathbf{V}(\mathbf{x})$ called the storage function. If along solutions there holds

$$\frac{d}{dt}\mathbf{V}(\mathbf{x}(\cdot)) \leq \mathbf{s}(\mathbf{u}(\cdot), \mathbf{y}(\cdot)),$$

the system is called **dissipative**. The construction of storage functions for linear systems with quadratic supply rates led to the **algebraic Riccati inequality** and to the acronym **LMI**.

Later, these ideas became important in robust control and in algorithmic methods. Thus dissipativity is:

a generalization of the concept of Lyapunov functions to open systems,

and has become a **concept** of its own.

• J.C. Willems, Dissipative dynamical systems, Part I: General theory, Arch. Rational Mech. Anal., 45: 321-351 (1972).

• J.C. Willems, Dissipative dynamical systems, Part II: Linear systems with quadratic supply rates, Arch. Rational Mech. Anal., 45: 352-393 (1972).

From storage to LMI's – Simple example:

(a) Consider Σ : $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with Lyapunov (storage) function $\mathbf{V}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}$, $\mathbf{K} = \mathbf{K}^T \geq \mathbf{0}$, and supply rate $\mathbf{s} = \mathbf{0}$. According to Lyapunov, the system is stable iff:

$$\frac{d}{dt}\mathbf{V}(\mathbf{x}) \leq \mathbf{s}(\mathbf{u}, \mathbf{y}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{A}^T\mathbf{K} + \mathbf{K}\mathbf{A} \leq \mathbf{0}.$$

(b) Consider now the **open** system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$. Let the supply rate be, for instance, $\mathbf{s}(\mathbf{u}, \mathbf{y}) = \|\mathbf{u}\|^2 - \|\mathbf{y}\|^2$, and the quadratic storage function $\mathbf{V}$ as above. The system is **dissipative** iff:

$$\frac{d}{dt}\mathbf{V}(\mathbf{x}) \leq \mathbf{s}(\mathbf{u}, \mathbf{y}) = \|\mathbf{u}\|^2 - \|\mathbf{y}\|^2 \quad \Rightarrow \quad \begin{bmatrix} \mathbf{A}^T\mathbf{K} + \mathbf{K}\mathbf{A} + \mathbf{C}^T\mathbf{C} & \mathbf{K}\mathbf{B} \\ \mathbf{B}^T\mathbf{K} & -\mathbf{I} \end{bmatrix} \leq \mathbf{0}.$$

$$\Rightarrow \quad \begin{cases} \mathbf{A}^T\mathbf{K} + \mathbf{K}\mathbf{A} + \mathbf{K}\mathbf{B}\mathbf{B}^T\mathbf{K} + \mathbf{C}^T\mathbf{C} \leq \mathbf{0} & \text{solvable?} \quad \text{ARIneq} \\ \mathbf{A}^T\mathbf{K} + \mathbf{K}\mathbf{A} + \mathbf{K}\mathbf{B}\mathbf{B}^T\mathbf{K} + \mathbf{C}^T\mathbf{C} = \mathbf{0} & \text{solvable?} \quad \text{ARE} \end{cases}$$

• Carsten Scherer and Siep Weiland, "Linear matrix inequalities in control", Lecture Notes, January 2015, 283 pages.

Geometric Approach

• One of the main research areas in control in the 1970s was the geometric theory, built around two central notions: controlled invariant and conditionally invariant subspaces.

• In particular at the end of the sixties (1969) Basile, Marro and Laschi published some results using geometric techniques in five papers, three in Italian and two in English, that analyzed the basic tools and presented solutions to some problems to which they could be applied: disturbance rejection, unknown-input observability and noninteraction. The first paper in english by Basile and Marro presented definitions and properties of what were called ”controlled and conditioned invariant subspaces”.

• In 1970 Wonham and Morse applied an algorithm similar to that for deriving the maximal controlled invariant subspace to the solution of decoupling and noninteracting control problems, but did not explicitly define either the controlled invariant or the conditioned invariant subspaces. The new name ”(A,B)-invariant” instead of ”(A,B)-controlled invariant”, was only used by Wonham in his book about five years afterwards, and, due to the circulation of this book, it was generally adopted in the literature. Wohnham’s book did not contain any reference to the contribution of the italian researchers, raising issues of fairness.

