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Lyapunov exponents for random continuous-time switched systems and stabilizability

Based on joined work with

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Introduction

Let $N, d \in \mathbb{N}$ and $A_1, \ldots, A_N \in \mathcal{M}_d(\mathbb{R})$ and consider the continuous-time linear switched (hybrid) system

$$\dot{x}(t) = A_{\alpha(t)} x(t),$$

where the switching signals α belong to

$$\mathcal{P} := \{\alpha : \mathbb{R}_+ \to \underline{N} = \{1, ..., N\} \text{ piecewise constant and right continuous}\}.$$

The solutions are denoted by $\varphi_c(t; x_0, \alpha)$, $t \geq 0$.

The **values** $\alpha(t)$ and also the **switching times** will be **random**.

Lyapunov exponents

We will analyze the **Lyapunov exponents**

$$\lambda(x_0, \alpha) = \limsup_{t\to\infty} \frac{1}{t} \log \left\| \varphi_c(t; x_0, \alpha) \right\| .$$

and the associated initial values $x_0 \neq 0$.

This will also be used for **stabilization** with arbitrary decay rates of random switched systems.

Contents

- Some motivation from persistently excited systems

- Facts on Lyapunov exponents

- Random switched systems

- The Oseledets' Multiplicative Ergodic Theorem (cf. L. Arnold, RDS)

- A formula for the maximal Lyapunov exponent and exponential almost sure stability

- Stabilizability with arbitrary decay rates

Motivation: Persistently excited systems

Definition. Let $T > \mu > 0$. A **$(\mathbf{T}, \boldsymbol{\mu})$-signal** is a measurable function $\alpha : \mathbb{R}_+ \to [0, 1]$ with

$$\int_t^{t+T} \alpha(s)ds \geq \mu \text{ for all } t \in \mathbb{R}_+,$$

Let $\mathcal{G}(T, \mu)$ be the set of all (T, μ)-signals.
Note: If $T = \mu$ the only (T, μ)-signal is $\alpha(t) \equiv 1$.

A **persistently excited (PE-)system** has the form

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t),\ \alpha \in \mathcal{G}(T, \mu).$$

If α only takes the values 0 and 1, the system switches between the uncontrolled system $\dot{x} = Ax$ and the controlled system $\dot{x} = Ax + Bu$.

Chitour, Sigalotti.

Persistent Excitation: a classical result

Morgan and Narendra (1977) showed: Consider

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)^\top \\ B(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with $A(t) + A(t)^\top$ negative definite.
This system is uniformly asymptotically stable iff there are $T_0, \mu, T > 0$ s.t. for all $t_1 \geq 0, \|w\| = 1$ there is $t \in [t_1, t_1 + T_0]$ with

$$\left\| \int_t^{t+T} B(s)^\top w ds \right\| \geq \mu.$$

Many applications in adaptive control and parameter identification.

The following will be independent of this result.

Pole-shifting for Persistently Excited (PE) systems

For linear feedbacks $u = Kx$ one is interested in exponential stability uniformly with respect to $\alpha \in \mathcal{G}(T, \mu)$: The solutions of

$$\dot{x} = Ax + \alpha(t)BKx = [A + \alpha(t)BK]x, \ x(0) = x_0,$$

should satisfy $\|x(t, x_0, \alpha, K)\| \leq Ce^{-\gamma t} \|x_0\|$ for $t \geq 0$.
It is known, that **controllability** does not guarantee that arbitrary decay rates γ can be achieved, hence controllability $\not\Rightarrow$ stabilizability.

Chaillet, Chitour, Loria, Sigalotti (2008), Chitour, Sigalotti (2010),

The proof is based on the construction of signals with very fast switching.
What about systems with random switching?
In fact, we will analyze **all Lyapunov exponents**, not only the maximal one.

Lyapunov exponents

For autonomous linear ODE

$$\dot{x}(t) = Ax(t),\ A \in \mathcal{M}_d(\mathbb{R}),$$

the limits

$$\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|x(t; x_0)\| .$$

exist for every $x_0 \neq 0$ (they are the real parts of the eigenvalues).
There are $1 < q \leq d$, numbers $\lambda_1 > ... > \lambda_q$ and invariant subspaces

$$\mathbb{R}^d = V_1 \supset V_2 \supset \ ... \supset V_q$$

such that $\lambda(x_0) = \lambda_i$ iff $x_0 \in V_i \setminus V_{i-1}$.
For **time-dependent linear ODE**, the limits may not exists:

$$\dot{x} = (\cos t - t \sin t - 2)x,\ x(t) = e^{t \cos t - 2t} x(0),$$

$$\limsup_{t\to\infty} \frac{1}{t} \log \|x(t; x_0)\| = -1 \text{ and } \liminf_{t\to\infty} \frac{1}{t} \log \|x(t; x_0)\| = -3.$$

Analogously for linear difference equations $x_{n+1} = Ax_n$.

