


# Path-Following for Nonlinear Systems Subject to Constraints

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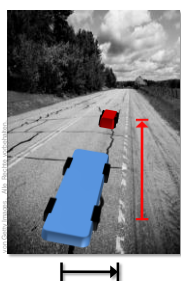
## Path-Following for Nonlinear Systems Subject to Constraints

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### Steering a Car as a Control Problem



**Stabilisation**

- control distance
- velocity?

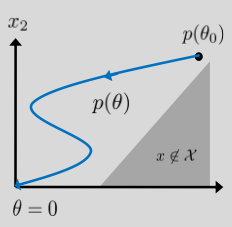
**Trajectory Tracking**

- determine reference offline
- design trajectory tracking controller
- robustness?

**Path-Following**

- geometric reference
- determine reference velocity online

### Path-Following Problem



$\dot{x} = f(x, u), \quad x(0) = x_0$

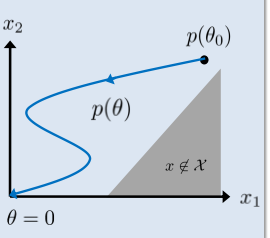
- path  $\subset$  regular, 1D curve  
 $\mathcal{P} = \{p(\theta) \in \mathbb{R}^n : \theta \in [\theta_0, 0] \mapsto p(\theta)\}$
- final point = origin  $\lim_{\theta \rightarrow 0} \|p(\theta)\| = 0$
- given initial point  $p(\theta_0)$

$\dot{x} = f(x, u), \quad x(0) = x_0$

**Control Task**

- path convergence  $\lim_{t \rightarrow \infty} \|x(t) - p(\theta(t))\| = 0$
- forward motion  $\forall \theta \in [\theta_0, 0) : \dot{\theta}(t) > 0$
- satisfaction of constraints  $x \in \mathcal{X}, \quad u \in \mathcal{U}$

### Existing Results



- linear systems [Aguilar et al. '05; Dacic & Kokotovic '06]
- nonlinear systems [Aguilar et al. '08]
- robust path-following [Skjetne, Kokotovic et al. '05; Do et al. '06]
- passive systems [El-Hawary '08]
- feedforward control of robots [Shin & McKay '85; Verscheure et al. '09]

**Usually, no consideration of constraints**

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### Trajectory Tracking and Path-Following

reference trajectory  $t \mapsto r(t) \in \mathbb{R}^n$

tracking error  
 $e(t) = x(t) - r(t)$   
 $\dot{e}(t) = f(x, u) - \dot{r}(t)$

time variant problem  
 performance limits?

reference path  $\theta \mapsto p(\theta) \in \mathbb{R}^n$   
 $t \mapsto \theta(t)? \Rightarrow \dot{\theta} = g(\theta, v)$

path-following error  
 $e(t) = x(t) - p(\theta(t))$   
 $\dot{e}(t) = f(x, u) - \nabla p \cdot g(\theta, v)$

time invariant problem  
 additional design parameter

**Consideration of constraints?**  
**Optimization-based approaches**

### Principle of Model Predictive Control

predictive control = repeated optimisation

1. State measurement  $x(t_i)$  at  $t_i$
2. Solve  

$$\min_u \int_{t_i}^{t_i+T_p} F(x(\tau), u(\tau)) d\tau + E(x(t_i + T_p))$$

$$\dot{x} = f(x, u), x(t_i) \text{ measured } x \in \mathcal{X}, u \in \mathcal{U}$$
3. Apply  $u(t) : t_i < t < t_i + T_p$

stability by choice of  $E(x(t_i + T_p))$

satisfaction of constraints

**Applicable to path-following?**

### Principle of Predictive Path-Following

- expanded system dynamics  $\dot{x} = f(x, u)$
- path parameter and additional input  $\dot{\theta} = g(\theta, v)$
- costs penalize path-following error  $F(\cdot) = F(x - p(\theta), \theta, u, v)$

**Optimal Control Problem**

$$\min_{u, v} \int_{t_i}^{t_i+T_p} F(x - p(\theta), \theta, u, v) d\tau + E(x(t_i + T_p), \theta(t_i + T_p))$$

$\dot{x} = f(x, u), x(t_i) \text{ measured}$   
 $\dot{\theta} = g(\theta, v), \dot{\theta} \geq 0$

$(u, x)^T \in \mathcal{U} \times \mathcal{X}$   
 $(v, \theta)^T \in \mathcal{V} \times [\theta_0, 0]$   
 $(x, \theta)^T|_{t_i+T_p} \in \mathcal{E} \subseteq \mathcal{X} \times [\theta_0, 0]$

**Stability? Robustness? Existence of solutions?**

### Stability

**Theorem**

- i.  $\forall (x, \theta)^T \in \mathcal{E}$  exists  $(u_{\mathcal{E}}, v_{\mathcal{E}})^T \mapsto \mathcal{U} \times \mathcal{V}$  s.t.  

