Exponential estimates for time delay systems

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February 23, 2004

Abstract

In this talk we demonstrate how Lyapunov-Krasovskii functionals can be used to obtain exponential bounds for the solutions of time-invariant linear delay systems.

Keywords: time delay system, exponential estimate, Lyapunov-Krasovskii functional.
1 Introduction

The objective of this talk is to describe a systematic procedure of constructing quadratic Lyapunov functionals for exponentially stable linear time delay systems in order to obtain exponential estimates for their solutions.
Delay free case

The procedure is a counterpart to the well known Lyapunov based method of deriving exponential estimates for the stable system

\[ \frac{dx}{dt} = Ax; \quad (\mathbb{R}) \]

Consider a quadratic function

\[ w(x) = x^T W x; \quad W > 0; \]

and define the quadratic Lyapunov function

\[ v(x) = x^T V x \]

such that

\[ \frac{d}{dt} v(x) = -w(x) \]

Matrix \( V > 0 \) is the solution of the Lyapunov matrix equation

\[ A^T V + VA = -W; \quad (\mathbb{R}) \]
If $\frac{3}{4} > 0$ is such that

$$2^{\frac{3}{4}} = W;$$

then

$$k\chi(t; x_0) = \varphi^t e^{\frac{3}{4} t} x_0; \quad t \geq 0.$$ 

The decay rate

$$\frac{3}{4} \max_{1 \leq j \leq n} \operatorname{Re}_{a_j}(A);$$

The $\varphi$-factor depends on some geometric properties of matrix $A$. 


It is surprising that a similar constructive method does not exist for delay systems. It is true, there exists an operator theoretic version of Lyapunov’s equation in the abstract semigroup theory of infinite dimensional time-invariant linear systems, but this does not provide us with a constructive procedure. For constructive purposes more specific Lyapunov functions must be considered.
2 Preliminaries

In this paper we consider time delay systems of the following form

\[ \frac{dx(t)}{dt} = A_0x(t) + \sum_{k=1}^X A_kx(t - h_k) \]  \( (1) \)

where \( A_0; A_1; \ldots; A_m \in \mathbb{R}^{n \times n} \) are given matrices and \( 0 < h_1 < \ldots < h_m = h \) are positive delays.

For any continuous initial function

\[ x(\mu; \cdot) \coloneqq \mu; \mu \in [-h; 0] \]

there exists the unique solution, \( x(t; \cdot) \), of (1) satisfying the initial condition

\[ x(\mu; \cdot) = \mu; \mu \in [-h; 0] \]

If \( t > 0 \) we denote by \( x_t(\cdot) \) the trajectory segment

\[ x_t(\cdot) = f(x(t + \mu; \cdot)); \mu \in [0; 0] \]
Stability definition

We will use the Euclidean norm for vectors and the induced matrix norm for matrices. The space of continuous initial functions $C([\mu; h; 0]; \mathbb{R}^n)$ is provided with the uniform norm

$$k' k_h = \max_{\mu \in [\mu; h; 0]} k' (\mu) k.$$

**Definition 1:** The system (1) is said to be exponentially stable if there exist $\gamma > 0$ and $\mu > 1$ such that the following exponential estimate

$$k x(t;') k \leq e^{\gamma t} k' k_h; \quad t \geq 0; \quad (2)$$

holds for every solution of (1).

For simplicity we will call an exponentially stable system just stable.
Fundamental matrix

The matrix-valued function $K(t)$ which solves the matrix differential equation

$$\frac{d}{dt}K(t) = A_0 K(t) + \sum_{j=1}^{\infty} A_j K(t - h_j); \quad t \geq 0;$$

with initial condition

$$K(t) = 0 \quad \text{for} \quad t < 0; \quad K(0) = I_n;$$

is called the fundamental matrix of the system (1).

