Digit(al)ization of nonautonomous control systems

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References


Nonautonomous dynamical systems

Let \((X, d_X)\) and \((P, d_P)\) are complete metric spaces.

We define a nonautonomous dynamical system \((\theta, \phi)\) in terms of a cocycle mapping \(\phi\) on a state space \(X\) which is driven by an autonomous dynamical system \(\theta\) acting on a base or parameter space \(P\).

Specifically, \(\theta = \{\theta_t : t \in \mathbb{R}\}\) is a dynamical system on \(P\), i.e., a group of homeomorphisms under composition on \(P\) with the properties that

1) \(\theta_0(p) = p\) for all \(p \in P\);
2) \(\theta_{s+t} = \theta_s(\theta_t(p))\) for all \(s, t \in \mathbb{R}\);
3) the mapping \((t, p) \mapsto \theta_t(p)\) is continuous,

and the cocycle mapping \(\phi : \mathbb{R}^+ \times P \times X \to X\) satisfies

1) \(\phi(0, p, x) = x\) for all \((p, x) \in P \times X\);
2) \(\phi(s + t, p, x) = \phi(s, \theta_t(p), \phi(t, p, x))\) for all \(s, t, \in \mathbb{R}^+, (p, x) \in P \times X\);
3) the mapping \((t, p, x) \mapsto \phi(t, p, x)\) is continuous.
Examples

Example 1. An autonomous differential equations with the triangular form

\[
\frac{dx}{dt} = f(x, p), \quad \frac{dp}{dt} = g(p)
\]

generates an NDS with the uncoupled component for \( p \) being considered as “driving” the coupled or “driven” system for \( x \). Here \( \theta_t(p_0) = p(t, p_0) \).

Example 2. A general nonautonomous differential equation

\[
\frac{dx}{dt} = f(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d,
\]

generates an NDS with \( P = \mathbb{R} \) and \( \theta_t(t_0) = t + t_0 \).

Example 3. A scalar nonautonomous differential equation

\[
\frac{dx}{dt} = x - p(t)x^3;
\]

where \( p : \mathbb{R} \to \mathbb{R} \) is almost periodic, generates an NDS with the shift operators

\[
\theta_t p(\cdot) := p(t + \cdot) \quad \text{for all} \quad t \in \mathbb{R},
\]

and the base space

\[
P := \text{cls} \{ p(t + \cdot) : t \in \mathbb{R} \},
\]

which is a compact metric space with the metric corresponding to the norm \( \| \cdot \|_\infty \).
Attractors of nonautonomous dynamical systems

Consider a family \( \hat{A} = \{ A_p : p \in P \} \) of nonempty compact subsets \( A_p \) of \( X \), which is invariant under the cocycle mapping, i.e.,

\[
\phi(t, p, A_p) = A_{\theta_t(p)} \quad \text{for all} \quad t \geq 0
\]

\( \hat{A} \) is called a pullback attractor of \((\theta, \phi)\) if it is pullback attracting, i.e.,

\[
\lim_{t \to \infty} H_X^*(\phi(t, \theta_{-t}(p), D), A_p) = 0
\]

for any nonempty bounded subset \( D \) of \( X \) and \( p \in P \).

\( \hat{A} \) is called a forward attractor if the forward convergence

\[
\lim_{t \to \infty} H_X^*(\phi(t, p, D), A_{\theta_t(p)}) = 0
\]

holds instead of pullback convergence.

The existence of a pullback attractor follows from the existence of a uniform absorbing set \( B \), which is a nonempty compact subset of \( X \) and absorbs nonempty bounded subsets \( D \) of \( X \), i.e.,

\[
\exists T_{D,p} \geq 0 \quad \text{such that} \quad \phi(t, p, D) \subset B \quad \text{for all} \quad t \geq T_{D,p}.
\]

It is uniformly absorbing if \( T_{D,p} \) is independent of \( p \in P \).

