Stabilitätsanalyse von modellprädiktiven Reglern für zeitvariante Systeme

Lars Grüne, Simon Pirkelmann

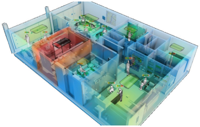
Universität Bayreuth, Lehrstuhl für Angewandte Mathematik

13. Elgersburg Workshop

24. - 28. Februar 2019

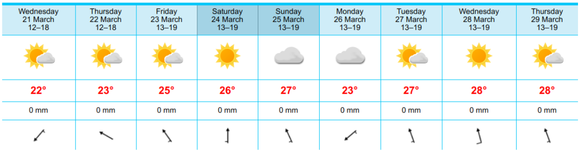
Gefördert durch DFG Grant GR 1569/16-1

Motivation: Energy Efficient Building Operation



1

2



Long term forecast

Wednesday 21 March 12–18	Thursday 22 March 12–18	Friday 23 March 13–19	Saturday 24 March 13–19	Sunday 25 March 13–19	Monday 26 March 13–19	Tuesday 27 March 13–19	Wednesday 28 March 13–19	Thursday 29 March 13–19
22°	23°	25°	26°	27°	23°	27°	28°	28°
0 mm	0 mm	0 mm	0 mm	0 mm	0 mm	0 mm	0 mm	0 mm

3

- Control temperature and airflow of a building using **H**eating **V**entilation and **A**ir**C**onditioning (HVAC)
- Focus on energy efficiency

[1] Image: Yang Liu http://news.usc.edu/40390/

[2] Image: http://www.knaufinsulation.com.sg/en/how-it-works

[3] Image: https://yr.no/

Outline

Setting

Consider a discrete-time, time-varying system

$$x(k+1) = f(k, x(k), u(k)),\ x(0) = x_0$$

with $x(k) \in X$, $u(k) \in U$.

Notation: $x_u(\cdot, x_0)$ solution trajectory

Example: Simplified building model:

$$x(k+1) = \qquad x(k) \qquad + \qquad u(k) \qquad + \qquad w(k)$$

Goal: Keep temperature (x) within a certain range $\mathbb{X}(k)$, using as little energy (u) as possible.

Setting

Consider a discrete-time, time-varying system

$$x(k+1) = f(k, x(k), u(k)),\ x(0) = x_0$$

with $x(k) \in X$, $u(k) \in U$.

Notation: $x_u(\cdot, x_0)$ solution trajectory

Example: Simplified building model:

$$x(k+1) = \underbrace{x(k)}_{\text{inside temperature}} + \underbrace{u(k)}_{\text{heating/cooling}} + \underbrace{w(k)}_{\text{outside temperature}}$$

Goal: Keep temperature (x) within a certain range $\mathbb{X}(k)$, using as little energy (u) as possible.

Setting

Infinite horizon optimal control problem

$$\min_{u \in \mathbb{U}^\infty(k,x_0)} J_\infty(k, x_0, u) = \sum_{j=0}^{\infty} \ell(k+j, x_u(j, x_0), u(j)) \\ \text{s.t.} \quad x(k+1) = f(k, x(k), u(k)), \quad x(0) = x_0 \tag{1}$$

with stage cost $\ell : \mathbb{N}_0 \times X \times U \to \mathbb{R}$, and where $\mathbb{U}^\infty(k, x_0)$ is the set of admissible control sequences.

Setting

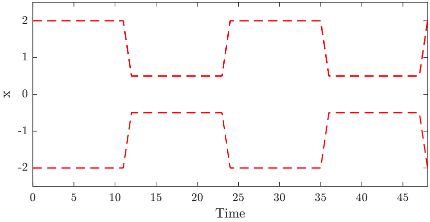
Infinite horizon optimal control problem

$$\min_{u \in \mathbb{U}^\infty(k,x_0)} J_\infty(k, x_0, u) = \sum_{j=0}^{\infty} \ell(k+j, x_u(j, x_0), u(j)) \tag{1}$$
$$\text{s.t.} \quad x(k+1) = f(k, x(k), u(k)), \quad x(0) = x_0$$

with stage cost $\ell : \mathbb{N}_0 \times X \times U \to \mathbb{R}$, and where $\mathbb{U}^\infty(k, x_0)$ is the set of admissible control sequences.

In the example:

$$x(k+1) = x(k) + u(k) + w(k)$$
$$\ell(k, x, u) = u^2$$
$$w(k) = -2\sin(\frac{k\pi}{12}) + \text{ randomness}$$



Setting

Infinite horizon optimal control problem

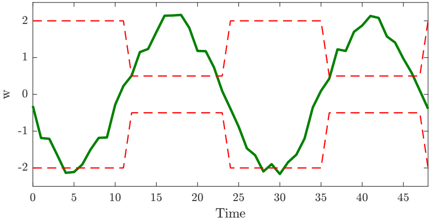
$$\min_{u \in \mathbb{U}^{\infty}(k,x_0)} J_{\infty}(k, x_0, u) = \sum_{j=0}^{\infty} \ell(k+j, x_u(j, x_0), u(j)) \quad (1)$$

$$\text{s.t.} \quad x(k+1) = f(k, x(k), u(k)), \quad x(0) = x_0$$

with stage cost $\ell : \mathbb{N}_0 \times X \times U \to \mathbb{R}$, and where $\mathbb{U}^{\infty}(k, x_0)$ is the set of admissible control sequences.

In the example:

$$x(k+1) = x(k) + u(k) + w(k)$$
$$\ell(k, x, u) = u^2$$
$$w(k) = -2\sin\left(\frac{k\pi}{12}\right) + \text{ randomness}$$



What is optimal?

Problem: $J_\infty(k, x_0, u) = \infty$ for all feasible $u \in \mathbb{U}^\infty(k, x_0)$.

