Robust Stability, Numerical Dynamics and Sampled Data Control

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We consider ordinary differential equations in $\mathbb{R}^d$

$$\dot{x}(t) = f(x(t))$$

where

$$f : \mathbb{R}^d \to \mathbb{R}^d, \text{Lipschitz in } x$$

We denote the solution trajectories by $\Phi(t, x_0)$ and assume existence and uniqueness for all $t \geq 0$
Numerical Approximation

**Goal:** investigate stability properties via numerical approximations, e.g. one step methods $\Phi_h$ for step size $h > 0$

$$x(t + h) = \tilde{\Phi}_h(x(t)),$$

with solution trajectories $\tilde{\Phi}_h(t, x_0)$

The classical error bounds for numerical schemes tend to infinity for $t \to \infty$

→ useless for stability investigations

**Idea:** use robust stability concepts from control theory
(1) robust stability concepts for nonlinear systems

(2) application of (1) to numerical dynamics

(3) application of (1) and (2) to sampled–data systems
Comparison functions [Hahn 67]

\[ \mathcal{K} := \{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ | \text{continuous and strictly increasing with } \alpha(0) = 0 \} \]

\[ \mathcal{K}_\infty := \{ \alpha \in \mathcal{K} | \text{unbounded} \} \]

\[ \mathcal{KL} := \{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ | \text{continuous, } \beta(\cdot, r) \in \mathcal{K} \text{ and } \beta \text{ strictly decreasing to } 0 \text{ in } 2\text{nd argument} \} \]

Lemma: \( \beta(r, t) \leq \mu(\sigma(r), t) \) for \( \sigma \in \mathcal{K}_\infty \) and \( \mu \in \mathcal{KL} \) with \( \mu(r, 0) = r \) and \( \mu(r, t + s) = \mu(\mu(r, t), s) \)

(1d dynamical system) [Sontag 98, Gr. 02]
Asymptotic Stability

For a compact set $A \subset \mathbb{R}^d$ denote by $\| \cdot \|_A$ the Euclidean distance from $A$.

$A$ is called asymptotically stable with neighborhood $B$ and attraction rate $\beta \in KL$, if for all $x \in B$

$$\| \Phi(t, x) \|_A \leq \beta(\|x\|_A, t), \quad t \geq 0$$

Consider now the perturbed solutions $\Psi(t, x, w)$ of

$$\dot{x}(t) = \bar{f}(x(t), w(t)) \quad \text{with} \quad \|\bar{f}(x, w) - f(x)\| \leq C\|w\|$$

with time varying perturbations $w : \mathbb{R} \to W \subseteq \mathbb{R}^l$. 
$A$ is asymptotically stable if

$$\| \psi(t, x, 0) \|_A \leq \beta(\|x\|_A, t)$$

for $\beta \in \mathcal{KL}$
equivalently, \( A \) is asymptotically stable if

\[
\|\Psi(t, x, 0)\|_A \leq \mu(\sigma(\|x\|_A), t)
\]

for \( \sigma \in \mathcal{K}_\infty \) and \( \mu \in \mathcal{KL} \) with
\[
\mu(r, t + s) = \mu(\mu(r, t), s) \quad \text{and} \quad \mu(r, 0) = r
\]
$A$ is called input–to–state stable (ISS) if

$$\|\Psi(t, x, w)\|_A \leq \max\{\mu(\sigma(\|x\|_A), t), \gamma(\|w\|_{\infty})\}$$

for $\gamma, \sigma \in \mathcal{K}_\infty$ and $\mu \in \mathcal{KL}$ with $\mu(r, t + s) = \mu(\mu(r, t), s)$ and $\mu(r, 0) = r$. 
A is called input–to–state dynamically stable (ISDS) if

\[
\| \Psi(t, x, w) \|_A \leq \max \{ \mu(\sigma(\|x\|_A), t), \ \nu(w, t) \}
\]