If \( B \) is also \( \phi \)-positively invariant i.e.,

\[
\phi(t, p, B) \subset B \quad \text{for all} \quad t \geq 0, \ p \in P,
\]

then the nonautonomous dynamical system \((\theta, \phi)\) has a pullback attractor \( \hat{A} = \{ A_p : p \in P \} \) with component sets given by

\[
A_p = \bigcap_{t \geq 0} \phi(t, \theta_{-t}(p), B) \quad \text{for each} \quad p \in P.
\]
Example of pullback attractors

Consider again the scalar nonautonomous differential equation

\[ \frac{dx}{dt} = x - p(t)x^3, \]

where \( p : \mathbb{R} \to \mathbb{R} \) is almost periodic with \( 0 < p_0 \leq p(t) \leq p_1 \).

The solution of this Bernoulli equation with initial value \( x(t_0) = x_0 \) is

\[ x(t, t_0, x_0) = \frac{\text{sgn}(x_0)}{\sqrt{\frac{1}{x_0} e^{-2(t-t_0)} + 2 \int_{t_0}^t p(s)e^{-2(t-s)} ds}}. \]

The pullback limit gives

\[ x(t, t_0, x_0) \to \text{sgn}(x_0) \tilde{\phi}_p(t) := \frac{\text{sgn}(x_0)}{\sqrt{2 \int_{-\infty}^t p(s)e^{-2(t-s)} ds}} \text{ as } t_0 \to -\infty, \]

where \( \tilde{\phi}_p \) is almost periodic.

The pullback attractor in this process formalism has component sets

\[ A(t) := [-\tilde{\phi}_p(t), \tilde{\phi}_p(t)], \quad \forall \ t \in \mathbb{R}, \]

and the setvalued mapping \( t \mapsto A(t) \) is almost periodic.

The above differential equation also generates an NDS with \( \theta_{\Delta p}(\cdot) := p(t + \cdot) \) on the compact base space \( P := \text{cls} \{ p(t + \cdot) : t \in \mathbb{R} \} \) in the topology of the norm \( \| \cdot \|_\infty \).

The pullback attractor in this NDS formalism then has component sets

\[ A(p) := [-\tilde{\phi}_p(0), \tilde{\phi}_p(0)], \quad \forall \ p \in P. \]
Parametric dependence: an example

Consider the scalar nonautonomous differential equation

\[ \frac{dx}{dt} = \nu x - p(t)x^3, \]

where \( p : \mathbb{R} \to \mathbb{R} \) is almost periodic with \( 0 < p_0 \leq p(t) \leq p_1 \) and \( \nu > 0 \) is a parameter.

The pullback attractor \( \hat{A}^{(\nu)} \) in this process formalism has component sets

\[ A^{(\nu)}(t) := [-\bar{\phi}_{p,\nu}(t), \bar{\phi}_{p,\nu}(t)], \quad \forall \ t \in \mathbb{R}. \]

where

\[ \bar{\phi}_{p,\nu}(t) := \frac{\text{sgn}(x_0)}{\sqrt{2 \int_{-\infty}^{t} p(s)e^{-2\nu(t-s)} ds}}. \]

The above differential equation also generates an NDS with \( \theta_t p(\cdot) := p(t + \cdot) \) on the compact base space \( P := \text{cls} \{ p(t + \cdot) : t \in \mathbb{R} \} \) in the topology of the norm \( \| \cdot \|_\infty \).