It is known that $K(t)$ also satisfies the differential equation

$$\frac{d}{dt}K(t) = K(t)A_0 + \sum_{j=1}^{\infty} K(t - h_j)A_j; \quad t \geq 0;$$
Cauchy formula

The following result is known as the Cauchy formula for the solutions of system (1)

\[ x(t'; ) = K(t)'(0) \]

\[ + \sum_{j=1}^{X} \int_{h_j}^{Z} K(t; h_j, \mu)A_j'(\mu)d\mu \quad t \geq 0 \]  

(3)
3 Lyapunov-Krasovskii functionals

We construct Lyapunov-Krasovskii quadratic functionals for system (1) in a similar way as for the case of delay free systems.

Delay free case
Given a stable linear system

\[
\frac{dx}{dt} = Ax: \quad (\text{x})
\]

For an arbitrarily quadratic function

\[
w(x) = xWx \text{ with } W > 0
\]

there exists quadratic function \( v(x) = xVx, V > 0 \) such that

\[
\frac{dv(x(t;x_0))}{dt} = w(x(t;x_0));
\]

for every solution of (\( x \)).
Function $v(x)$ satisfies

$$v(x_0) = Z \int_0^T w(x(t; x_0)) \, dt$$

$$= x_0^T e^{AT} \int_0^T w e^{At} \, dt x_0.$$ 

Matrix

$$V = Z \int_0^T e^{AT} w e^{At} \, dt$$

is the (uniquely determined, positive definite) solution of the Lyapunov equation

$$A^T V + VA = \mu Z \int_0^T e^{AT} W e^{At} \, dt x_0.$$
Time delay system

Given positive definite \( n \times n \) matrices \( W_0, W_1, ..., W_{2m} \), consider the quadratic functional

\[
w(x_t) = x(t)W_0x(t) + \sum_{k=1}^{Z} x(t - h_k)W_kx(t - h_k) + \sum_{k=1}^{Z} x(t + \mu)W_{m+k}x(t + \mu)d\mu; \quad (4)
\]

If the system (1) is exponentially stable, then there exists a quadratic functional \( v(x_t) \) such that

\[
\frac{dv(x_t)}{dt} = w(x_t); \quad t \geq 0 \quad (5)
\]

along solutions of (1).

Functional \( v(x_t) \) is called the complete type Lyapunov-Krasovskii functional generated by (4).
The functional is given by

\[ v(x_t) = x(t)U(0)x(t) \quad (6) \]

\[ + \sum_{k=1}^{Z} 2x(t)U(i \ h_k i \ \mu)A_kx(t + \mu)d\mu \]

\[ + \sum_{k=1}^{X} x \sum_{j=1}^{Z} x(t + \mu_2)A_k \]

\[ \left[ \sum_{k=1}^{X} x(t + \mu) \left[ W_k + (h_k + \mu)W_{m+k} \right] x(t + \mu)d\mu \right] \]

where

\[ U(\dot{z}) = \int_0^Z K(t)W(t + \dot{z})dt; \quad (7) \]

and

\[ W = W_0 + \sum_{k=1}^{X} (W_k + h_kW_{m+k}) \quad (8) \]

We call \( U(\dot{z}) \) the Lyapunov matrix function for the system (??) associated with (8).
Note that the first \((1 + m + m^2)\) terms in (6) are completely determined by \(U(\xi)\) and hence only depend upon the weighted sum \(W\) of the positive definite matrices \(W_k\). However, the last \(m\) terms in (6) depends on the individual \(W_k\)'s. We will see later that all these terms are needed in order to derive exponential estimates for (1) by means of the above quadratic functionals.

For the case of a single delay in (1) a similar Lyapunov-Krasovskii functional has been considered by Infante and Castelan. However, in their paper the matrix

\[ W_k + (h_k + \mu)W_{m+k} \]

in each one of the last \(m\) terms of (6) is replaced by a constant positive definite matrix. This is due to the fact that Infante and Castelan did not include the integral term in their definition of the functional \(w(\phi)\), see [Infante, Castelan: J. Diﬀ. Eqs 1978]. However, we will see that this integral term is an essential ingredient for deriving exponential estimates.
Remark 1: If \((1)\) is without delays \((h_k = 0; k = 1; \cdots; m)\) then the interval \([i; h; 0]\) is reduced to \(f0g\), and we have

\[ K(t) = e^{At}. \]

The quadratic functional \((4)\) is given now by

\[ w(x) = xWx; \]

where \(W\) is defined by \((8)\), and \(v\) is given by

\[ v(x) = xU(0)x. \]

Note that in this case \(U(0)\) is equal to

\[ U = \int_{0}^{\infty} e^{At}We^{At}dt \]

so that \(V = U(0)\) satisfies the Lyapunov equation \((\Xi)\).