$\rightsquigarrow$ What does $J_\infty(k, x_0, u^*) \leq J_\infty(k, x_0, u)$ mean?

[4] Joël Blot and Naïla Hayek. *Infinite-horizon optimal control in the discrete-time framework*. Springer, 2014.

[5] David Gale. "On Optimal Development in a Multi-Sector Economy". In: *Rev. Econ. Studies* 34.1 (1967), pp. 1–18.

What is optimal?

Problem: $J_\infty(k, x_0, u) = \infty$ for all feasible $u \in \mathbb{U}^\infty(k, x_0)$.

$\rightsquigarrow$ What does $J_\infty(k, x_0, u^*) \leq J_\infty(k, x_0, u)$ mean?

Overtaking optimality:
(x_{u^*}, u^*) is called *overtaking optimal*[4][5] if for all pairs (x_u, u)

$$\liminf_{K \to \infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x_{u^*}(k, x_0), u^*(k)) \geq 0. \qquad (2)$$

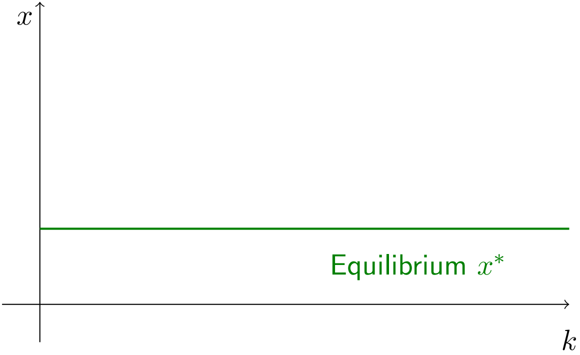
[4] Joël Blot and Naïla Hayek. *Infinite-horizon optimal control in the discrete-time framework*. Springer, 2014.

[5] David Gale. "On Optimal Development in a Multi-Sector Economy". In: *Rev. Econ. Studies* 34.1 (1967), pp. 1–18.

Optimal operation

Assumption: optimal reference trajectory (x^*, u^*) exists:

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x^*(k), u^*(k)) \geq 0 \qquad (3)$$

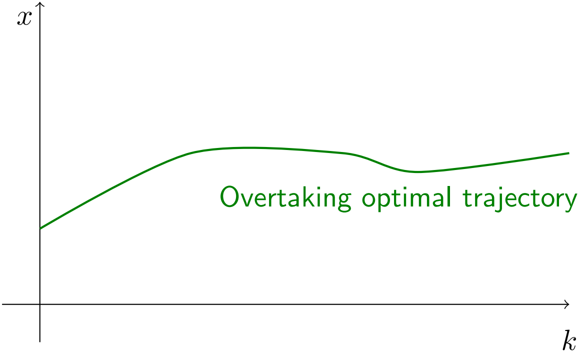


Remark: the optimal trajectory does not have to be unique

Optimal operation

Assumption: optimal reference trajectory (x^*, u^*) exists:

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x^*(k), u^*(k)) \geq 0 \qquad (3)$$

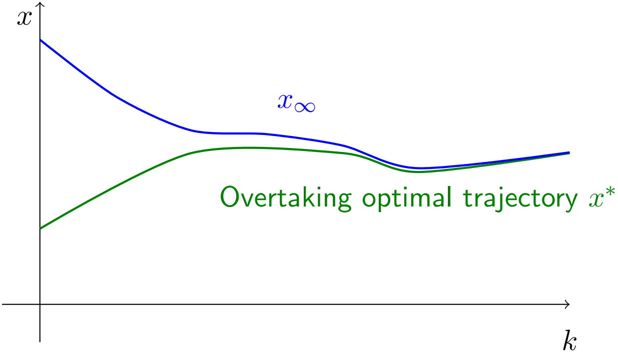


Remark: the optimal trajectory does not have to be unique

Optimal operation

Assumption: optimal reference trajectory (x^*, u^*) exists:

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x^*(k), u^*(k)) \geq 0 \tag{3}$$

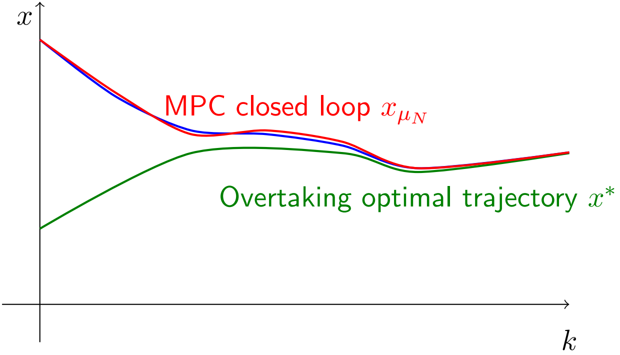


Remark: the optimal trajectory does not have to be unique

Optimal operation

Assumption: optimal reference trajectory (x^*, u^*) exists:

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x^*(k), u^*(k)) \geq 0 \qquad (3)$$



Remark: the optimal trajectory does not have to be unique

(Economic) Model predictive control

- Goal: Solve $\min_{u \in \mathbb{U}^{\infty}(k,x_0)} J_\infty(k, x_0, u)$
- **MPC**:
  Instead repeatedly solve *finite horizon* optimization problems

$$\min_{u \in \mathbb{U}^{N}(k,x_0)} J_N(k, x_0, u) = \sum_{j=0}^{N-1} \ell(k + j, x_u(j, x_0), u(j)) \tag{4}$$

where ℓ is the stage cost and $N \in \mathbb{N}$ the *horizon length*.