for \( \gamma, \sigma \in \mathcal{K}_\infty \) and \( \mu \in \mathcal{KL} \) with \( \mu(r, t + s) = \mu(\mu(r, t), s) \) and \( \mu(r, 0) = r \), where

\[
\nu(w, t) := \text{ess sup}_{\tau \in [0, t]} \mu(\gamma(\|w(\tau)\|), t - \tau)
\]
**Theorem** [Gr. 02]: \( A \) is ISDS with \( \mu(r,t) \) satisfying

\[
\dot{\mu}(r,t) = -g(\mu(r,t))
\]

if and only if there exists an ISDS Lyapunov function, i.e., a function \( V : \mathbb{R}^d \to \mathbb{R}^+ \) with

\[
\|x\|_A \leq V(x) \leq \sigma(\|x\|_A)
\]

and

\[
\sup_{\gamma(\|w\|) \leq V(x)} DV(x) \bar{f}(x,w) \leq -g(V(x))
\]
Properties of ISDS

**Theorem** [Gr. 02]:

(i) ISDS $\Rightarrow$ ISS with same gain $\gamma$

(ii) ISS $\Rightarrow$ ISDS for any gain $\tilde{\gamma} > \gamma$

**Theorem** [Gr. 02]: Every asymptotically stable set for $\Phi$ is ISDS for $\Psi$ (for sufficiently small $\|w(t)\|$)

Attraction rate $\beta \sim$ upper bound for $\gamma$

e.g., $\beta(r, t) = Ce^{-\lambda t}r \sim \gamma(r) \leq Kr$
By embedding

- of the numerical approximation into the “inflated” exact system
  \[ \dot{x}(t) = \bar{f}(x(t), w(t)) := f(x(t)) + w(t) \]

- of the exact system into the “inflated” numerical system
  \[ x(t + h) = \tilde{\Psi}_h(x(t), w) := \tilde{\Phi}_h(x(t)) + hw(t) \]

the ISDS property can be used in order to investigate the numerical long time behavior.
Embedding of the Approximation

Numerical $\tilde{\Phi}_h(t, x_0)$ and exact solution $\Phi(t, x_0)$
Embedding of the Approximation

Embedding of $\Phi_h(t, x_0)$ into $\bigcup_w \Psi(t, x_0, w)$
Embedding of the Exact Solution

Embedding of $\Phi(t, x_0)$ into $\bigcup_w \tilde{\Psi}_h(t, x_0, w)$
Embedding

For a numerical scheme with consistency order $p$

$$\| \tilde{\Phi}_h(x) - \Phi(h, x) \| \leq C h^{p+1}$$

this embedding is satisfied with time varying perturbations $w(t)$ from

$$\mathcal{W} := \{ w : \mathbb{R} \text{ (or } h\mathbb{Z}) \rightarrow \mathbb{R}^n \mid \| w(t) \| \leq K h^{p} \}$$

for suitable $K > 0$.

Application to numerical dynamics:

Analysis of asymptotically stable sets and attractors under discretization in time
**Theorem:** [Kloeden/Lorenz 1986] Let $A$ be an asymptotically stable set for $\Phi$, and let $\tilde{\Phi}_h$ be an approximation by a numerical one step method.

Then $\tilde{\Phi}_h$ has “numerical” asymptotically stable sets $A_h$ with $A_h \to A$ (in the Hausdorff metric) for $h \to 0$.

**But:**

In general the Hausdorff limit $\lim_{h \to 0} A_h$ of numerically asymptotically stable sets $A_h$ is not a “real” asymptotically stable set for $\Phi$.