The pullback attractor in this NDS formalism then has component sets

\[ A^{(\nu)}_p := [-\bar{\phi}_{p,\nu}(0), \bar{\phi}_{p,\nu}(0)], \quad \forall \ p \in P. \]

The component sets of a pullback attractor depend upper semicontinuously on a parameter, i.e.

\[ H^* \left( A^{(\nu)}(t), A^{(\nu_0)}(t) \right) \to 0 \quad \text{as} \quad \nu \to \nu_0, \quad \forall t \in \mathbb{R} \]

and

\[ H^* \left( A^{(\nu)}_p, A^{(\nu_0)}_p \right) \to 0 \quad \text{as} \quad \nu \to \nu_0, \quad \forall p \in P \]
Synchronization of dissipative systems

Consider two dissipative nonautonomous dynamical systems in $\mathbb{R}^d$

$$\frac{dx}{dt} = f(p, x), \quad p \in P,$$

with driving system $\theta_t : P \to P$, and

$$\frac{dy}{dt} = g(q, x), \quad q \in Q,$$

with driving system $\psi_t : Q \to Q$.

Suppose both systems satisfy a uniform dissipativity condition

$$\langle x, f(p, x) \rangle \leq K - L|x|^2, \quad p \in P,
\langle x, g(q, x) \rangle \leq K - L|x|^2, \quad q \in Q,$$

In both cases the closed ball

$$B_d[0, \sqrt{(K + 1)/L}] := \{ x \in \mathbb{R}^d ; |x|^2 \leq (K + 1)/L \}$$

is uniformly absorbing and positively invariant.

Thus both systems have pullback attractors in $\mathbb{R}^d$

$$\hat{A}^{(f)} = \{ A_p^{(f)} : p \in P \}, \quad \hat{A}^{(g)} = \{ A_q^{(g)} : q \in Q \}.$$
Synchronized systems

Let $\nu > 0$ and consider the dissipatively coupled system

\[
\frac{dx}{dt} = f(p, x) + \nu(y - x), \quad \frac{dy}{dt} = g(q, x) + \nu(x - y)
\]

with the product driving system $(\theta_t, \psi_t) : P \times Q \to P \times Q$.

By the uniform dissipativity condition we have

\[
\frac{d}{dt} \left( |x|^2(t) + |y|^2(t) \right) \leq 4K - 2L \left( |x(t)|^2 + |y(t)|^2 \right)
\]

so the closed ball

\[
B_{2d}[0, \sqrt{(2K + 1)/L}] := \{ x \in \mathbb{R}^{2d} ; |x|^2 \leq (2K + 1)/L \} \quad \text{in } \mathbb{R}^{2d}
\]

is uniformly absorbing and positively invariant for the coupled system, which thus has a pullback attractor

\[
\widehat{A}^{(\nu)} = \left\{ A_{(p,q)}^{(\nu)} : (p, q) \in P \times Q \right\}
\]

in $\mathbb{R}^{2d}$ for each $\nu > 0$. 
Write
\[ (x^{(\nu)}(t), y^{(\nu)}(t)) = (x^{(\nu)}(t, p, q, x_0, y_0), y^{(\nu)}(t, p, q, x_0, y_0)) \]
for the solution of the coupled system with initial parameter value \((p, q)\) and initial state \((x_0, y_0)\).

**Theorem 1** For all finite \(T_2 \geq T_1 > 0\), all \((x_0, y_0) \in B_{2d}(0, \sqrt{(2K + 1)/L})\) and all \((p, q)\)
\[
\lim_{\nu \to \infty} |x^{(\nu)}(t) - y^{(\nu)}(t)| = 0 \quad \text{uniformly in } t \in [T_1, T_2].
\]

**Theorem 2** Let \(\text{Diag } (\mathbb{R}^d \times \mathbb{R}^d) = \{ (x, x) \ ; \ x \in \mathbb{R}^d \}\). Then
\[
\lim_{\nu \to \infty} H^*_2 \left( A^{(\nu)}_{(p,q)}, \text{Diag } (\mathbb{R}^d \times \mathbb{R}^d) \bigcap B_{2d}(0, \sqrt{(2K + 1)/L}) \right) = 0.
\]