Description of the random switched system

Consider

$$\dot{x} = A_{\alpha(t)}x \text{ with } \alpha(t) \in \underline{N} = \{1, ..., N\}.$$

(i) **Choice of the matrix** $A_i, i \in \{1, ..., N\}$:

Determined by a row stochastic matrix M, i.e., $M_{ij} \geq 0$ with $\sum_{j=1}^{N} M_{ij} = 1$ for all i, and $p \in [0, 1]^N$ with

$$p_1 + ... + p_N = 1 \text{ and } pM = p.$$

(ii) Random **interarrival times** t_i:

They are determined by probability measures $\mu_1, ..., \mu_N$ on $(0, \infty)$ with finite expectation $\int_0^\infty t d\mu_i(t) < \infty$.

The associated Markov process

Define transition probabilities $P : \underline{N} \times (0, \infty) \to \mathrm{Pr}(\underline{N} \times (0, \infty))$ by

$$P(i, t)(\{j\} \times U) = M_{ij}\mu_j(U) \text{ for } i, j \in \underline{N} \text{ and } U \subset (0, \infty).$$

This defines a Markov process on $\underline{N} \times (0, \infty)$.
A **probability measure** $\mathbb{P}$ on $\Omega = (\underline{N} \times (0, \infty))^{\mathbb{N}}$ is determined by

$$\mathbb{P}\left(\{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N} \setminus \underline{n}}\right)$$
$$= p_{i_1}\mu_{i_1}(U_1)M_{i_1 i_2}\mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n}\mu_{i_n}(U_n)$$

for $n \in \mathbb{N}$, $i_1, ..., i_n \in \underline{N}$ and $U_1, ..., U_n \subset (0, \infty)$

Recall that

$$\mathcal{P} := \{\alpha : \mathbb{R}_+ \to \underline{N} \text{ piecewise constant and right continuous}\}.$$

For $\omega = (i_n, t_n)_n \in \Omega$ with values $i_n \in \underline{N}$ and interarrival times $t_n > 0$ let the switching times $s_n(\omega)$ be

$$s_0 = 0, s_n = \sum_{k=1}^{n} t_k, n \geq 1,$$

and define **random signals** using $\alpha : \Omega \to \mathcal{P}$

$$[\alpha(\omega)](t) = i_n \text{ for } t \in [s_{n-1}, s_n) \text{ and } n \in \mathbb{N}.$$

This is well defined on a subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$.
We get solutions $\varphi_c(t, x_0, \alpha(\omega)), t \geq 0$.

In addition to the random continuous-time system we consider a **discrete-time system** looking only at the switching times:

Recall that the random switching times are $s_n(\omega), \omega \in \Omega$, and define $\varphi_{\mathrm{d}} : \mathbb{N}_0 \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ as the solution of

$$x_{n+1} = e^{A_{\alpha(s_n(\omega))}(s_{n+1}(\omega) - s_n(\omega))} x_n$$

with initial value x_0 and $i_0 \in \underline{N}$.
By construction

$$\varphi_{\mathrm{d}}(n, x_0, \omega) = \varphi_{\mathrm{c}}(s_n, x_0, \alpha(\omega)).$$

The **Lyapunov exponents** of the systems in continuous and discrete time are

$$\lambda_{\mathrm{c}}(x_0, \alpha(\omega)) = \limsup_{t\to\infty} \frac{1}{t} \log \left\| \varphi_{\mathrm{c}}(t, x_0, \alpha(\omega)) \right\|,$$
$$\lambda_{\mathrm{d}}(x_0, \omega) = \limsup_{n\to\infty} \frac{1}{n} \log \left\| \varphi_{\mathrm{d}}(n, x_0, \omega) \right\|.$$

They are related by

$$\boxed{\lambda_{\mathrm{d}}(x_0, \omega) = m(\omega) \lambda_{\mathrm{c}}(x_0, \alpha(\omega))}$$

with

$$m(\omega) := \lim_{n\to\infty} \frac{s_n(\omega)}{n}$$

This limit exists with $\mathbb{P}$-probability 1 (by Birkhoff's Ergodic Theorem).

A **linear random dynamical system** (with discrete or continuous time set $\mathbb{T}$) is given by a measurable dynamical system

$$(\theta, \varphi) : \mathbb{T} \times \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d,$$

where
- $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ is a dynamical system preserving $\mathbb{P}$, i.e., for all $t, s \in \mathbb{T}$

$$\theta_{t+s} = \theta_t \circ \theta_s, \ \mathbb{P}(\theta_t^{-1}(A)) = \mathbb{P}(A) \text{ for all } A \in \mathfrak{F},$$

- $\varphi : \mathbb{T} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a **linear cocycle**, i.e.,

$$\varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x)), t, s \in \mathbb{T}, \omega \in \Omega,$$

and $\varphi(t, \omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is linear.