Recall the definition of the inflated numerical system

$$x(t + h) = \tilde{\Psi}_h(x(t), w) := \tilde{\Phi}_h(x(t)) + hw(t)$$
**Theorem:** [Gr. 03] $A := \lim_{h \to 0} A_h$ is as. stable

$\iff$

the sets $A_h$ are ISDS for $\Psi_h$ with comparison functions $\mu_h \in \mathcal{KL}$, $\sigma_h \in \mathcal{K}_\infty$, $\gamma_h \in \mathcal{K}_\infty$ satisfying

$\mu_h \to \mu \in \mathcal{KL}$, $\sigma_h \to \sigma \in \mathcal{K}_\infty$ and $\gamma_h \to \gamma \in \mathcal{K}_\infty$ for $h \to 0$

$\iff$

the sets $A_h$ have attraction rates $\beta_h \in \mathcal{KL}$ with

$\beta_h \to \beta \in \mathcal{KL}$ for $h \to 0$

For attractors one also obtains error estimates of the type $d_H(A_h, A) \leq \gamma(ch^q)$ ($q =$ order of scheme)

For hyperbolic attractors one obtains $d_H(A_h, A) \leq Kh^q$
Aim: Stabilize the system

\[ \dot{x}(t) = f(x(t), u(t)) \]

via feedback, i.e., find \( u : \mathbb{R}^d \to U \), such that \( A = \{0\} \) is an asymptotically stable set for

\[ \dot{x}(t) = f(x(t), u(x(t))). \]
We use a digital or sampled-data design approach:

Choose a sampling rate $T > 0$, consider the solutions $\Phi(T, x, u)$ for $u$ constant on $[0, T]$ and stabilize the discrete time system

$$x(k + 1) = \Phi_T(x(k), u) := \Phi(T, x(k), u),$$

via feedback, i.e., look for $u_T : \mathbb{R}^d \to U$, such that $A = \{0\}$ is an asymptotically stable set for

$$x(k + 1) = \Phi_T(x(k), u_T(x(k))).$$
Why Sampled Data Design?

- Implementation of nonlinear controllers usually via digital devices → sampling necessary

- Sampling of continuous time controllers may result in bad performance, design on discrete time level yields better results

Example: Consider the 2d system

\[
\begin{align*}
\dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\
\dot{x}_2 &= -u
\end{align*}
\]

(simplified model of a jet engine) with controller

\[u(x) = -7x_1 + 5x_2\]

cf. [Krstić, Kanellakopoulos and Kokotović, 1995]
Why Sampled Data Design?

Continuous time trajectory
Sampled continuous time controller, $T = 0.1$
Sampled discrete time controller, $T = 0.1$
In order to illustrate the application of numerical dynamics methods we investigate the construction of $u_T$ via optimal control techniques:

Compute

$$\min_u J_T(x, u)$$

with

$$J_T(x, u) := \sum_{k=0}^{\infty} l_T(\Phi_T(k, x, u), u_k),$$

If $\Phi_T$ is stabilizable, then for suitable cost $l_T$ this problem is solvable and the optimal feedback control $u_T$ is stabilizing.
**Approximate Models**

**Problem:** The map $\Phi_T$ is not known in general

→ solve the optimal control problem for an approximate model $x(k+1) = \Phi_T(x(k), u(k))$, e.g. a numerical approximation

**Desired result:** If

- $\tilde{u}_T$ stabilizes $\tilde{\Phi}_T$
- $\tilde{\Phi}_T$ is sufficiently accurate

then $\tilde{u}_T$ stabilizes $\Phi_T$ — at least approximately.

This is not true in general!
Problems with Approximate Models

Consider \( \dot{x} = x + u \) and its Euler approximation

\[ x(k+1) = (1 + T)x(k) + Tu(k). \]

For \( Q_T = T \) and \( R_T = T^3(1 - T)^3 \) we minimize

\[ \tilde{J}_T(x, u) = \sum_{k=0}^{\infty} (Q_T x^2(k) + R_T u^2(k)) \]

\[ \rightarrow \text{stabilizing controller } \tilde{u}_T(x) = \left(-1 - \frac{5T^2}{2} + O(T^3)\right)x \]
Stability via Approximate Models

Reasons:

• **non–uniform attraction rate**
  (here due to ill–parametrized cost function).