• The geometric theory looked promising both from the theoretical and practical viewpoints; for instance, there were a number of applications: disturbance decoupling, noninteracting control, the internal model principle, and tracking, all usually combined with stabilization. In addition generalizations to nonlinear and distributed systems appeared. Research in this area came to a complete standstill.

A typical problem treated by geometric control

Consider the plant equations given by

$$\Sigma: \quad \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{Gd}, \quad \mathbf{y} = \mathbf{Cx}, \quad \mathbf{z} = \mathbf{Hx},$$

with state $\mathbf{x} \in \mathbb{R}^n$, input $\mathbf{u} \in \mathbb{R}^m$, disturbance $\mathbf{d} \in \mathbb{R}^q$ and output $\mathbf{y} \in \mathbb{R}^p$.

The DDPM (Disturbance Decoupling with Measurement Feedback) problem is to find feedback matrices $\mathbf{F}, \mathbf{E}, \mathbf{M}, \mathbf{N},$ defining the feedback processor

$$\Sigma_f: \quad \dot{\mathbf{w}} = \mathbf{Fw} + \mathbf{Ey}, \quad \mathbf{u} = \mathbf{Mw} + \mathbf{Ny},$$

with $\mathbf{w} \in \mathbb{R}^k$, the state of the feedback processor, such that the closed loop system

$$\Sigma_{cl}: \quad \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{BNC} & \mathbf{BM} \\ \mathbf{EC} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix} \mathbf{d}, \ \ \mathbf{z} = [\mathbf{H} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix},$$

which may be written compactly as

$$\dot{\mathbf{x}}^e = \mathbf{A}^e \mathbf{x}^e + \mathbf{G}^e \mathbf{d}, \ \ \mathbf{z} = \mathbf{H}^e \mathbf{x}^e,$$

has zero transfer function $\mathbf{H}^e(s\mathbf{I} - \mathbf{A}^e)^{-1}\mathbf{G}^e = \mathbf{0}$, i.e., the controlled output $\mathbf{z}$ is influenced only by the initial conditions and not by the disturbances $\mathbf{d}$.

• A main drawback of geometric control solutions was the lack of robustness. Jan Willems tried to remedy the situation by introducing ”almost” concepts of controlled and conditioned invariant subspaces. But, the H_∞ framework which was a few years in the future, would render geometric control meaningless!

Polynomial approach to geometric control and linear systems

Given $\mathbf{Z}(s) = \mathbf{D}^{-1}(s)\mathbf{N}(s)$, where $\mathbf{D} \in \mathbb{R}^{p\times p}[s]$, $\mathbf{N} \in \mathbb{R}^{p\times m}[s]$, are coprime, a minimal realization of $\mathbf{Z}$ in the polynomial setting advocated by Kalman and implemented by Fuhrmann, is:

State space:	$\mathbf{X_D} = \{\mathbf{x} \in \mathbb{R}^p[s] : \ \mathbf{D}^{-1}\mathbf{x} \ \textit{is spr}\}$
Dynamics:	$\mathbf{A} : \ \mathbb{R}^p[s] \mapsto \mathbb{R}^p[s] : \quad \mathbf{x} \mapsto \underbrace{\mathbf{D}\pi\mathbf{D}^{-1}}_{\pi_\mathbf{D}} \cdot s \cdot \mathbf{x}$
Input map:	$\mathbf{B} : \ \mathbb{R}^m \mapsto \mathbf{X_D} : \quad \xi \mapsto \mathbf{N}\cdot\xi$
Output map:	$\mathbf{C} : \ \mathbf{X_D} \mapsto \mathbb{R}^p : \quad \mathbf{x} \mapsto (\mathbf{D}^{-1}\cdot\mathbf{x})_{-1}$

The Fuhrmann Realization

where π picks the strictly proper part of $(\cdot)$ and $(\cdot)_{-1}$ is the coefficient of s^{-1} of $(\cdot)$.

• P. A. Fuhrmann, "Algebraic system theory: An analyst's point of view", J. Franklin Inst., vol. 301, pages 521-540 (1976).

• A.C. Antoulas, "New results on the algebraic theory of linear systems: The solution of the cover problems", Linear Algebra and its Applications, vol. 50, pages 1-43 (1983).

• P.A. Fuhrmann, "A polynomial approach to linear algebra", Springer Verlag (1996).