We can say much more about the dynamics inside the pullback attractor \(\tilde{A}^{(\nu)}\) of the coupled system as \(\nu \to \infty\).
Theorem 3 For any entire trajectory $(x^{(\nu)}(t), y^{(\nu)}(t))$ of the coupled system inside the pullback attractor $\hat{A}^{(\nu)}$ there exists convergent subsequences

$$\lim_{\nu \to \infty} x^{(\nu)}(t) = z(t), \quad \lim_{\nu \to \infty} y^{(\nu)}(t) = z(t)$$

uniformly on compact time subintervals in $\mathbb{R}$, where $z(t)$ is a solution of the nonautonomous differential equation

$$\frac{dz}{dt} = \frac{1}{2} (f(p, z) + g(q, z)) \quad (*)$$

with the product driving system $(\theta_t, \psi_t) : P \times Q \to P \times Q$.

It follows from the uniform dissipativity condition that the closed ball

$$B_d[0, \sqrt{(K + 1)/L}] := \{ x \in \mathbb{R}^d ; \ |x|^2 \leq (K + 1)/L \}$$

is uniformly absorbing and positively invariant for the limiting system $(*)$, which thus has a pullback attractor

$$\hat{A}^{(\infty)} = \left\{ A_{(p,q)}^{(\infty)} : (p, q) \in P \times Q \right\}$$

in $\mathbb{R}^d$.

Corollary 1 Let $\text{Diag} \left( A_{(p,q)}^{(\infty)} \times A_{(p,q)}^{(\infty)} \right) = \left\{ (x, x) ; x \in A_{(p,q)}^{(\infty)} \right\}$. Then

$$\lim_{\nu \to \infty} H_2^* \left( A_{(p,q)}^{(\nu)}, \text{Diag} \left( A_{(p,q)}^{(\infty)} \times A_{(p,q)}^{(\infty)} \right) \right) = 0.$$
Pullback attractors under digitization: an example

The scalar nonautonomous differential equation

$$\frac{dx}{dt} = x - p(t)x^3,$$

where $p$ is almost periodic with $0 < p_0 \leq p(t) \leq p_1$, has a pullback attractor with component sets (process, NDS formalisms)

$$A(t) := \left[-\bar{\phi}_p(t), \bar{\phi}_p(t)\right], \quad A_p := \left[-\bar{\phi}_p(0), \bar{\phi}_p(0)\right]$$

Suppose we replace $p$ by a function $p_\delta : \mathbb{R} \to \mathbb{R}$ such that

$$\|p_\delta - p\| := \sup_{t \in \mathbb{R}} |p_\delta(t) - p(t)| \to 0 \quad \text{as} \quad \delta \to 0.$$

For example, $p_\delta$ could be a quasi-periodic approximation of $p$ or given by

$$p_\delta(t) = \begin{cases} p(n\delta) & t \in [n\delta, (n+1)\delta), \ n \in \mathbb{Z}. \end{cases}$$

The perturbed nonautonomous differential equation

$$\frac{dx}{dt} = x - p_\delta(t)x^3$$

has a pullback attractor with component sets (process, NDS formalisms)

$$A_\delta(t) := \left[-\bar{\phi}_{p_\delta}(t), \bar{\phi}_{p_\delta}(t)\right], \quad A_{p_\delta} := \left[-\bar{\phi}_{p_\delta}(0), \bar{\phi}_{p_\delta}(0)\right]$$

The component sets depend (upper semi) continuously on the parameter $\delta$, i.e.

$$H(A_\delta(t), A(t)) \to 0 \quad \text{as} \quad \delta \to 0, \ \forall t \in \mathbb{R}$$

and

$$H(A_{p_\delta}, A_p) \to 0 \quad \text{as} \quad \delta \to 0, \ \forall p \in P.$$
Digitization of nonautonomous systems

Consider the nonautonomous ODE

\[ x' = f(t, x), \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \]

where \( f \in \mathcal{F} \).