The above construction of the Lyapunov-Krasovskii functional \(v\) from \(w\) generalizes this construction procedure via the Lyapunov equation \((\Xi)\) in the delay free case.
Remark 2: In the delay free case the success of quadratic Lyapunov functions rely on the fact that for a given \( w(x) \) the corresponding \( v(x) \) is not obtained via the integral expression but can be computed from the linear Lyapunov equation (16).

Similarly, the above construction of the Lyapunov-Krasovskii functional (6) would not be practical if it required the evaluation of the integral (7) (and so, in particular, the knowledge of the fundamental matrix \( K(t) \) on \([0; 1)\).
Lyapunov matrix for time delay system (1)

It is not difficult to show that for \( \dot{\xi} \in [0; h] \) matrix \( U(\dot{\xi}) \) solves the following matrix delay differential equation

\[
\frac{d}{d\dot{\xi}} U(\dot{\xi}) = U(\dot{\xi})A_0 + \sum_{k=1}^{X} U(\dot{\xi} - h_k)A_k; \quad (9)
\]

and additionally the following conditions

\(^2\) the symmetry condition

\[
U(\xi) = U(\dot{\xi}); \quad \xi \in [\dot{\xi}; h]; \quad (10)
\]

Remark 1 \(^2\) the Lyapunov type linear matrix equation

\[
iW = U(0)A_0 + A_0U(0) + \sum_{k=1}^{X} [U(h_k)A_k + A_kU(h_k)]; \quad (11)
\]

A systematic study of the equations (9), (10), (11) has not yet been accomplished, but we know that if system (1) is exponentially stable, then \( U(\dot{\xi}) \) is the unique solution of this set of equations.
4 Main results

In this section we show how one can use functionals (4) and (6) in order to obtain exponential estimates for solutions of system (1).

We first state the following result.

**Proposition 1:** Suppose that functionals (4) and (6) satisfy the following conditions:

1. $\mathcal{K}_1 k(x) k^2 v(x_t) \mathcal{K}_2 x_t \leq \mathcal{K}_1 > 0$ and $\mathcal{K}_2 > 0$;

2. $2\bar{\varepsilon} v(x_t) w(x_t)$, for some $\bar{\varepsilon} > 0$;

3. $\frac{d}{dt} v(x_t) = \frac{1}{w(x_t)}$ for all $t \geq 0$.

**Proposition 2** then

$$ r \frac{\mathcal{K}_2}{\mathcal{K}_1} e^{-\varepsilon t} k(x_t) \leq \mathcal{K}_1 k_n; \quad t \geq 0; \quad (12) $$
Proof: Given initial function \( y \), conditions 2. and 3. imply that

\[
\frac{d}{dt} v(x_t(y)) \leq 2 \bar{v}(x_t(y)); \quad t \geq 0.
\]

This inequality implies that

\[
v(x_t(y)) \leq v(y) e^{2t}; \quad t \geq 0.
\]

Then conditions 1. yields

\[
\otimes_1 k(x(t'; y)) k^2 v(x_t(y))
\]

\[
v(y) e^{2t} \leq \otimes_2 k t' k^2 e^{2t}; \quad t \geq 0.
\]

Comparing the left and the right hand sides, the exponential estimate (12) follows.
We will now show that conversely, if the system (1) is exponentially stable, then such positive constants $\hat{\alpha}_1; \hat{\alpha}_2; \hat{\beta}_1$ do exist.

As a consequence we obtain an exponential estimate of the form (12) for every set of positive definite $n \times n$ matrices $W_0; \ldots; W_{2m}$.