More recently: Uwe Helmke and Knut Hüper in Würzburg showed interest in the cover problems.

Simple example illustrating the Fuhrmann realization. Consider $\mathbf{Z}(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0} = \frac{\mathbf{n}(s)}{\mathbf{d}(s)}$.

The state space is $\mathbf{X_d} = \{\mathbf{x} \in \mathbb{R}^2[s],$ such that $\frac{\mathbf{x}}{\mathbf{d}} = \mathrm{spr}\} = span_{\mathbb{R}}\{1, s\}$. Hence

$$\mathbf{A} : \begin{cases} 1 & \mapsto & \pi_{\mathbf{d}}(s) = s, \\ s & \mapsto & \pi_{\mathbf{d}}(s^2) = -\alpha_1 s - \alpha_0, \end{cases}$$

$$\mathbf{B} : \quad \mathbb{R} \to \mathbf{X_d} : \ \xi \mapsto \mathbf{n}(s)\xi = \beta_1 s \xi + \beta_0 \xi,$$

$$\mathbf{C} : \quad \mathbf{X_d} \to \mathbb{R} : \begin{cases} 1 \mapsto (1 \cdot \mathbf{d}^{-1})_{-1} = 0, \\ s \mapsto (s \cdot \mathbf{d}^{-1})_{-1} = 1. \end{cases}$$

Therefore in the basis $(1, s)$ of $\mathbf{X_d}$, the matrix representation of these operators is:

$$\Rightarrow \ \mathbf{A} = \begin{bmatrix} 0 & -\alpha_0 \\ 1 & -\alpha_1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \ \mathbf{C} = [0, \quad 1].$$

Further developments:

• M. Vidyasagar, Control System Synthesis: A Factorization Approach, MIT Press, Cambridge, MA, 1985. (stable proper rational factorizations).

Such factorizations circumvent stability requirements, and can be applied for instance, for balanced truncation or Hankel norm approximation of unstable systems.

Behavioral systems

The basic concept in the behavioral approach, is the behavior of a dynamical system. This is the set of trajectories which are compatible with the equations of the system. Equivalence of models, representations of models, properties of models, approximation of models, symmetries, all refer to the behavior. Dynamic modeling and system identification aim at coming up with a specification of the behavior. Control comes down to restricting the behavior.

Control need not be sensor-output-to-actuator-input feedback. Many useful practical control devices, as dampers for vibration attenuation, heat fins, strips, and grooves to control turbulence, insulation equipment for heat or noise, stabilizers on ships, etc., do not function through sensing and actuation.

Interconnection means variable sharing, not input-to-output assignment.

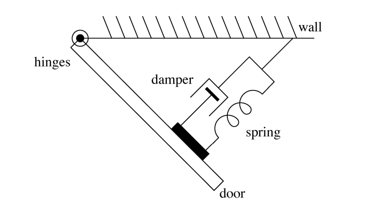
In this context, it is worth mentioning the **behavioral definition of controllability**. A system is controllable if any two trajectories in the behavior are **patchable**, that is, if for any two trajectories in the behavior, there is a third trajectory in the behavior that has the past of the first one as its past, and the future of the second one as its sometime-future. This definition has the classical state definition as a special case. But it is simpler more general and independent of the representation. The system $\mathbf{p}(d/dt)\mathbf{y} = \mathbf{q}(d/dt)\mathbf{u}$, is controllable in the sense of behaviors if and only if $\mathbf{p}$ and $\mathbf{q}$ have no common factors.

Willems: the problem with common factors between $\mathbf{p}(s)$ and $\mathbf{q}(s)$, is now finally understood: it is a **lack of controllability**.

An example of interconnection and control in the behavioral setting

Model for the door (plant):

$$\mathbf{M}' \frac{d^2\theta}{dt^2} = \mathbf{F}_c + \mathbf{F}_e,$$



where $\mathbf{F}_c$ is the force exerted by the door closing device, and $\mathbf{F}_e$ is an exogenous force.