Here \( \mathcal{F} \) denotes the vector space of functions \( f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) satisfying

i) For each compact set \( K \subset \mathbb{R}^d \), \( f \) is uniformly continuous on \( \mathbb{R} \times K \);

ii) For each compact set \( K \subset \mathbb{R}^d \), there exists a constant \( L_K \) (also depending on \( f \)) so that

\[ ||f(t, x) - f(t, y)|| \leq L_K ||x - y|| \quad \text{for all } x, y \in K, t \in \mathbb{R}. \]

Put the metrizable topology of uniform convergence on compact sets on \( \mathcal{F} \).

Consider the Bebutov flow \( \{ \theta_t : t \in \mathbb{R} \} \) on \( \mathcal{F} \) defined by translation of the \( t \)-variable, i.e.,

\[ \theta_t(f)(s, x) := f(t + s, x) \]

for all \( f \in \mathcal{F}, x \in \mathbb{R}^d \) and \( s, t \in \mathbb{R} \).

Then \( P := \text{cls}\{ \theta_t(f) : t \in \mathbb{R} \} \subset \mathcal{F} \) is compact for \( f \in \mathcal{F} \).

The above nonautonomous ODE generates a NDS with base space \( P \) and the Bebutov flow as its driving system.
Digitization

A digitization is a procedure which assigns to each \( f \in \mathcal{F} \) and to each real number \( \delta > 0 \) the following:

1) There is a collection \( \mathcal{I}^\delta = \{ I^\delta_j : j \in \mathbb{Z} \} \) of nonempty half-open intervals in \( \mathbb{R} \) such that \( \bigcup_{j=-\infty}^{\infty} I^\delta_j = \mathbb{R} \), and such that each interval \( I^\delta_j \) has length \( \leq \delta \) and (say) \( \geq \delta/2 \).

2) To each \( f \in \mathcal{F} \) there is associated a collection \( \{ f^\delta_j : \delta > 0, j \in \mathbb{Z} \} \) of autonomous vector fields. There is a positive function \( \omega = \omega(\epsilon) \), defined for positive values of \( \epsilon \) and tending to zero as \( \epsilon \to 0^+ \), such that for each interval \( I^\delta_j \in \mathcal{I}^\delta \) and each \( x \in \mathbb{R}^d \) the following property holds: if \( \epsilon_x = \sup \{ \| f(r, x) - f(s, x) \| : r, s \in I^\delta_j \} \), then

\[
\| f^\delta_j(x) - f(t, x) \| \leq \omega(\epsilon_x), \quad t \in I^\delta_j.
\]

3) There is a positive function \( \omega_1 = \omega_1(M) \), which is defined for positive values of \( M \) and which depends only on \( M \), such that, if \( x, y \in \mathbb{R}^d \) satisfy \( \| f(t, x) - f(t, y) \| \leq M \) for all \( t \) in some interval \( I^\delta_j \), then

\[
\| f^\delta_j(x) - f^\delta_j(y) \| \leq \omega_1(M)\| x - y \|
\]

for all \( \delta > 0 \).

4) There is a positive function \( \omega_2 = \omega_2(\eta) \), defined for positive values of \( \eta \) and tending to zero as \( \eta \to 0^+ \), such that, if \( J \subset \mathbb{R} \) is an interval and if \( x \in \mathbb{R}^d \) is a point, and if \( f, \tilde{f} \in \mathcal{F} \) satisfy \( \| f(t, x) - \tilde{f}(t, x) \| \leq \eta \) for all \( t \in J \), then

\[
\| f^\delta_j(x) - \tilde{f}^\delta_j(x) \| \leq \omega_2(\eta)
\]

for all \( \delta > 0 \) and all \( j \) such that \( I^\delta_j \subset J \).
The digitized system

Replace \( f \in \mathcal{F} \) by \( f_\delta \) defined by digitization

\[
f_\delta(t, x) = f_j^\delta(x) \quad \text{for} \quad t \in I_j^\delta, \ j \in \mathbb{Z},
\]

and consider the nonautonomous differential equation

\[
x' = f_\delta(t, x).
\]

To handle temporally discontinuous \( f_\delta \), we need to enlarge the space \( \mathcal{F} \).