Observations from simulations with MPC

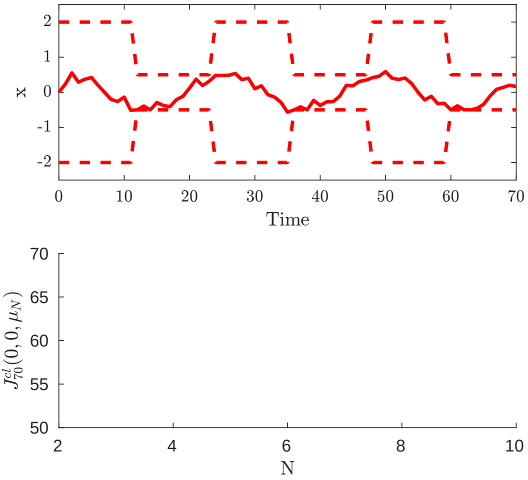


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

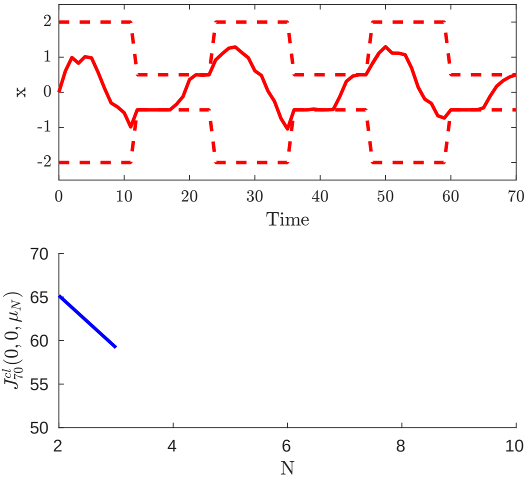


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

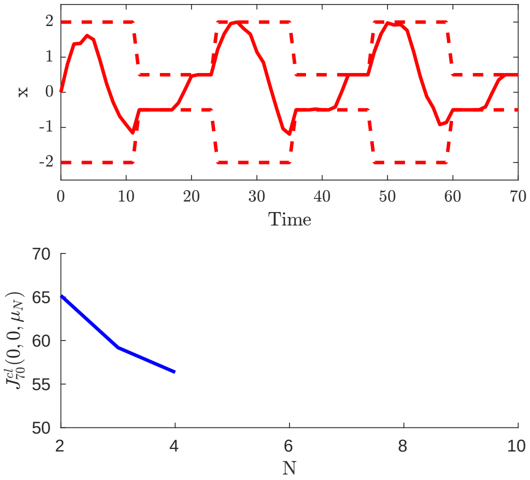


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

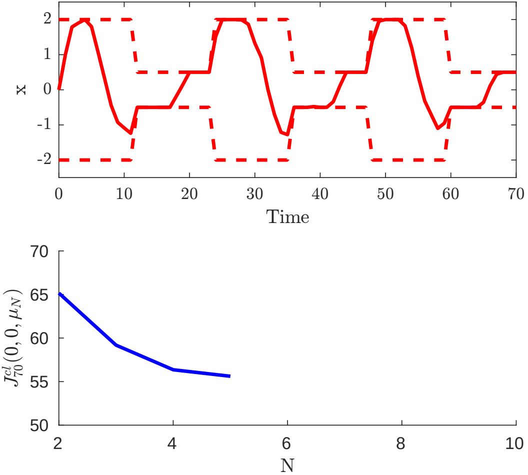
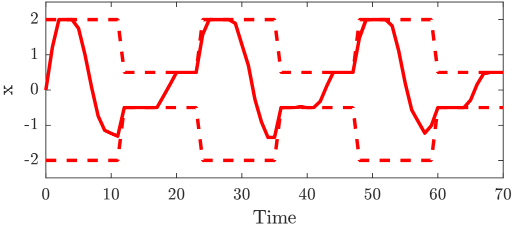