The door closing mechanism modeled as mass-spring-damper combination (the controller):

$$\mathbf{M}'' \frac{d^2\theta}{dt^2} + \mathbf{D}\frac{d\theta}{dt} + \mathbf{K}\theta = -\mathbf{F}_c.$$

Controlled dynamical system

$$(\mathbf{M}' + \mathbf{M}'')\frac{d^2\theta}{dt^2} + \mathbf{D}\frac{d\theta}{dt} + \mathbf{K}\theta = \mathbf{F}_e.$$

Specifications on the controlled system:

small overshoot, fast settling, not-too-high gain from $\mathbf{F}_e \mapsto \theta$.
Finding a suitable controller: suitable values for $\mathbf{M}'$, $\mathbf{K}$, and $\mathbf{D}$.
Note: Plant: second order;
Controller: second order;
Controlled plant: second **(not fourth)** order.

H_∞ control

• The last 15 years of the 20th century the field of systems and control was dominated by H_∞-theory and by model reduction. H_∞ was the fourth variation of the disturbance attenuation problem to be considered, after (i) LQG (or H_2-optimal disturbance attenuation), (ii) bounding the effect of disturbances via pole placement and stabilization, and (iii) exact disturbance decoupling of the geometric theory.

• The solution of the H_∞-optimal disturbance attenuation problem is subtle, especially because of the coupling condition between two algebraic Riccati equations.

H_∞, in contrast to H_2, leads to robustness. This robustness feature was reinforced by a number of new ideas on how to deal with uncertainties using linear fractional transformations, so that the small gain paradigm became applicable to a wider class of problems.

• Finally, robust control brought algorithmic issues, as LMI's, convex programming, and complexity questions to the forefront. The combination of H_∞, robust control, and LMI's in all their facets became a successful research area.

Model Reduction

Willems: because of its relevance to modeling, model reduction is one of the most valuable contributions of mathematical system theory to applied mathematics.

• In the mid 1960s Kalman showed that the problem of constructing a minimal realizations from a discrete time convolution leads to rank determination of a Hankel matrix.

• The optimal model reduction in the 2-norm of the convolution operator is not solvable with the current state of knowledge. However the same problem in the 2-induced norm of the Hankel operator, known as the Hankel norm approximation problem, has been solved.

• Another idea – not involving optimality – is to use a balanced realization, to make the system equally controllable as observable. Combined with rank reduction using SVD and H_∞- bounds, this leads to algorithms for model simplification known as balanced truncation (BT).

• Finally, interpolatory-based reduction methods lead to a rich family of model reduction procedures including H_2 optimal model reduction and data-driven or Loewner based model reduction. These are numerically efficient and applicable to very high order systems.

The Adamjan-Arov-Krein result and optimal Hankel norm approximation

AAK Theorem. Given the sequence of $p \times m$ matrices $\mathbf{h} = (\mathbf{h}(k))_{k>0}$, such that the associated Hankel matrix $\mathcal{H}$ has finite rank n, there exists a sequence of $p \times m$ matrices $\hat{\mathbf{h}} = (\hat{\mathbf{h}}(k))_{k>0}$, such that the associated Hankel matrix $\hat{\mathcal{H}}$ has rank $r < n$ and in addition

$$\|\mathcal{H} - \hat{\mathcal{H}}\|_{\text{2-ind}} = \sigma_{r+1}(\mathcal{H}).$$

• V.M. Adamjan, D.Z. Arov, and M.G. Krein, Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem, Math. USSR Sbornik, 15: 31-73 (1971).

• V.M. Adamjan, D.Z. Arov, and M.G. Krein, Infinite block Hankel matrices and related extension problems, American Math. Society Transactions, 111: 133-156 (1978).

Complete linear algebraic treatment of the optimal Hankel norm approximation problem:

• K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, International Journal of Control, 39: 1115-1193 (1984).

Approximation by balanced truncation (BT)

Seminal paper: introduced concept of balanced state representation.

• B.C. Moore, Principal component analysis in linear systems: Controllability, observability and model reduction, IEEE Trans. Automatic Control, AC-26: 17-32 (1981).

Given linear system $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$, $\det \mathbf{E} \neq \mathbf{0}$, $(\mathbf{A}, \mathbf{E})$ stable, use **state** and **output**. This implies the computation of the gramians which satisfy the generalized **Lyapunov equations**:

$$\mathbf{APE}^* + \mathbf{EPA}^* + \mathbf{BB}^* = \mathbf{0}, \;\; \mathbf{P} > \mathbf{0}, \;\; \mathbf{A}^*\mathbf{QE} + \mathbf{E}^*\mathbf{QA} + \mathbf{C}^*\mathbf{C} = \mathbf{0}, \;\; \mathbf{Q} > \mathbf{0} \quad \Rightarrow$$

$$\boxed{\sigma_i = \sqrt{\lambda_i(\mathbf{PE}^*\mathbf{QE})}}$$

Hankel singular values: provide **trade-off** between **accuracy** and **complexity**.