The random system in discrete-time

For an application of the classical Oseledets theorem, we construct a **random dynamical system**: Recall

$$\Omega = (\underline{N} \times (0,\infty))^{\mathbb{N}}$$

and the probability measure $\mathbb{P}$ on Ω given by

$$\mathbb{P}\left(\{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N} \setminus \underline{n}}\right)$$
$$= p_{i_1} \mu_{i_1}(U_1) M_{i_1 i_2} \mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n} \mu_{i_n}(U_n)$$

for $n \in \mathbb{N}$, $i_1, \dots, i_n \in \underline{N}$ and $U_1, \dots, U_n \subset (0,\infty)$.

Define the **shift** $\theta : \Omega \to \Omega$ by

$$\theta(\omega) = \theta((i_n, t_n)_{n=1}^{\infty}) = (i_{n+1}, t_{n+1})_{n=1}^{\infty}.$$

Then $\mathbb{P}$ is invariant under the shift θ and

$$\varphi_{\mathrm{d}}(n+m, x_0, \omega) = \varphi_{\mathrm{d}}(n, \varphi_{\mathrm{d}}(m, x_0, \omega), \theta\omega).$$

Note: The analogous property in continuous time is **not valid**.

Theorem. Suppose that M is irreducible. Then $\mathbb{P}$ is ergodic and there exists an invariant subset Ω_0 of Ω with $\mathbb{P}$-measure 1 such that for every $\omega \in \Omega_0$
(i) there are subspaces $\mathbb{R}^d = V_1(\omega) \supset ... \supset V_q(\omega)$ with dimensions $d = d_1 > ... > d_q$ with

$$A(\omega)V_i(\omega) = V_i(\theta\omega)$$

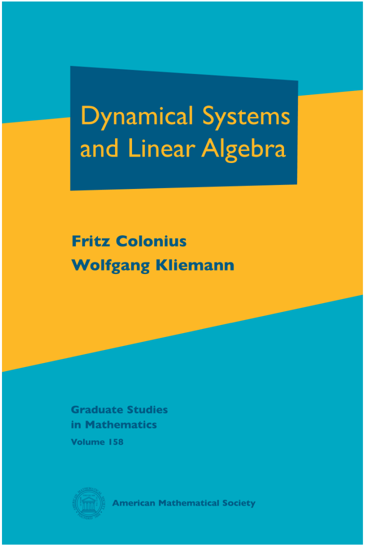
(ii) and real numbers $\lambda_1^{\mathrm{d}} > ... > \lambda_q^{\mathrm{d}}$ and $\lambda_1^{\mathrm{c}} > ... > \lambda_q^{\mathrm{c}}$ such that for every $1 \leq i \leq q$

$$\lambda_{\mathrm{d}}(x_0, \omega) = \lambda_i^{\mathrm{d}} \Longleftrightarrow \lambda_{\mathrm{c}}(x_0, \omega) = \lambda_i^{\mathrm{c}} \Longleftrightarrow x_0 \in V_i(\omega) \setminus V_{i+1}(\omega).$$

The **proof** applies the Multiplicative Ergodic Theorem to the discrete random dynamical system and then deduces the result for continuous time.

If $\mathbb{P}$ is not ergodic, the numbers above depend on ω.

A short break for this:



The maximal Lyapunov exponent

Corollary. Suppose M is irreducible and there is $r > 1$ such that $\int_0^\infty t^r d\mu_i(dt) < \infty$ for $i \in \underline{N}$. Let

$$\Phi(n,\omega) := \varphi_{\mathrm{d}}(n,\cdot,\omega) : \mathbb{R}^d \to \mathbb{R}^d.$$

Then

$$\lambda_{\max}^{\mathrm{d}} = \inf_n \frac{1}{n} \int_\Omega \log \|\Phi(n,\omega)\| \, d\mathbb{P}(\omega)$$

and

$$\lambda_{\max}^{\mathrm{c}} = \frac{1}{\sum_{i=1}^N p_i \int_0^\infty t d\mu_i(t)} \lambda_{\max}^{\mathrm{d}}.$$