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC



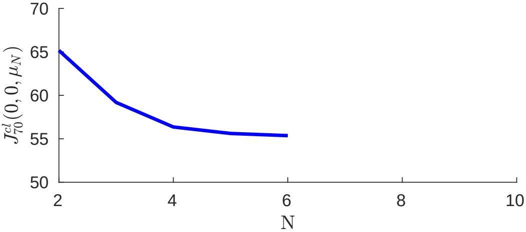


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

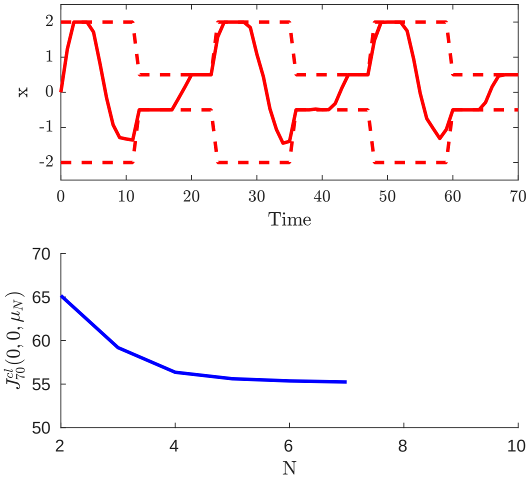


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

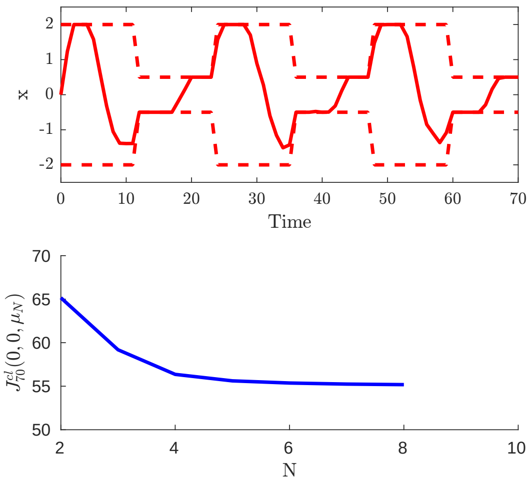


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

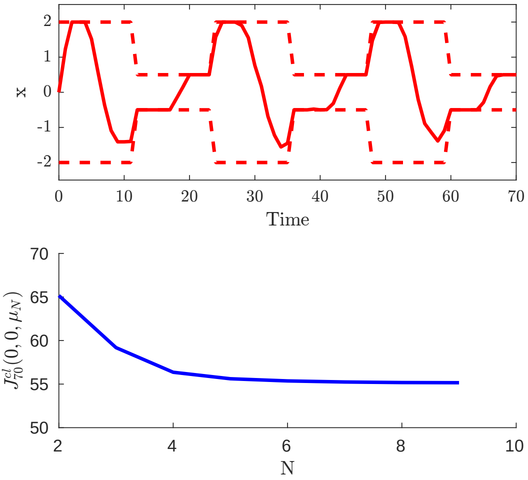


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

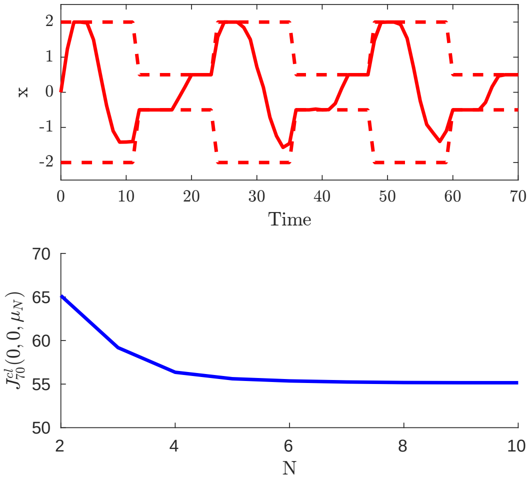


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

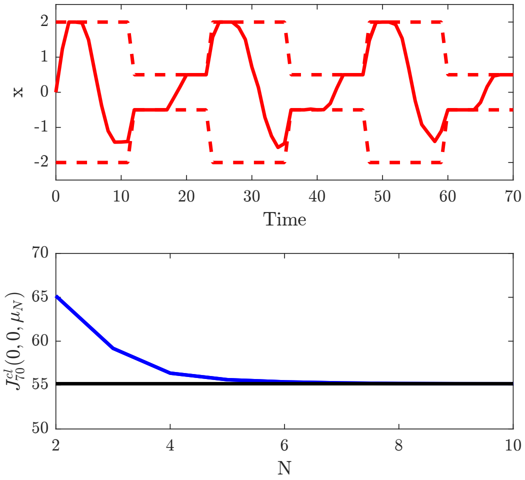


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

Talk of Elgersburg Workshop 2017:

Theorem 1

[Grüne, P., 2017] If the following assumptions hold:

- *Turnpike property*
- *Continuity of optimal value function*

Then an estimate for the closed-loop cost is given by:

$$\hat{J}_L^{cl}(k, x, \mu_N) + \hat{V}_\infty(k + L, x_{\mu_N}(L, x)) \leq \hat{V}_\infty(k, x) + L\delta(N)$$

for some function $\delta \in \mathcal{L}$.

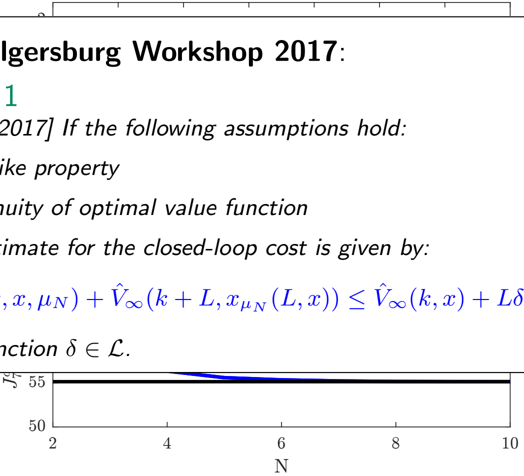


Figure: MPC closed loop cost for varying horizon length

Observations from simulations with MPC

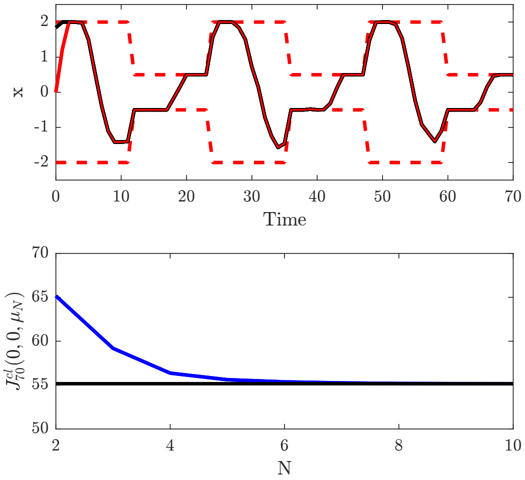


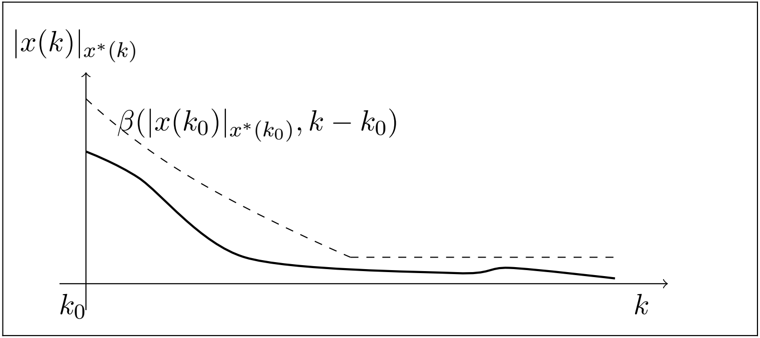
Figure: MPC closed loop cost for varying horizon length

MPC trajectory convergence

Goal: Show that not only cost converges, but also trajectory.
(Uniform) P-practical asymptotic stability of MPC closed loop:

$$|x(k;k_0,x_0)|_{x^*(k)} \leq \beta(|x_0|_{x^*(k_0)}, k-k_0)$$

with $\beta \in \mathcal{KL}$ for all $x(k;k_0,x_0) \notin P(k)$.