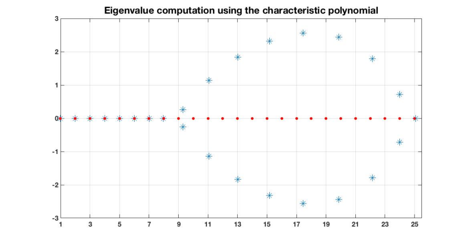
Properties

1. Stability is preserved
2. Global error bound: $\sigma_{k+1} \leq \| \mathbf{H}(s) - \hat{\mathbf{H}}(s) \|_\infty \leq 2(\sigma_{k+1} + \cdots + \sigma_n)$
3. Numerically expensive requiring $\mathcal{O}(n^3)$ operations.

Illustrative example: the eigenvalue problem.

Given $\mathbf{A} \in C^{n\times n}$, the conventional solution consists of 2 steps: (a) compute the characteristic polynomial $\chi_{\mathbf{A}}(s) = \det(s\mathbf{I} - \mathbf{A})$, to obtain the eigenvalues $\lambda_i \in C$, (b) solve $\ker(\lambda_i\mathbf{I} - \mathbf{A})$, to obtain the corresponding eigenvectors.

This may work for small matrices. Here is the counter-example of a 25×25 matrix with eigenvalues $1,\ 2,\ 3,\ \cdots,\ 24,\ 25$.



Thus, for large matrices only a few eigenvalues and the corresponding eigenvectors can be computed.

Methods used: **iterative eigenvalue computations**.

1. Dense computations, matrix factorizations and inversions $\Rightarrow$ may be ill-conditioned; number of operations $\mathcal{O}(n^3)$

2. Bottleneck: solution of Lyapunov equations:

$$\mathbf{APE}^* + \mathbf{EPA}^* + \mathbf{BB}^* = \mathbf{0}.$$

For large **A** such equations cannot be solved exactly.
Instead, since $\mathbf{P} > 0 \Rightarrow$ **square root L exists**: $\mathbf{P} = \mathbf{LL}^*$.

Hence compute **approximations** **V** to **L**: $\hat{\mathbf{P}} = \mathbf{VV}^*$: $\operatorname{rank}\mathbf{V} = k \ll n$:

$$P = \mathbf{L}\;\mathbf{L}^* \approx \mathbf{V}\;\mathbf{V}^* = \hat{\mathbf{P}}$$

Iterative solution: ADI, modified Smith with multi-shift strategy (guaranteed convergence).

Approximate solution of Lyapunov equations (Beattie-Embree-Sabino)

n	matrix dimension
teigs	time required to estimate the spectrum, seconds
tshifts	time required to compute real Nelder-Mead shifts, seconds
tsmith	time required for the modified Smith method, seconds
k	number of distinct shifts selected
s	number of times each shift is applied
bk	number of distinct shifts actually used
r	rank of computed solution

n	teigs	tshifts	tsmith	k	s	bk	r
7396	1	1.0	3	4	33	3	37
29796	6	1.0	17	4	58	3	59
67196	13	1.0	50	4	77	3	80
119596	25	1.0	121	4	105	3	103
269396	52	1.1	345	7	80	5	143
366796	77	5.3	591	5	135	4	165

Convergence criterion: relative residual norm $\leq 10^{-6}$
Sun Ultra 20, 2.2 GHz AMD Opteron 148; 3 GB RAM

Summary. The study of interpolatory reduction methods can be subdivided in three areas:

(1) Interpolatory projections,

(2) the Loewner Framework, and

(3) $\mathcal{H}_2$ Optimal Reduction known as IRKA.