• **coupling** of sampling rate $T$ and numerical accuracy

  $\rightsquigarrow$ loss of robustness **dominates** gain of accuracy
  for $T \rightarrow 0$

Solution: guarantee **uniform robustness of as. stability** with respect to accuracy of the approximate model by assuming suitable **regularity properties**

easier if **sampling rate $T$ and numerical accuracy** are **decoupled**
Theorem: Let $\varepsilon > 0$ be a measure the accuracy of the approximation $\tilde{\Phi}_T$ and assume the following conditions uniformly w.r.t. $\varepsilon$:

- Local Lipschitz conditions for $l_T$ and $\tilde{\Phi}_T$
- Local boundedness of $\tilde{u}_T(x)$

Then the feedback–controlled exact system satisfies

$$\|\Phi_T(k, x, \tilde{u}_T)\| \leq \beta(\|x\|, Tk) + \delta \text{ for all } \|x\| \leq \Delta$$

- $\delta \to 0$ for $\varepsilon \to 0$ and fixed $T > 0$ (practical stability)
- $\Delta \to \infty$ for $T \to 0$ and $\varepsilon \to 0$ sufficiently fast (semiglobal stability)
Idea of Proof

• the optimal value functions $\tilde{W}_T$ are Lyapunov functions for the approximate system

• the assumptions imply suitable uniformity of the properties of $\tilde{W}_T$ w.r.t. $\varepsilon$
  $\Rightarrow$ the value functions are uniform (in $\varepsilon$) ISDS–Lyapunov functions for
  \[ \tilde{\Psi}_T(x, w) = \tilde{\Phi}_T(x, \tilde{u}_T(x)) + Tw \]

• accuracy and uniform boundedness of $\tilde{u}_T$
  $\Rightarrow$ embedding $\Rightarrow$ assertion
Stabilization via Approx. Optimal Control

If the numerical accuracy depends on $T$, e.g.,

$$\|\tilde{\Phi}_T(x, u) - \Phi_T(x, u)\| \leq C T^{p+1},$$

then we need additional regularity assumptions uniformly in $T$:

- $\sigma_1(\|x\|) \leq \tilde{W}_T(x) \leq \sigma_2(\|x\|)$ for $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$
- $\tilde{W}_T$ is H"older continuous with exponent $\eta > \frac{1}{p+1}$
- $l_T/T$ is suitably bounded from below

Then the assertion remains true, i.e.,

$$\|\Phi_T(k, x, \tilde{u}_T)\| \leq \beta(\|x\|, Tk) + \delta$$

for all $\|x\| \leq \Delta$

with $\delta \to 0$ and $\Delta \to \infty$ for $T \to 0$

(practical semiglobal asymptotic stability)
Hölder continuity implies that $\tilde{W}_T$ is an ISDS Lyapunov function for

$$\tilde{\Psi}_T(x, w) = \tilde{\Phi}_T(x, \tilde{u}_T(x)) + T w$$

with robustness gain

$$\gamma_T(r) = \alpha(r^n T^{\eta-1})$$

for some $\alpha \in \mathcal{K}_\infty$ independent of $T$.

This provides just enough robustness to compensate for the approximation error, because

$$\gamma_T(C T^p) = \alpha(C^n T^{\eta p} T^{\eta-1}) = \alpha(C^n T^{\eta(p+1)-1}) \to 0$$

as $T \to 0$, because $\eta(p + 1) - 1 > 0$.
Conclusions and Outlook

- **ISDS–techniques** provide a framework for the analysis of stability under numerical errors.
- Provides tighter **quantitative information** than ISS.
- Particularly suitable for the analysis of **sampled–data systems**.

Future Work:

- Optimization of **quantitative** properties, like attraction rates, transient behavior... 
- First results on the **improvement** of **sampled data performance** of continuous time controllers via high order correction terms ("Re–Design") are available.