**Lemma 1:** If system (1) is exponentially stable and $W_0; W_1; \ldots; W_{2m}$ are positive definite real $n \times n$ matrices then there exist positive constants $\hat{\alpha}_1; \hat{\alpha}_2$, such that the quadratic Lyapunov-Krasovskii functional (6) satisfies the inequalities

$$\hat{\alpha}_1 \| x(t) \|^2 \leq v(x_t) \leq \hat{\alpha}_2 \| x_t \|^2.$$
Proof: Let

\[ \lambda_{\min} = \min_{k=0, \ldots, 2m} \lambda_{\min}(W_k); \]

\[ \lambda_{\max} = \max_{k=0, \ldots, 2m} \lambda_{\max}(W_k); \]

where \( \lambda_{\min}(W_k) \) and \( \lambda_{\max}(W_k) \) denote the smallest and the largest eigenvalue of the positive definite matrix \( W_k \), respectively.

Exponential stability of (1) implies that matrix \( U(\xi) \) is well defined on \([i; h; h]\). So, we can compute

\[ 1 = \max_{\xi \in [i; h]} kU(\xi)k; \]

and

\[ a = \max_{k=1, \ldots, m} kA_kk; \]

Now,

\[ x(t)U(0)x(t) \leq kx(t)k^2; \]
and

\[ Z \left( 2x(t) \prod_{i \leq k} \mu A \right) x(t + \mu) d\mu \]

\[ 1 \prod_{i \leq k} k x(t)^2 + 1 \prod_{i \leq k} k x(t + \mu) k^2 d\mu; \]

In a similar way

\[ Z \prod_{i \leq k} (W_k + (h_k + \mu) W_{m+k}) x(t + \mu) d\mu \]

\[ (1 + h_k \max) \prod_{i \leq k} k x(t + \mu) k^2 d\mu; \]

for \( k = 1; \ldots; m \).
In order to find an upper estimate of the double integrals in (6) we make use of the fact that by the Cauchy-Schwartz inequality we have

\[ \mu Z \text{ } \| Z \text{ } \begin{array}{c}
\int_{i}^{h_i}
\int_{h_i}
kx(t + \mu)k d\mu
\end{array}
\]

\[ \int_{i}^{h_i}
\int_{h_i}
kx(t + \mu)k^2 d\mu
\]

for \( i = 1; 2; \ldots; m \).

So, for all \( k, j = 1; 2; \ldots; m \)

\[ Z \text{ } Z \text{ } \begin{array}{c}
\int_{i}^{h_k}
\int_{h_k}
x(t + \mu_2)A_k
\end{array}
\]

\[ U(\mu_2 i \mu_1 h_k h_j) \]

\[ \begin{array}{c}
\int_{i}^{h_1}
\int_{h_1}
A_j x(t + \mu_1) d\mu_1 d\mu_2
\end{array}
\]

\[ \begin{array}{c}
\int_{i}^{h_2}
\int_{h_2}
kx(t + \mu_2)k d\mu_2
\end{array}
\]

\[ \begin{array}{c}
\int_{i}^{h_1}
\int_{h_1}
kx(t + \mu_1)k d\mu_1
\end{array}
\]

\[ \begin{array}{c}
\int_{i}^{h_k}
\int_{h_k}
kx(t + \mu)k d\mu
\end{array}
\]

\[ \begin{array}{c}
\int_{i}^{h_j}
\int_{h_j}
kx(t + \mu)k d\mu
\end{array}
\]

\[ \begin{array}{c}
\int_{i}^{h_k}
\int_{h_k}
kx(t + \mu)k^2 d\mu
\end{array}
\]

\[ \begin{array}{c}
\int_{i}^{h_j}
\int_{h_j}
kx(t + \mu)k^2 d\mu
\end{array}
\]
As a consequence we obtain the following upper bound for $v(x_t)$:

$$v(x_t) \pm_0 k x(t) k^2 + \sum_{k=1}^{\infty} k x(t + \mu) k^2 d\mu$$

where

$$\pm_0 = 1 + \frac{1}{1 + a} \prod_{k=1}^{\infty} h_k$$

$$\pm_k = 1 + (1 + h_k)_{\text{max}} + \frac{m+1}{2} \frac{1}{h_k a^2}; \ k = 1; \ldots; m$$