Set-up. To start with, we consider linear, time-invariant systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{1}$$

assumed to be scalar for simplicity, i.e.

$$\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{n\times n},\ \mathbf{B},\ \mathbf{C}^T \in \mathbb{R}^n.$$

This is a descriptor realization ($\mathbf{C}$, $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$). The associated transfer function is

$$\mathbf{H}(s) = \mathbf{C}\boldsymbol{\Phi}(s)\,\mathbf{B},$$

where $\boldsymbol{\Phi}$ denotes the resolvent

$$\boldsymbol{\Phi}(s) = (s\,\mathbf{E} - \mathbf{A})^{-1} \in \mathbb{C}^{n\times n}. \tag{2}$$

Interpolatory projectors (Skelton, Grimme)

Consider two sets of complex numbers namely μ_i, $i = 1, \cdots, q$, and λ_j, $j = 1, \cdots, k$, which we will refer to as *left* and *right* interpolation points:

$$\mathcal{R} = \left[\begin{array}{ccc} \mathbf{\Phi}(\lambda_1)\mathbf{B} & \cdots & \mathbf{\Phi}(\lambda_k)\mathbf{B} \end{array} \right] \in \mathbb{C}^{n \times k} \quad \text{and} \quad \mathcal{O} = \left[\begin{array}{c} \mathbf{C\Phi}(\mu_1) \\ \vdots \\ \mathbf{C\Phi}(\mu_q) \end{array} \right] \in \mathbb{C}^{q \times n}. \quad (3)$$

These are called *the generalized controllability* and the *generalized observability* matrices.

Construction. For arbitrary k and q, the following relationships hold:

$$\hat{\mathbf{E}} = \mathcal{O}\mathbf{E}\mathcal{R} = -\begin{bmatrix} \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_k)}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_1)}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_k)}{\mu_q-\lambda_k} \end{bmatrix} = -\mathbb{L} \in \mathbb{C}^{q\times k}, \quad (4)$$

$$\hat{\mathbf{A}} = \mathcal{O}\mathbf{A}\mathcal{R} = -\begin{bmatrix} \frac{\mu_1\mathbf{H}(\mu_1)-\lambda_1\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{H}(\mu_1)-\lambda_k\mathbf{H}(\lambda_k)}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{H}(\mu_q)-\lambda_1\mathbf{H}(\lambda_1)}{\mu_q-\lambda_1} & \cdots & \frac{\mu_q\mathbf{H}(\mu_q)-\lambda_k\mathbf{H}(\lambda_k)}{\mu_q-\lambda_k} \end{bmatrix} = -\mathbb{L}_\sigma \in \mathbb{C}^{q\times k}, \quad (5)$$

$$\hat{\mathbf{B}} = \mathcal{O}\mathbf{B} = \begin{bmatrix} \mathbf{H}(\mu_1) \\ \vdots \\ \mathbf{H}(\mu_q) \end{bmatrix} = \mathbf{V} \in \mathbb{C}^{q\times 1}, \quad \text{and} \quad (6)$$

$$\hat{\mathbf{C}} = \mathbf{C}\mathcal{R} = \begin{bmatrix} \mathbf{H}(\lambda_1) & \cdots & \mathbf{H}(\lambda_k) \end{bmatrix} = \mathbf{W} \in \mathbb{C}^{1\times k}. \quad (7)$$

$(\mathbf{W}, \mathbb{L}, \mathbb{L}_\sigma, \mathbf{V})$ is the *Loewner quadruple*. (The **Loewner framework** was introduced by Mayo-Antoulas.)

Given:

row array $(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i),\ \mathbf{v}_j = \mathbf{g}(\mu_j),$ are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.

Then $k, q \geq \deg \mathbf{g} \ \Rightarrow\ \operatorname{rank} \mathbb{L} = \deg \mathbf{g}.$

Lemma

1. For $q = k \leq n$, define the transfer function $\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$. Then

$$\hat{\mathbf{H}}(\mu_i) = \mathbf{H}(\mu_i) \text{ and } \hat{\mathbf{H}}(\lambda_i) = \mathbf{H}(\lambda_i) \text{ for } i = 1, \cdots, k.$$

If $k = q = n$, the Loewner quadruple is equivalent to the original quadruple $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$.