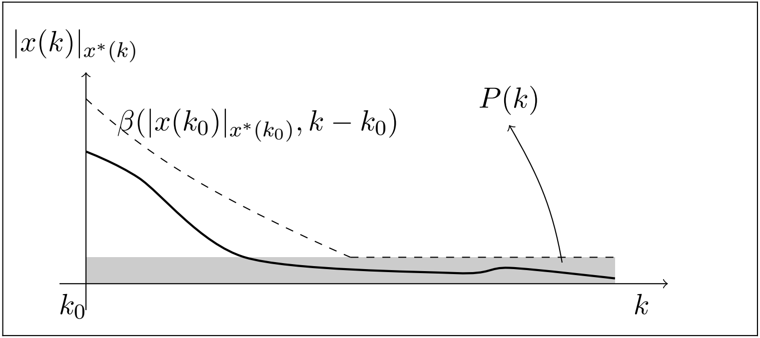


MPC trajectory convergence

Goal: Show that not only cost converges, but also trajectory.
(Uniform) P-practical asymptotic stability of MPC closed loop:

$$|x(k;k_0,x_0)|_{x^*(k)} \leq \beta(|x_0|_{x^*(k_0)}, k-k_0)$$

with $\beta \in \mathcal{KL}$ for all $x(k;k_0,x_0) \notin P(k)$.



MPC trajectory convergence

Goal: Show that not only cost converges, but also trajectory.
(Uniform) P-practical asymptotic stability of MPC closed loop:

$$|x(k; k_0, x_0)|_{x^*(k)} \leq \beta(|x_0|_{x^*(k_0)}, k - k_0)$$

with $\beta \in \mathcal{KL}$ for all $x(k; k_0, x_0) \notin P(k)$.

Idea: Show the existence of a (uniform time-varying) **Lyapunov function**:

1. $$\alpha_1(|x|_{x^*(k)}) \leq V(k, x) \leq \alpha_2(|x|_{x^*(k)})$$

   with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.
2. $$V(k+1, f(k, x, \mu_N(x))) \leq V(k, x) - \alpha_V(|x|_{x^*(k)})$$

   with $\alpha_V \in \mathcal{K}$.

Definitions

- Shifted cost function:

$$\hat{\ell}(k, x, u) = \ell(k, x, u) - \ell(k, x^*(k), u^*(k)) \tag{5}$$

$$\hat{J}_N(k, x_0, u) = \sum_{j=0}^{N-1} \hat{\ell}(k + j, x_u(j, x_0), u(j)) \tag{6}$$

for $N \in \mathbb{N}_0 \cup \{\infty\}$.

- Optimal value function

$$\hat{V}_N(k, x_0) := \inf_{u \in \mathbb{U}^N(k, x_0)} \hat{J}_N(k, x_0, u) \tag{7}$$

for $N \in \mathbb{N}_0 \cup \{\infty\}$.

Assumption: Dissipativity

A system is strictly dissipative if the following inequality holds

$$\underbrace{\lambda(k+1, f(k,x,u)) - \lambda(k,x)}_{\text{change of stored energy}} \leq \underbrace{s(k,x,u)}_{\text{external energy added to the system}} - \alpha(|(x,u)|_{(x^*(k),u^*(k))}) \tag{8}$$

for all $k \in \mathbb{N}_0, (x,u) \in \mathbb{X}(k) \times \mathbb{U}(k)$.

- $\lambda : \mathbb{N}_0 \times Y \to \mathbb{R}$ storage function
- $s : \mathbb{N}_0 \times Y \times U \to \mathbb{R}$ supply rate
- $\alpha \in \mathcal{K}_\infty$
- In our case: dissipativity w.r.t. supply rate

$$s(k,x,u) = \ell(k,x,u) - \ell(k,x^*(k),u^*(k)) \tag{9}$$

Define **modified cost**

$$\tilde{\ell}(k,x,u) = \hat{\ell}(k,x,u) + \lambda(k,x) - \lambda(k+1, f(k,x,u))$$

$$\tilde{J}_N(k,x,u) = \sum_{j=0}^{N-1} \tilde{\ell}(k+j, x_u(j;k,x), u(j))$$

Optimal value function as a candidate for Lyapunov function:

$$\tilde{V}_N(k,x) := \inf_{u \in \mathbb{U}^N(k,x)} \tilde{J}_N(k,x,u)$$

Special property (assuming strict dissipativity and additional upper bound):

$$\tilde{\ell}(k, x^*(k), u^*(k)) = 0$$

$$\Rightarrow \tilde{V}_N(k, x^*(k)) = 0 \qquad \text{and} \qquad \tilde{V}_N(k,x) \geq 0$$

This implies upper and lower bounds for Lyapunov function. ✓

Further Assumptions

- **Turnpike property**[6]: Optimal trajectories satisfy

$$|(x_{u_N^*}(j,x), u_N^*(j))|_{(x^*(k+j),u^*(k+j))} \leq \sigma(P)$$
$$\text{for all } j \in \{0,\dots,N\} \text{ with } j \notin \mathcal{Q}(k,x,P,N).$$

for some $\sigma \in \mathcal{L}$.

- **Continuity** of optimal value functions

$$|\hat{V}_N(k,x) - \hat{V}_N(k,x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|)$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.

[6] Emmanuel Trélat and Enrique Zuazua. "The turnpike property in finite-dimensional nonlinear optimal control". In: *Journal of Differential Equations* 258 (2015), pp. 81–114.