Let \( \mathcal{G} \) be the class of jointly Lebesgue measurable mappings \( g \in \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) satisfying:

\[
a) \quad \text{For each compact set } K \subset \mathbb{R}^d, \text{ one has } \sup_{x \in K} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s, x)\| \, ds < \infty;
\]

\[
b) \quad \text{For each compact set } K \subset \mathbb{R}^d \text{ there is a constant } L_K \text{ (depending on } g) \text{ so that, for almost all } t \in \mathbb{R}:
\]

\[
\|g(t, x) - g(t, y)\| \leq L_K \|x - y\|, \quad x, y \in K,
\]

with the metric

\[
d(g_1, g_2) = \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{d_r(g_1, g_2)}{1 + d_r(g_1, g_2)},
\]

where

\[
d_r(g_1, g_2) = \sum_{r=1}^{\infty} \frac{1}{2^N} \frac{d_{r,N}(g_1, g_2)}{1 + d_{r,N}(g_1, g_2)}, \quad d_{r,N}(g_1, g_2) = \sup_{\|x\| \leq r} \int_{-N}^{N} \|g_1(s, x) - g_2(s, x)\| \, ds
\]

for \( r = 1, 2, \ldots \) and \( N = 1, 2, \ldots \).

Then \( P_\delta := \text{cls}\{\theta_t(f_\delta) : t \in \mathbb{R}\} \subset \mathcal{G} \) is compact for \( f_\delta \in \mathcal{G} \).
Pullback attractors under digitization: general case

Suppose that $f \in \mathcal{F}$ satisfies the dissipativity condition:

$$\langle f(t, x), x \rangle < 0$$

for all $t \in \mathbb{R}$ and $\|x\| \geq R$ for some $R > 0$.

Then the nonautonomous differential equation

$$x' = f(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d,$$

generates an NDS on $\mathbb{R}^d$ with the compact base space

$$P := \text{cls}\{\theta_t(f) : t \in \mathbb{R}\} \subset \mathcal{F}$$

which has a pullback attractor $\hat{A} = \{A_p : p \in P\}$.

Consider a digitization $f_\delta \in \mathcal{G}$ of $f$. Then the nonautonomous differential equation

$$x' = f_\delta(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d,$$

generates an NDS on $\mathbb{R}^d$ with the compact base space

$$P_\delta := \text{cls}\{\theta_t(f_\delta) : t \in \mathbb{R}\} \subset \mathcal{G}$$

which has a pullback attractor $\hat{A}_\delta = \{A_{p_\delta} : p_\delta \in P_\delta\}$.

**Theorem 4**

$$H^*_\mathcal{G}(P_\delta, P) \to 0 \quad \text{as} \quad \delta \to 0$$

**Theorem 5**

$$H^*_\mathbb{R}^d(A_{p_\delta}, A_p) \to 0 \quad \text{as} \quad \delta \to 0, \quad \text{uniformly in} \quad p \in P.$$
Controllability of nonautonomous control systems

Let $A(\cdot)$ and $B(\cdot)$ be bounded measurable matrix–valued functions of size $n \times n$ and $n \times m$ and consider the linear nonautonomous control system

$$x' = A(t)x + B(t)u \quad (x \in \mathbb{R}^n, \ u \in \mathbb{R}^m), \quad (**)$$

where the control functions $u(\cdot)$ are measurable and take values in a compact convex subset $U$ of $\mathbb{R}^m$, which contains the origin.