2. If $k,\ q \geq n$, let $[\mathbf{U}_1, \mathbf{S}_1, \mathbf{V}_1] = \text{svd}([\mathbb{L} \quad \mathbb{L}_\sigma])$, $[\mathbf{U}_2, \mathbf{S}_2, \mathbf{V}_2] = \text{svd}([\mathbb{L}; \quad \mathbb{L}_\sigma])$; with $\mathbf{X} = \mathbf{U}_1(:, 1:r)$, $\mathbf{Y} = \mathbf{V}_2(:, 1:r)$, where r is the *numerical rank*, we have

$$\tilde{\mathbf{E}} = \mathbf{X}^T\mathbb{L}\mathbf{Y} \in \mathbb{C}^{r\times r},\ \tilde{\mathbf{A}} = \mathbf{X}^T\mathbb{L}_\sigma\mathbf{Y} \in \mathbb{C}^{r\times r},\ \tilde{\mathbf{B}} = \mathbf{X}^T\mathbf{V} \in \mathbb{C}^{r},\ \tilde{\mathbf{C}} = \mathbf{W}\mathbf{Y} \in \mathbb{C}^{1\times r}$$

Then the following approximate interpolation conditions are satisfied:

$$\tilde{\mathbf{H}}(\mu_i) \approx \mathbf{H}(\mu_i),\ i = 1, \cdots, q, \text{ and } \tilde{\mathbf{H}}(\lambda_j) \approx \mathbf{H}(\lambda_j),\ j = 1, \cdots, k.$$

• C. de Villemagne and R.E. Skelton, Model reduction using a projection formulation, International Journal of Control, 46: 2141-2169 (1987).

• E.J. Grimme, Krylov projection methods for model reduction, Ph.D. Thesis, ECE Dept., University of Illinois, Urbana-Champaign, (1997). Thesis supervisor: Paul van Dooren.

• A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).

Optimal $\mathcal{H}_2$ reduction

The $\mathcal{H}_2$ norm of a stable system $\boldsymbol{\Sigma} = (\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$, is:

$$\|\boldsymbol{\Sigma}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace } [\mathbf{H}(i\omega)\mathbf{H}^*(-i\omega)] \; d\omega \right)^{1/2},$$

where $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, is the system transfer function. The goal is to construct a reduced system $\boldsymbol{\Sigma}_k$ of order k, such that

$$\boldsymbol{\Sigma}_k = \underset{\deg(\hat{\boldsymbol{\Sigma}})=k}{\arg\min} \left\| \boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}} \right\|_{\mathcal{H}_2}.$$

This optimization problem is **nonconvex**. We seek therefore reduced models that satisfy **first-order necessary optimality conditions**. These turn out to be interpolatory conditions. Let the rational function $\mathbf{H}_k$ solve the optimal $\mathcal{H}_2$ problem and let $\hat{\lambda}_i$ denote its poles. Assuming that $m = p = 1$, the following **interpolation conditions** hold:

$$\mathbf{H}(-\hat{\lambda}_i^*) = \mathbf{H}_k(-\hat{\lambda}_i^*) \quad \text{and} \quad \left. \frac{d}{ds}\mathbf{H}(s) \right|_{s=-\hat{\lambda}_i^*} = \left. \frac{d}{ds}\mathbf{H}_k(s) \right|_{s=-\hat{\lambda}_i^*}.$$

Thus the (locally) optimal reduced system with transfer function $\mathbf{H}_k$ matches the first two moments of the original system at the **mirror image of its poles**.

IRKA (Beattie-Gugercin-Antoulas)

Fix the dimension of the reduced system to $k < n$, and pick left and right interpolation points to be the same: $\lambda_i = \mu_i \in \mathbb{C}$, $i = 1, \cdots, k$. Then, repeat:

1. Define the generalized controllability and observability matrices $\mathcal{R}$ and $\mathcal{O}$ by means of the scalars $\lambda_1, \cdots, \lambda_k$.

2. Define $\hat{\mathbf{E}}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{C}}$ as in the projection lemma above, and compute new scalars $\bar{\lambda}_i$, $i = 1, \cdots, k$, as follows

$$\{\bar{\lambda}_i\} = -\,\text{eig}\,(\hat{\mathbf{A}}, \hat{\mathbf{E}}).$$

3. If $\{\lambda\} = \{\bar{\lambda}\}$, the system $(\hat{\mathbf{C}}, \hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}})$ is a (locally) optimal $\mathcal{H}_2$ approximant of the original system. Otherwise go to step 2., and repeat until convergence.

• S. Gugercin, A.C. Antoulas, C.A. Beattie: Optimal $\mathcal{H}_2$ model reduction, SIAM J. Matrix Anal. Appl. (2008).

• S. Gugerin, C.A. Beattie, A.C. Antoulas, ”Interpolatory methods for model reduction”, SIAM book, Philadelphia (late 2018).