$$\nabla E(x, \theta) \cdot \begin{pmatrix} f(x, u_{\mathcal{E}}) \\ g(\theta, v_{\mathcal{E}}) \end{pmatrix} + F(x, \theta, u_{\mathcal{E}}, v_{\mathcal{E}}) \leq 0,$$
- ii. Optimal control problem solvable at  $t_0$   
 $\lim_{t \rightarrow \infty} \|x(t) - p(\theta(t))\| = 0$   
 Optimal control problem solvable for all  $t_i$

[Faulwasser & Findeisen '09]

- direct formulation of path-following problem
- suff. stability conditions, through  $E(x, \theta)$  and  $\mathcal{E} \subseteq \mathcal{X} \times [\theta_0, 0]$
- satisfaction of constraints
- calculation of terminal weights?

### Calculation of Stabilising Terminal Weights

**Idea**

- path as terminal region  $\mathcal{E} = \mathcal{P} \times [\theta_0, 0]$
- path exactly followable  $\forall \theta \exists u, v : f(x, u) = \nabla p \cdot g(\theta, v)$
- terminal weight = costs along path  $E(x, \theta) = E(\theta)$

**Corollary**

- quadratic cost  $F(\cdot) = \left\| \begin{matrix} x - p(\theta) \\ \theta \end{matrix} \right\|_Q^2 + \left\| u \right\|_R^2$
- input signals  $\forall \theta \exists u_{\mathcal{E}}, v_{\mathcal{E}} : f(x, u_{\mathcal{E}}) = \nabla p \cdot g(\theta, v_{\mathcal{E}})$
- if an  $\epsilon \in \mathbb{R}$  exists, s.t.  $\forall \theta \in [\theta_0, 0]$ 

$$\epsilon > \frac{\left\| \begin{matrix} \dot{q}\theta^2 + \frac{|u_{\mathcal{E}}|}{v_{\mathcal{E}}} \right\|_R}{-\nabla g(\theta, v_{\mathcal{E}}) \cdot \theta} \geq 0.$$

Then  $E(\theta) = \frac{1}{2} \epsilon \cdot \theta^2$  guarantees stability and  $\lim_{t \rightarrow \infty} \|x(t) - p(\theta(t))\| = 0$

[Faulwasser & Findeisen '09]

### Example: Path-Following for an Autonomous Helicopter

**Prototype system ARTIS**

- on-board image processing
- I/O-linearising flight stabilisation
- task: accurate path following

joint project with Institute for Flight Systems, DLR (S. Lorenz)

**Comparison existing controller and path-following**

### Steering a Car as a Control Problem

**Predictive Path-Following**

- 1D paths
- consideration of constraints

**Extensions**

- multi-dimensional path corridors?
- path-following for output paths?

### Corridor Path-Following

$\dot{x} = f(x, u), \quad x(0) = x_0$

- path = regular, kD surface  $\mathcal{P}_k \subset \mathbb{R}^n$   
 $\mathcal{P}_k = \{p_k(\Theta) : \Theta \in \mathbb{R}^k \mapsto p_k(\Theta)\}$
- endpoint = origin  $\lim_{\Theta \rightarrow 0} \|p_k(\Theta)\| = 0$
- given initial point  $p_k(\Theta_0)$

**Control Task**

- convergence to corridor  $\lim_{t \rightarrow \infty} \|x(t) - p_k(\Theta(t))\| = 0$
- forward motion  $\forall \Theta \in [\Theta_0, \Theta] : \dot{\Theta}(t) > 0$
- satisfaction of constraints  $x \in \mathcal{X}, \quad u \in \mathcal{U}$

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### Predictive Corridor Path-Following

- expanded system dynamics
 
$$\dot{x} = f(x, u)$$
- k path parameters, k add. inputs
 
$$\dot{\Theta} = g(\Theta, v)$$

$$F(\cdot) = F(x - p_k(\Theta), \Theta, u, v)$$

**Optimal Control Problem**

$$\min_{u, v} \int_{t_i}^{t_i+T_p} F(x - p_k(\Theta), \Theta, u, v) d\tau + E(x(t_i+T_p), \Theta(t_i+T_p))$$

$\dot{x} = f(x, u), x(t_i)$  measured

$\dot{\Theta} = g(\Theta, v), \dot{\Theta}_1 \geq 0$

$(u, x)^T \in \mathcal{U} \times \mathcal{X}$

$(v, \Theta)^T \in \mathcal{V} \times [\underline{\Theta}, \bar{\Theta}]$

$(x, \Theta)^T|_{t_i+T_p} \in \mathcal{E} \subseteq \mathcal{X} \times [\underline{\Theta}, \bar{\Theta}]$