Therefore we can select

$$\bar{\varrho}_2 = \pm_0 + \sum_{k=1}^{\infty} \pm_k h_k.$$
Now we find $\mathbb{C}_k$. To this end we consider functional

$$v_1(x_t) = v(x_t) \mathbb{C}_k x(t) k^2$$

By direct calculations we have

$$\frac{d}{dt} v_1(x_t) = w_1(x_t);$$

where

$$w_1(x_t) = w(x_t) \mathbb{C}_k x(t) \mathbb{H}_k$$

$$= \mathbf{X}(t) \mathbb{X}(t; \mathbb{H}_m) \mathbf{A} \mathbf{x}(t; \mathbb{H}_m)$$

$$+ \sum_{k=1}^{\infty} \mathbf{x}(t; \mathbb{H}_k) W_{m+k} x(t; \mathbb{H}_k) d\mu;$$
Here

\[ L = \text{diag}(W_0; W_1; \ldots; W_m) \]

\[
\begin{pmatrix}
0 & A_0 + A_1 & A_1 & \cdots & A_m & 1 \\
A_0^T & 0 & A_1 & \cdots & A_m & 0 \\
A_1^T & 0 & 0 & \cdots & A_m & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
A_m^T & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

It is evident that there exists \( \@ = \@_1 > 0 \) such that

\[ L \succ 0 \]

For this value of \( \@ \)

\[ w_1(x_t) \succ 0; \]

and

\[ Z \]

\[ v(\cdot) = \int_0 w(x_t(\cdot)) \, dt \succ 0; \]

As a result,

\[ \@_2 k(x)(k^2 v(x_t): \]

In particular one can get \( \@_2 \) as the largest positive for which \( L \succ 0 \).
Lemma 2: If system (1) is exponentially stable and $W_0, W_1, \ldots, W_{2m}$ are positive definite real $n \times n$ matrices then there exist positive constant $\bar{\gamma}$, such that

$$2^\gamma v(x_t) \leq w(x_t).$$

Proof: We already know that

$$v(x_t) \leq \pm_k k x(t) k^2$$

$$+ \sum_{k=1}^{\infty} \pm_{i_k} k x(t + \mu) k^2 \, d\mu.$$ 

If we select $\bar{\gamma}$ in such a way that

$$2^\gamma \leq \pm_k \min(W_k)$$

and

$$2^\gamma \leq \pm_k \min(W_{m+k})$$

for $k = 1, 2, \ldots, m$. Then

$$2^\gamma v(x_t) \leq w(x_t).$$
Remark 3: The above proof shows that every quadratic functional $w$ of the form (4) with positive definite matrices $W_0, \ldots, W_{2m}$ satisfies the second condition of Proposition 1 with the constant $\bar{\gamma}$ given by Lemma 2. The first $(m + 1)$ terms in (4) were needed to prove that the corresponding functional $v$ (6) satisfies the left inequality in the first condition of Proposition 1. The last $m$ terms in (4), along with the first term, were used in the proof to derive the second inequality. So, $w$ defined by (4) can be viewed as a quadratic functional with a minimum number of quadratic terms to yield a Lyapunov-Krasovskii functional $v$ satisfying the first condition in Proposition 1.

As mentioned above the integral terms in (4) are missing in the definition of $w$ in [Infante-Castelan, 1978]. This is compensated by an additional exponential factor $e^{\pm t}; \pm > 0$ in the definition of $v$, where $|\pm|$ is supposed to be strictly greater than the exponential growth rate of the delay system. Therefore the construction of Infante and Castelan requires some a priori knowledge about the spectral abscissa of the system.
Remark 4: Clearly the exponential estimate obtained by Theorem 1 depends on the choice of the matrices $W_k > 0, k = 0; 1; \ldots; 2m$. These matrices may serve as free parameters in an optimization of the estimate. In this note we do not try to obtain tight estimates, we only wish to demonstrate that the above Lyapunov-Krasovskii approach yields a systematic procedure for determining exponential estimates for an exponentially stable delay system (1) without any additional a priori information.
5 Example