The role and necessity of considering **E** *terms*

The semi-discretized **Oseen equation** has the form:

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix}.$$

Let: $\boldsymbol{\Delta} = \left[\mathbf{A}_{12}^T \mathbf{A}_{12}\right]^{-1}$. Thus $\mathbf{p} = -\boldsymbol{\Delta}\mathbf{A}_{12}^T\mathbf{A}_{11}\mathbf{v} - \boldsymbol{\Delta}\mathbf{A}_{12}^T\mathbf{B}_1\mathbf{u} - \mathbf{B}_2\dot{\mathbf{u}}$.

Eliminating **p** from the above equations we get:

$$\begin{aligned} \dot{\mathbf{v}} &= \underbrace{[\mathbf{I} - \mathbf{A}_{12}\boldsymbol{\Delta}\mathbf{A}_{12}^T]\mathbf{A}_{11}}_{\mathbf{F}}\,\mathbf{v} + \underbrace{[\mathbf{I} - \mathbf{A}_{12}\boldsymbol{\Delta}\mathbf{A}_{12}^T]\,\mathbf{B}_1}_{\mathbf{G}_0}\mathbf{u} \underbrace{-\mathbf{A}_{12}\mathbf{B}_2}_{\mathbf{G}_1}\dot{\mathbf{u}}, \\ \mathbf{y} &= \underbrace{\left[\mathbf{C}_1 - \mathbf{C}_2\boldsymbol{\Delta}\mathbf{A}_{12}^T\mathbf{A}_{11}\right]}_{\mathbf{H}}\mathbf{v} \underbrace{-\,\mathbf{C}_2\boldsymbol{\Delta}\mathbf{A}_{12}^T\mathbf{B}_1}_{\mathbf{J}_0}\,\mathbf{u} \underbrace{-\mathbf{C}_2\mathbf{B}_2}_{\mathbf{J}_1}\dot{\mathbf{u}}. \end{aligned}$$

The associated transfer function shows that the behavior at infinity is characterized by a constant and a linear term. Therefore **E** terms should be intcorporated in the usual system theoretic considerations.

$$\boxed{\mathbf{T}(s) = \underbrace{\mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}(\mathbf{G}_0 + \mathbf{F}\mathbf{G}_1)}_{\mathbf{T}_{\rm spr}(s)} + \underbrace{\mathbf{H}\mathbf{G}_1 + \mathbf{J}_0 + s\mathbf{J}_1}_{\mathbf{T}_{\rm poly}(s)}}$$

• M. Heinkenschloss, D.C. Sorensen and K. Sun, Balanced truncation model reduction for a class of descriptor systems with application to the Oseen equations, SIAM J. Sci. Comput., **30**: 1038-1063 (2008).

Open Versus Closed Systems (J.C. Willems)

Question: are closed systems, as flows on manifolds and $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a good mathematical starting point from which to embark on the study of dynamics? (Planetary motion, the n-body problem, Hamiltonian dynamics, population dynamics and economic evolution, etc.)

In the 1970s and 1980s it was hoped that system theory, with its emphasis on open systems, would by now have been incorporated and accepted as the new starting point for dynamical systems in mathematics. Better, more general, more natural, more apt for modeling, offering interesting new concepts as controllability, observability, dissipativity, model reduction, and with a rich, well developed, domain as linear system theory. What seemed like an intellectual imperative did not even begin to happen. Mathematicians and physicists invariably identify dynamical systems with closed systems!

Final remarks

• The field of systems and control has come a long way in the last 50 years.

• The mathematical methods used have expanded.

• The techniques that have been developed for modeling, stabilization, model reduction, robustness etc. are deep and relevant.

• The paradigm of open systems, combined with interconnection, make it into an area that fits modern technological developments well.

• We have recently seen a strong growth in the number of applications. Especially model predictive control appears to be a leading circle of ideas. MPC is an area where essentially all aspects of the field, e.g. modeling, optimal control, observers, identification and adaptation, are in synergy with computer control and numerical mathematics.

• The coherence of the field has weakened. Perhaps open dynamical systems, with model reduction and control is a good theme around which to build a core in the future.

Recommended reading:

• J.C. Willems, "In control almost from the beginning until the day after tomorrow", European Journal of Control, vol 13, pp. 71-81 (2007).