Further Assumptions

- **Turnpike property**[6]: Optimal trajectories satisfy

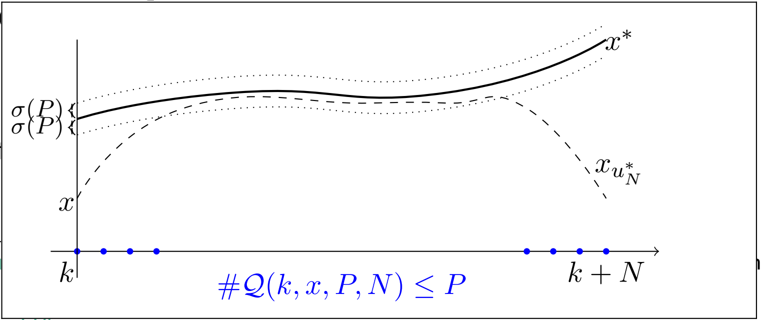
$$|(x_{u_N^*}(j,x), u_N^*(j))|_{(x^*(k+j),u^*(k+j))} \leq \sigma(P)$$

for all $j \in \{0,\ldots,N\}$ with $j \notin \mathcal{Q}(k,x,P,N)$.

for some $\sigma \in \mathcal{L}$.

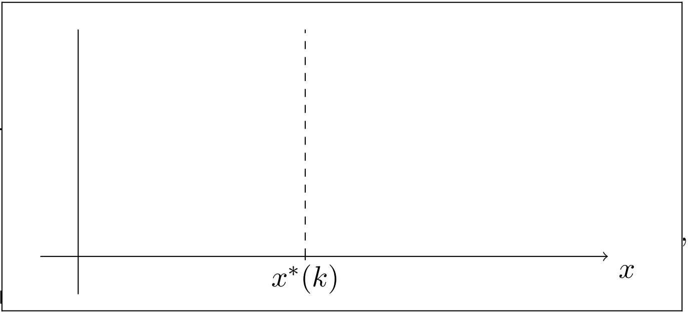
- (

t



[6]E ... nensional nonlin ... pp. 81–114.

- **Tu**



$, x, P, N).$

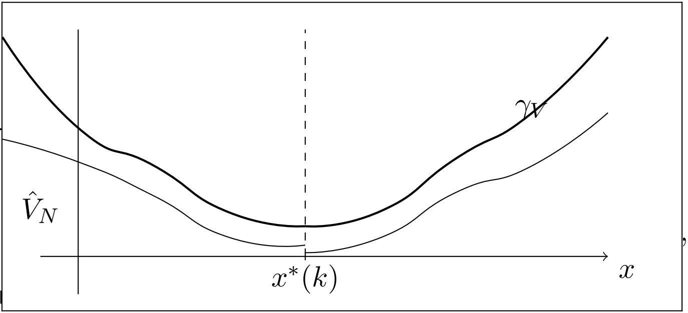
for

- **Continuity** of optimal value functions

$$|\hat{V}_N(k,x) - \hat{V}_N(k,x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|)$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.

[6] Emmanuel Trélat and Enrique Zuazua. "The turnpike property in finite-dimensional nonlinear optimal control". In: *Journal of Differential Equations* 258 (2015), pp. 81–114.



- **Tu**

  $, x, P, N).$

  for

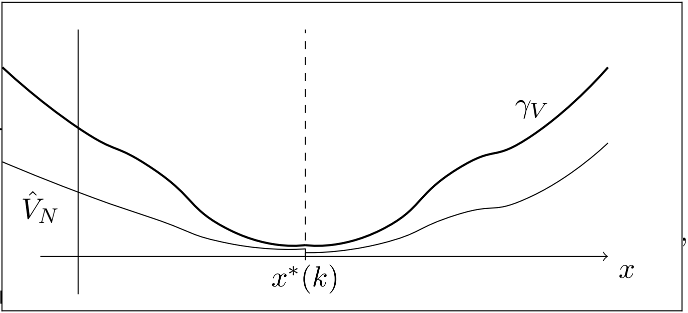
- **Continuity** of optimal value functions

$$|\hat{V}_N(k,x) - \hat{V}_N(k,x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|)$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.

[6] Emmanuel Trélat and Enrique Zuazua. "The turnpike property in finite-dimensional nonlinear optimal control". In: *Journal of Differential Equations* 258 (2015), pp. 81–114.

- **Tu**



$, x, P, N).$

fo

- **Continuity** of optimal value functions

$$|\hat{V}_N(k,x) - \hat{V}_N(k,x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|)$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.

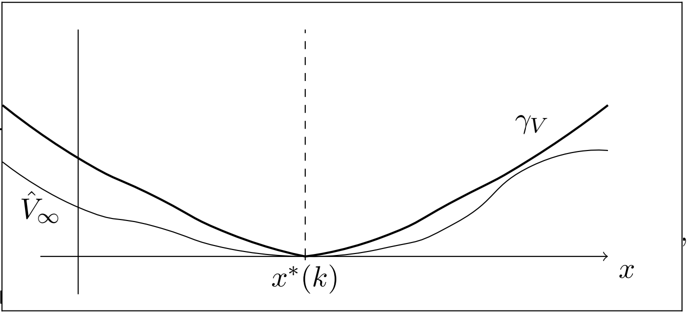
---

[6] Emmanuel Trélat and Enrique Zuazua. “The turnpike property in finite-dimensional nonlinear optimal control”. In: *Journal of Differential Equations* 258 (2015), pp. 81–114.

- **Tu**

$, x, P, N).$

fo



- **Continuity** of optimal value functions

$$|\hat{V}_N(k,x) - \hat{V}_N(k,x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|)$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.