**Definition 1** The linear control system (***) is said to be locally null controllable if there exists an open neighbourhood $V$ of the origin in $\mathbb{R}^n$ and a finite time $T > 0$ such that, to each $x_0 \in V$, there corresponds a measurable function $u : [0, T] \to U$ such that the solution $x(t)$ of (***) with $x(0) = x_0$ determined by this $u = u(t)$ satisfies $x(T) = 0$.

Let $H_U$ be the support function of $U$, i.e.,

$$H_U(x) = \sup_{u \in U} \langle u, v \rangle.$$

**Theorem 6 (Barmish–Schmitendorf criterion)** The linear control system (***) is locally null controllable if and only if there exists a positive number $\epsilon_0$ such that

$$\int_0^\infty H_U (B^\top(t)y(t)) \ dt \geq \epsilon_0$$

holds for every solution $y(t)$ of the adjoint system

$$y' = -A^\top(t)y \quad (y \in \mathbb{R}^n)$$

satisfying $|y(0)| = 1$. 
Let \( B_R = \left\{ (\tilde{A}, \tilde{B}) \in \mathcal{L}_{n,n} \times \mathcal{L}_{n,m} : |\tilde{A}|_\infty \leq R, |\tilde{B}|_\infty \leq R \right\} \) for each \( R > 0 \).

**Proposition 1** Suppose that the linear control system (**\( ** \)) is locally null controllable and that \( R > \max \{ |A|_\infty, |B|_\infty \} \). Then there is a weak* neighbourhood \( \mathcal{W} \) of \((A, B)\) in \( B_R \) such that the linear control system

\[
x' = \tilde{A}(t)x + \tilde{B}(t)u
\]

is locally null controllable for every \((\tilde{A}, \tilde{B}) \in \mathcal{W}\).

Consider a compact translation-invariant family \( P \) of control systems (**\( ** \))

\[
x' = A(\theta_t(p))x + B(\theta_t(p))u \quad (++)\tag{+++}
\]

for all \( p \in P \).

For example, \( P \) is the hull \( \text{cls} \left\{ (\theta_t(A), \theta_t(B)) : t \in \mathbb{R} \right\} \) of \((A, B)\) in \( \mathcal{L}_{n,n} \times \mathcal{L}_{n,m} \).

**Proposition 2** Suppose for each minimal subset \( M \subset P \) that there is at least one point \( p \in M \) such that the system (+++) is locally null controllable. Let \( R > 0 \) be a number such that \( P \subset B_R \). Then there is a weak* neighbourhood \( \mathcal{W} \subset B_R \) such that the family of control systems \( \{ (+++) : p \in \hat{P} \} \) is uniformly locally null controllable for any \( \hat{P} \) is a weak* compact translation–invariant subset of \( \mathcal{W} \).
Controllability and stabilization under digitization

We can apply the above Propositions to a digitization

\[ x' = A_\delta(\theta_t(p)) x + B_\delta(\theta_t(p)) u \quad (--) \]

of a control system system \((++),\) where the digitization preserves the linear character of the nonautonomous vector fields \(A(t)x\) and \(B(t)u.\)

**Theorem 7  Controllability**

*If system \((++\) with uniformly continuous coefficient matrices \(A(t)\) and \(B(t)\) is locally null controllable, then so is system \((-\) for \(\delta\) sufficiently small.*

*If the family \(P = \text{cls} \{\theta_t(A,B) : t \in \mathbb{R}\} \subset \mathcal{L}\) is uniformly locally null controllable, then so are the systems corresponding to the elements in the hull \(P_\delta\) of the digitized \(p_\delta\) in \(\mathcal{G}\) provided \(\delta\) is sufficiently small.*

**Theorem 8  Stabilization**

*If \(u(t,x)\) stabilizes the system \((++\) and \(u_\delta(t,x)\) stabilizes the system \((-\)) then there exists a constant \(L\) such that*

\[ \|u(t,x) - u_\delta(t,x)\| \leq L\delta\|x\| \]

*for all \(t \in \mathbb{R}\) and \(x \in \mathbb{R}^d.\)