**Stability?**

### Corridor Path-Following: Stability

**Theorem**

- $\forall (x, \Theta)^T \in \mathcal{E}$  exist  $(u_{\mathcal{E}}, v_{\mathcal{E}})^T \mapsto \mathcal{U} \times \mathcal{V}$  s.t.
 
$$\nabla E(x, \Theta) \cdot \begin{pmatrix} f(x, u_{\mathcal{E}}) \\ g(\Theta, v_{\mathcal{E}}) \end{pmatrix} + F(x, \Theta, u_{\mathcal{E}}, v_{\mathcal{E}}) \leq 0.$$
- optimal control solvable at  $t_0$ 

$$\lim_{t \rightarrow \infty} \|x(t) - p_k(\Theta(t))\| = 0$$

optimal control problem solvable for all  $t_i$  [Faulwasser & Findeisen '09]

- sufficient stability conditions
- satisfaction of constraints
- spacial deviation from 1d path
- online trajectory planning on path corridor

### Output Path-Following

$\dot{x} = f(x, u), x(0) = x_0$

$y = h(x), y \in \mathbb{R}^m$

- output path  $\subset$  regular, 1d curve
- $\mathcal{P} = \{p(\theta) \in \mathbb{R}^m : \theta \in [0, 0] \mapsto p(\theta)\}$

**Control Task**

- convergence of output
 
$$\lim_{t \rightarrow \infty} \|y(t) - p(\theta(t))\| = 0$$
- forward motion
 
$$\forall \theta \in [0, 0) : \dot{\theta}(t) > 0$$
- satisfaction of constraints
 
$$x \in \mathcal{X}, u \in \mathcal{U}$$

### Idea of Predictive Output Path-Following

**Challenge**

- natural cost function  $F(y - p(\theta), \theta, u, v)$  positive semi-definite

state space  $\mathbb{R}^n$       output space  $\mathbb{R}^{m < n}$

path consistent state set  $\Gamma$

$$\{\forall x \in \mathcal{X} : h(x) = p(\theta) \wedge \exists u \in \mathcal{U}, v \in \mathcal{V} \nabla h \cdot f(x, u) = \nabla p \cdot g(\theta, v)\}$$

**Path convergence instead of stability**

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### Example: Ship Control

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} w \cos x_3 - L u x_4 \cdot \sin x_3 \\ w \sin x_3 + L u x_4 \cdot \cos x_3 \\ x_4 \\ \frac{1}{J} (-x_4 + K u) \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \theta_1 + v_1 \\ -\lambda_2 \theta_2 + v_2 \end{pmatrix} \quad v_1 \in [0, \bar{v}_1] \\ v_2 \in [\underline{v}_2, \bar{v}_2]$$

**2D Output Path Corridor**

$$p(\theta_1, \theta_2) = \begin{pmatrix} \theta_1 \\ \alpha \sin(\beta \theta_1) \end{pmatrix} + \theta_2 \begin{pmatrix} -\alpha \beta \cos(\beta \theta_1) \\ 1 \end{pmatrix}^{-1} \begin{pmatrix} -\alpha \beta \cos(\beta \theta_1) \\ 1 \end{pmatrix}$$

### Summary

**new approach to constrained path-following problems**

- nonlinear path-following problems subject of constraints
- rigorous stability conditions
- extensions to path corridors and output paths
- outlook: optimal control on manifolds, robustness, ...

**Possible Applications**

- CNC-machines, robotics
- autonomous vehicles & planes
- crystallisation processes
- ...

Many applications → path-following problems

### Literatur

- Aguiar et al. (2005). Path-following for nonminimumphase systems removes performance limitations. *IEEE Transactions on Automatic Control* 50, 234-239.
- Dacic & Kokotovic (2006). Path-following for linear systems with unstable zero dynamics. *Automatica* 42, 1673-1683.
- Aguiar et al. (2008). Performance limitations in reference tracking and path-following for nonlinear systems. *Automatica* 44, 598-610.
- Skjetne et al. (2005). Robust output maneuvering for a class of nonlinear systems. *Automatica* 40, 373-383.
- Do & Pan (2006). Global robust adaptive path following of under actuated ships. *Automatica* 42, 1713-1722.
- Shin & McKay (1985). Minimum-time control of robotic manipulators with geometric path constraints. *IEEE Transactions on Automatic Control* 30, 531-541.
- Verschueren et al. (2009). Time-optimal path tracking for robots: A convex optimization approach. *IEEE Transactions on Automatic Control* 54, 2318-2327.
- Faulwasser & Findeisen (2009). Nonlinear model predictive path-following control. In Magni et al. (ed.) *Assessment and future directions of nonlinear model predictive control*, 335-343. Springer.
- Faulwasser et al. (2009). Model predictive path-following for constrained nonlinear systems. *Proc. of 48th CDC*, 8642-8647.
- Faulwasser & Findeisen (2010). Constrained output path-following for nonlinear systems using predictive control. Submitted to NOLCOS 2010.

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