Furthermore, the discrete- (continuous-)time system is **almost surely exponentially stable** iff there is $n \in \mathbb{N}$ with

$$\int_\Omega \log \|\Phi(n,\omega)\| \, d\mathbb{P}(\omega) < 0.$$

Stabilization of control systems

Consider linear control systems with switching of the form

$$\dot{x} = Ax(t) + B_{\alpha(t)} u_{\alpha(t)}$$

where $A \in \mathbb{R}^{d \times d}, \alpha : \mathbb{R}_+ \to \underline{N}$ is in $\mathcal{P}$ and for $j \in \underline{N}$ the control $u_j(t) \in \mathbb{R}^{m_j}$ and $B_j \in \mathbb{R}^{d \times m_j}$.
Thus we have a switched control system with dynamics given by the N equations

$$\dot{x} = Ax + B_j u_j.$$

We allow feedbacks $u_j = K_j x$ with matrices $K_j \in \mathbb{R}^{m_j \times d}, j \in N$, and analyze when the system can be stabilized with arbitrary exponential decay rate.
Special case :

$$\dot{x} = Ax(t) + \alpha(t) Bu$$

with $\alpha(t) \in \{0, 1\}$. Then $N = 2$ and

$$B_1 = B \text{ and } B_2 = 0.$$

The closed loop system

Let $M \in \mathbb{R}^{N \times N}$ be an irreducible stochastic matrix, and $(p_1, ..., p_N)$ its unique invariant probability vector, $\mu_1, ..., \mu_N$ probability measures on $(0, \infty)$.
Hence there are $(i_1, ..., i_r), r \geq N$, with $\{i_1, ..., i_r\} = \underline{N}$ and $M_{i_1 i_2} \cdots M_{i_{r-1} i_r} > 0$.
Consider, as before,

$$\Omega = (\underline{N} \times (0, \infty))^{\mathbb{N}}.$$

The **closed-loop random system** is

$$\dot{x}(t) = (A + B_{\alpha(\omega)(t)} K_{\alpha(\omega)(t)}) x(t)$$

where $\alpha(\omega) \in \mathcal{P}$ is the associated random signal.

For $j \in \underline{N}$ let

$$W_j := \mathrm{Range}\ (B_j, AB_j, ..., A^{d-1}B_j).$$

Theorem. Assume that

$$W_1 \oplus \ ... \ \oplus W_N = \mathbb{R}^d.$$

Then, for every $\gamma \in \mathbb{R}$ there are $K_j \in \mathbb{R}^{m_j \times d}, j \in \underline{N}$, such that

$$\lambda^{\mathrm{c}}_{\max}(\omega) \leq \gamma \text{ for almost all } \omega \in \Omega.$$

Hence arbitrary decay rates γ can be achieved!
The **proof** is based on the **Corollary** for the maximal Lyapunov exponent,

$$\lambda^{\mathrm{d}}_{\max}(\omega) = \inf_n \frac{1}{n} \int_\Omega \log \|\Phi(n,\omega)\| \, d\mathbb{P}(\omega).$$

Arbitrary decay rates: idea of the proof

The assumption guarantees that after coordinate change

$$A = \begin{bmatrix} A_1 & . & . & . & 0 \\ . & . & & & . \\ . & & A_j & & . \\ . & & & . & . \\ 0 & . & . & . & A_N \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ . \\ b_j \\ . \\ 0 \end{bmatrix}.$$

Thus we have N independent control systems such that, at each time, only one of them is controlled, while the others follow their uncontrolled dynamics.

By Cheng, Guo, Lin and Wang (2004) one finds for every $\gamma > 0$ a matrix k_j with

$$\left\| e^{(A_j + b_j k_j)t} \right\| \leq C\gamma^L e^{-\gamma t}, t \geq 0.$$

This allows us to estimate $\|\Phi(r, \omega)\|$ where $r \in \mathbb{N}$ comes from irreducibility of M.

In particular, one obtains that

$$\dot{x} = Ax(t) + \alpha(t)Bu$$

with $\alpha(t) \in \{0, 1\}$ and random switching according to

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $p_1 = p_2 = \frac{1}{2}$ yields arbitrary decay rates under feedback, if (A, B) is controllable.

The corresponding signals can be seen to be only **asymptotically persistently exciting** in the following sense: for almost every ω

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha(\omega)(s)ds \geq \rho$$

with

$$\rho = \frac{\int_0^\infty t d\mu_1(t)}{\int_0^\infty t d\mu_0(t) + \int_0^\infty t d\mu_1(t)} > 0.$$

They are **not** persistently exciting.

Final Remarks

We have analyzed the **Lyapunov exponents** of switched linear systems of the form

$$\dot{x} = A_{\alpha(t)}, \;\alpha(t) \in \{1, ..., N\}$$

with random choice of the values $\alpha(t)$ and random switching times.

The classical Oseledets Theorem (applied to an associated discrete time system) describes the initial points with the same exponential growth behavior.

This also gives some results on stabilization of linear control systems with arbitrary decay rate. This is in contrast to the case of persistently excited linear control systems.

Systems with random switching may be seen as special **Piecewise Deterministic Markov Processes**, **Davis (1990)**.

A. Teel and coworkers: Lyapunov function methods.