Consider the system

\[
\frac{dx(t)}{dt} = \begin{bmatrix} \mu & 1 & 0 \\ 0 & i & 2 \\ 0 & i & 2 \end{bmatrix} x(t) + \begin{bmatrix} \mu & 0 & 0.7 \\ 0 & 0 & 0 \end{bmatrix} x(t-i) + \begin{bmatrix} \mu & 0.49 \\ 0 & i & 0.49 \end{bmatrix} x(t-2i): (\pm)
\]

The characteristic quasipolynomial of the system is

\[
f(s) = s^2 + 3s + 2 + 0.98e^{i2s} + 0.98se^{i2s} + [0.49]^2 e^{i4s}.
\]

All the roots of this quasipolynomial lie in the open left half complex plane. The roots closest to the imaginary axis are

\[
s_{1,2} = i \ 0.582 \ \& \ j \ 0.766
\]

so that the spectral abscissa of the system is \( i \ 0.582 \).
Let us choose

\[ W_k = 1; \quad k = 0; 1; 2; 3; 4; \]

so that the functional (4) is given by

\[
w(x_t) = k\alpha(t)k^2 + k\alpha(t + 1)k^2 + k\alpha(t + 2)k^2
\]

\[
\begin{align*}
Z & \quad Z \\
+ & \quad k\alpha(t + \mu)k^2 d\mu + k\alpha(t + \mu)k^2 d\mu \\
& \quad i \quad 1 \\
& \quad i \quad 2
\end{align*}
\]

Obviously

\[
,_{\min} = ,_{\max} = 1:
\]

The Lyapunov matrix function

\[
U(\dot{\xi}) = 6 \int_0^Z K(t)K(t + \dot{\xi})d\dot{\xi}
\]

satisfies the equation

\[
\frac{dU(\dot{\xi})}{d\dot{\xi}} = \mu U(\dot{\xi}) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\]

\[
+ U(\dot{\xi}i) \begin{bmatrix} \mu & 0.49 \\ 0.7 & 0 \end{bmatrix} U(\dot{\xi}i) \begin{bmatrix} 0.7 & 0.49 \\ 0 & 0.49 \end{bmatrix}
\]
On Fig. 1 we plot the four components of a piecewise linear approximation of the Lyapunov matrix function.

Matrix \( U(\mathcal{E}) \)

From the plot we get

\[ l = 3.44 \] and \( a = 0.7 \):
Functional \( v \) satisfies the inequality

\[
v(x(t)) \pm_0 kx(t)k^2 + \pm_1 x(t+\mu) \mu \, \text{d} \mu + \pm_2 k^2 \mu \, \text{d} \mu:
\]

where

\[
\pm_0 = (1+3a) = 10:66
\]

\[
\pm_1 = a + 2 + \frac{3}{2}a^2 = 6:94
\]

\[
\pm_2 = a + 3 + 3\frac{1}{2}a^2 = 10:46
\]

The value \( \circ_2 = \pm_0 + \pm_1 h_1 + \pm_2 h_2 = 38:53 \)

The value \( \circ_1 = 0:23 \).

Now, the value \( - \) should be such that

\[
2^\pm_1 \pm_1 \mu; \quad k = 0; 1; 2;
\]

And we have

\[
- = 0:047
\]
As a result we obtain the following exponential estimate for the solutions of the system \((\pm)\) with the following constants

\[ r = \frac{\overline{\theta}_2}{\overline{\theta}_1} \approx 1.1294; \quad \overline{\eta}_4 = -1/4 \times 0.047; \]
Figure 1: Matrix $U(\hat{\zeta})$, piece wise linear approximation
In order to verify how well the piecewise linear approximation represents the Lyapunov matrix valued function we compute the solution of equation (±±) with the initial condition generated by the linear piecewise approximation as initial function on \([i; h; 0]\).

The four components of the corresponding solution are plotted on Fig. 2. A comparison Fig. 1 shows a good fit between the solution \(U(\hat{z})\) and the piecewise linear approximation on \([0; h]\).

Figure 2: Matrix \(U(\hat{z})\), numerical solution