[6] Emmanuel Trélat and Enrique Zuazua. "The turnpike property in finite-dimensional nonlinear optimal control". In: *Journal of Differential Equations* 258 (2015), pp. 81–114.

Monotone descent along the closed loop

We want to show:

$$\tilde{V}_N\ (k+1, f(k, x,\ \mu_N\ (x))) \leq\ \tilde{V}_N\ (k, x) - \alpha_V(|x|_{x^*(k)})$$

Monotone descent along the closed loop

We want to show:

$$\underbrace{\tilde{V}_N}_{\rightsquigarrow \tilde{u}_N^*}(k+1, f(k, x, \underbrace{\mu_N}_{\rightsquigarrow u_N^*}(x))) \leq \underbrace{\tilde{V}_N}_{\rightsquigarrow \tilde{u}_N^*}(k, x) - \alpha_V(|x|_{x^*(k)})$$

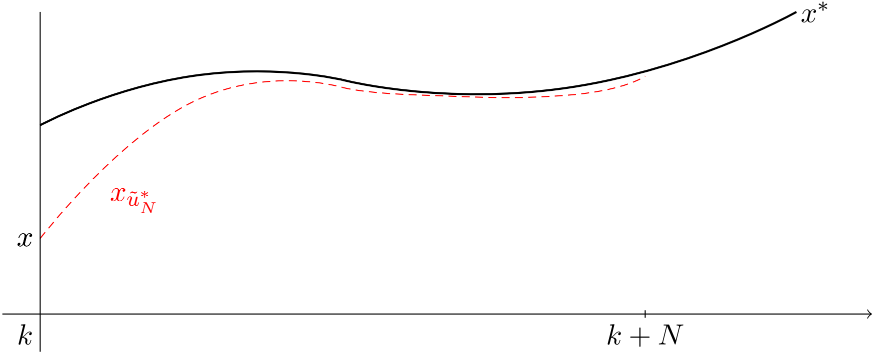
u_N^*: optimal solution of MPC problem
$\tilde{u}_N^*$: optimal solution of modified MPC problem

Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\ldots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

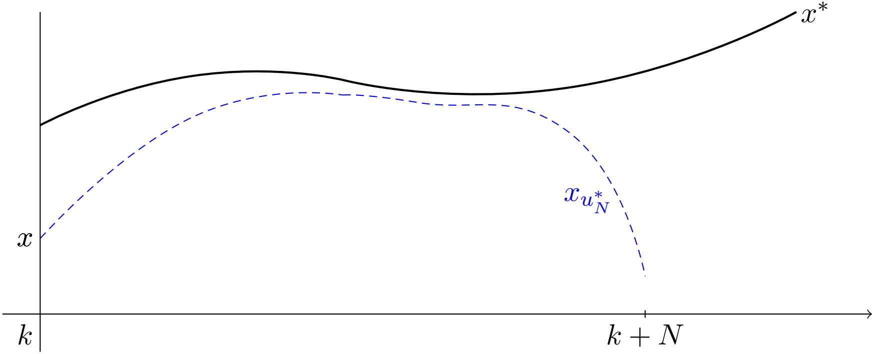


Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

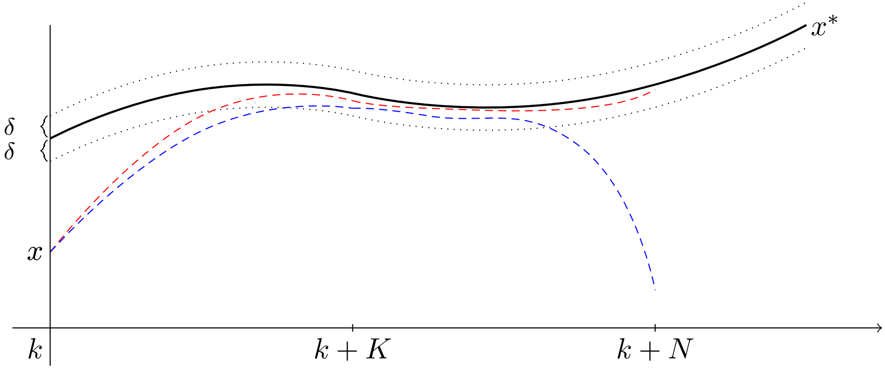


Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\ldots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

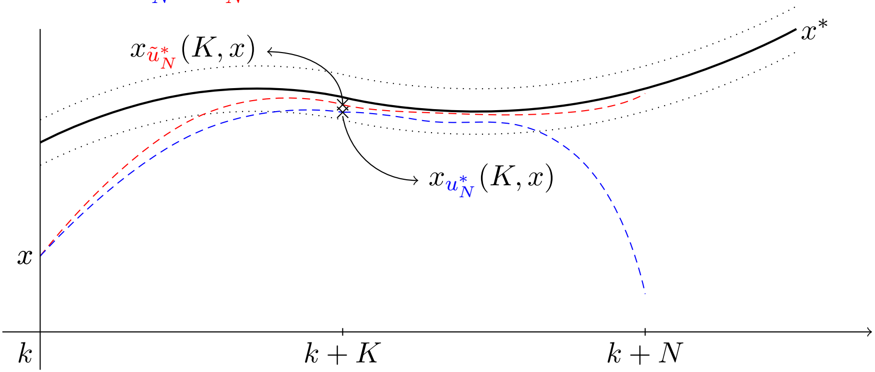


Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

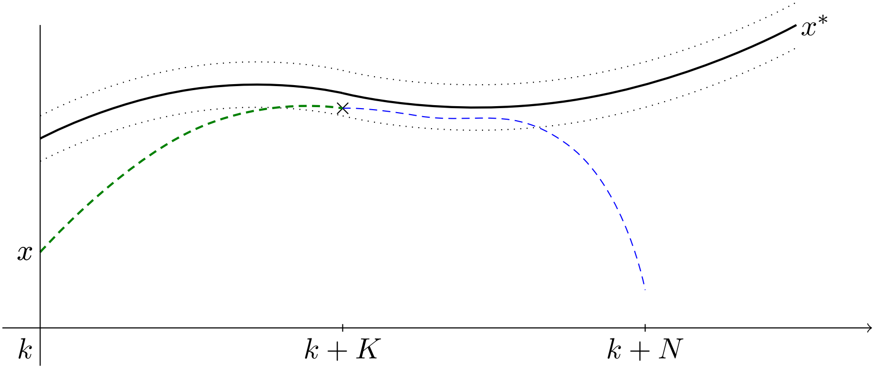


Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

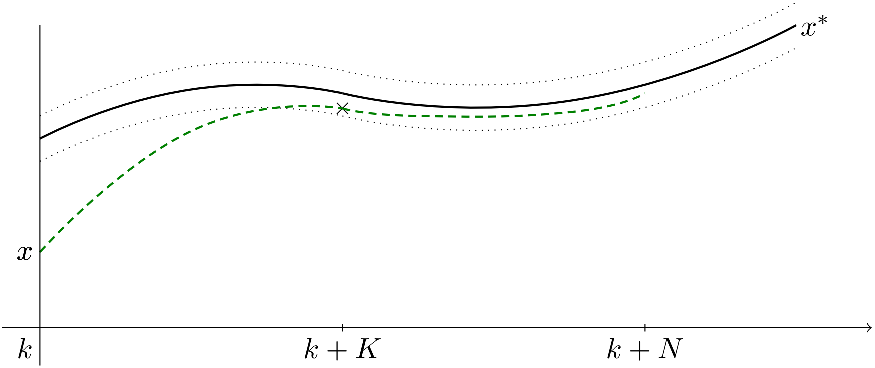


Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

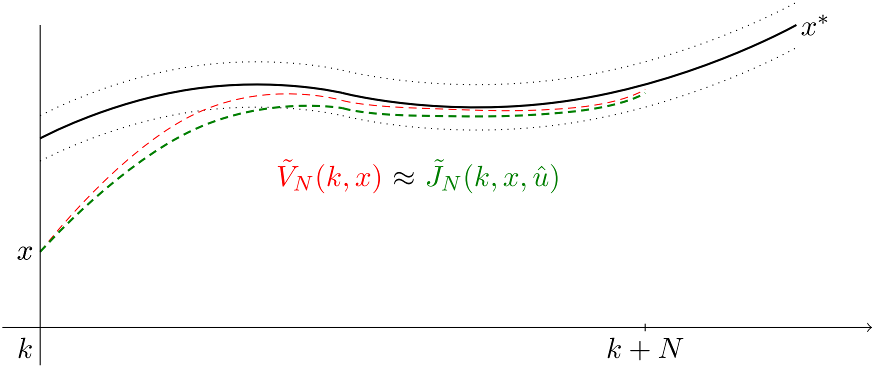


Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u^*_N \oplus \tilde{u}^*_N$"



Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

We know:

$$\tilde{\ell}(k, x, u) \quad + \; \tilde{J}_{N-1}(k + 1, f(k, x, u), u(\cdot + 1)) \; = \; \tilde{J}_N(k, x, u)$$

Key idea of the proof

Show that

$$\tilde{J}_N(k, x, \hat{u}) = \tilde{V}_N(k, x) + R(\dots)$$

where "$\hat{u} = u_N^* \oplus \tilde{u}_N^*$"

We know:

$$\underbrace{\tilde{\ell}(k, x, u)}_{\geq \alpha_v(|x|_{x^*(k)})} + \underbrace{\tilde{J}_{N-1}(k+1, f(k, x, u), u(\cdot + 1))}_{\approx \tilde{V}_{N-1}(k+1, f(k,x,\mu_N(x)))} = \underbrace{\tilde{J}_N(k, x, u)}_{\approx \tilde{V}_N(k,x)}$$

For specially chosen control sequence $u = \hat{u}$:

$$\Rightarrow \tilde{V}_{N-1}(k+1, f(k, x, \mu_N(x))) \leq \tilde{V}_N(k, x) - \alpha_v(|x|_{x^*(k)}) + \textit{error terms}$$

Conclusion

Summary:

- Convergence of MPC closed loop solution to an overtaking optimal trajectory
- Important tools: turnpike property and strict dissipativity

Open questions:

- How to check if continuity and turnpike hold?
  $\rightsquigarrow$ numerical verification in more recent research[7]
- What about dissipativity?

[7] Lars Grüne and Simon Pirkelmann. *Numerical Verification of Turnpike and Continuity Properties for Time-Varying PDEs*. Bayreuth, 2018. URL: https://eref.uni-bayreuth.de/46775/.

References I

Joël Blot and Naïla Hayek. *Infinite-horizon optimal control in the discrete-time framework*. Springer, 2014.

David Gale. "On Optimal Development in a Multi-Sector Economy". In: *Rev. Econ. Studies* 34.1 (1967), pp. 1–18.

Lars Grüne and Simon Pirkelmann. *Closed-loop performance analysis for economic model predictive control of time-varying systems*. Proceedings of the 56th IEEE Conference on Decision and Control (CDC). Bayreuth, 2017.

Lars Grüne and Simon Pirkelmann. *Economic Model Predictive Control for Time-Varying System: Performance and Stability Results*. Bayreuth, 2018. URL: https://eref.uni-bayreuth.de/45325/.

Lars Grüne and Simon Pirkelmann. *Numerical Verification of Turnpike and Continuity Properties for Time-Varying PDEs*. Bayreuth, 2018. URL: https://eref.uni-bayreuth.de/46775/.

Emmanuel Trélat and Enrique Zuazua. "The turnpike property in finite-dimensional nonlinear optimal control". In: *Journal of Differential Equations* 258 (2015), pp. 81–114.