On Linear-Quadratic Control Theory of Implicit Difference Equations

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Linear IDEs

- Consider the IDE (implicit difference equation)

$$E\sigma x_j = Ax_j + Bu_j, \qquad Ex_0 = Ex^0, \tag{1}$$

  - σ denotes the shift operator, i. e., $\sigma x_j = x_{j+1}$,
  - $(x_j) \in (\mathbb{K}^n)^{\mathbb{N}_0}$, $(u_j) \in (\mathbb{K}^m)^{\mathbb{N}_0}$ are some sequences.
  - x^0 is some consistent shift variable.
  - The IDE is assumed to be regular, i. e., $\det(zE - A) \not\equiv 0$.
- Given x^0, minimize

$$\mathcal{J}^\sigma(x,u) := \sum_{j=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}$$

subject to (1) with $Q = Q^*$, $R = R^*$ and $\lim_{j\to\infty} Ex_j = 0$.

Motivation

- Optimal control problems have a variety of applications, e. g.,
  - Trajectory-following problems, e. g., in Robotics
  - Reaching a goal with minimal energy consumption
- Discrete-time models are obtained by discretization methods from the continuous-time models.

Discretized Stokes Problem

- Solve

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \sigma \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} K + M & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u, \quad Mv_0 = Mv^0.$$

  - v describes the velocity
  - p describes the pressure
  - M is the mass matrix
  - K is the stiffness matrix

- Minimize

$$\sum_{j=0}^{\infty} \|u_j\|,$$

i. e., $Q = 0, S = 0, R = I_m$, such that $\lim_{j \to \infty} Mv_j = 0$.

Discrete Time Algebraic Riccati Equations

- Solve

$$A^*XA - X + Q - (A^*XB + S)(B^*XB + R)^{-1}(B^*XA + S^*) = 0$$

  for Hermitian X fulfilling $B^*XB + R \succ 0$.
- Under some circumstances invertibility of $B^*XB + R$ is not guaranteed, e. g., when R is indefinite (e. g., differential games)
- Can be rewritten as

$$\begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}.$$

Discrete Time Algebraic Riccati Equations

- Solve

$$A^* X A - X + Q - (A^* X B + S)(B^* X B + R)^{-1}(B^* X A + S^*) = 0$$

  for Hermitian X fulfilling $B^* X B + R \succ 0$.
- Under some circumstances invertibility of $B^* X B + R$ is not guaranteed, e. g., when R is indefinite (e. g., differential games)
- In the IDE case

$$\begin{bmatrix} A^* X A - E^* X E + Q & A^* X B + S \\ B^* X A + S^* & B^* X B + R \end{bmatrix} =_{\mathcal{V}^\Sigma} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix},$$

  where $\mathcal{V}^\Sigma$ denotes the system space, i. e., the space where all solutions (x, u) evolve.

Discrete Time Algebraic Riccati Equations

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for Hermitian X fulfilling $B^*XB + R \succ 0$.

- Under some circumstances invertibility of $B^*XB + R$ is not guaranteed, e. g., when R is indefinite (e. g., differential games)
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$$(V^\Sigma)^* \begin{bmatrix} A^*XA - E^*XE + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} V^\Sigma = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix},$$

where V^Σ is a basis matrix of $\mathcal{V}^\Sigma$.

Discrete-Time KYP Lemma

- Solve (discrete-time) Kalman-Yakubovich-Popov (KYP) inequality [Kalman 1963; Popov 1961; Yakubovich 1962]

$$\begin{bmatrix} A^*PA - E^*PE + Q & A^*PB + S \\ B^*PA + S^* & B^*PB + R \end{bmatrix} \succeq_{\mathcal{V}^\Sigma} 0, \quad P = P^*$$

- Popov function

$$\Phi(z) := \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix}^{\sim} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}^{m \times m}(z),$$

where $G(z)^{\sim} := G(\overline{z}^{-1})^*$

Discrete-Time KYP Lemma

Theorem (KYP Lemma for IDEs)

Let $(E, A, B, Q, S, R) \in \Sigma^{w}_{m,n}(\mathbb{K})$ and the system space $\mathcal{V}^{\Sigma}$ be given with corresponding Popov function $\Phi(z) \in \mathbb{K}^{m\times m}(z)$.

- *If there exists some $P \in \mathbb{K}^{n\times n}$ that is a solution of the KYP inequality, then $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in \mathbb{R}$ with $\det(\mathrm{e}^{\mathrm{i}\omega}E - A) \neq 0$.*
- *If on the other hand (E, A, B) is $\mathcal{R}$-controllable and $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in \mathbb{R}$ with $\det(\mathrm{e}^{\mathrm{i}\omega}E - A) \neq 0$, then there exists a solution $P \in \mathbb{K}^{n\times n}$ of the KYP inequality.*

$\mathcal{R}$-controllability

$\mathcal{R}$-controllability means controllability on the set of reachable states $\mathcal{R}$. In the IDE case $\mathcal{R} \neq \mathbb{K}^n$.

Lur'e Equations

- For $q := \mathrm{rk}_{\mathbb{K}(z)}\,\Phi(z)$ find $X = X^* \in \mathbb{K}^{n\times n}$, $K \in \mathbb{K}^{q\times n}$, and $L \in \mathbb{K}^{q\times m}$ such that

$$\begin{bmatrix} A^*XA - E^*XE + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} =_{\mathcal{V}^\Sigma} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}.$$

- If X is a solution of the KYP inequality, we can always find $K \in \mathbb{K}^{p\times n}$ and $L \in \mathbb{K}^{p\times m}$ solving the Lur'e equation, $p \geq q$.
- From now on $E = I_n$, i. e., solve

$$\begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}.$$

Construction of Deflating Subspace

- Consider the pencil

$$z\mathcal{E} - \mathcal{A} = z\begin{bmatrix} 0 & I_n & 0 \\ A^* & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ I_n & Q & S \\ 0 & S^* & R \end{bmatrix} \in \mathbb{K}^{2n+m \times 2n+m}[z].$$

- Find deflating subspace, i. e., $Y \in \mathbb{K}^{2n+m \times n+m}$, $Z \in \mathbb{K}^{2n+m \times n+q}$, $\tilde{E}, \tilde{A} \in \mathbb{K}^{n+q \times n+m}$ with $\mathrm{rk}_{\mathbb{K}(z)}(z\tilde{E} - \tilde{A}) = n + q$ such that

$$(z\mathcal{E} - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A}).$$

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$$(z\mathcal{E} - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A}).$$

- Ansatz:

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -A^*X & -K^* \\ -B^*X & -L^* \end{bmatrix},$$

and

$$z\tilde{E} - \tilde{A} = \begin{bmatrix} zI_n - A & -B \\ K & L \end{bmatrix}.$$

Application to Infinite Horizon Optimal Control

Suppose, we have a solution (X, K, L) of the Lur'e equation, $j_2 \geq j_1$ and optimal (x, u) with $\lim_{j\to\infty} x_j = 0$. Then

$$
\begin{aligned}
&x_{j_2}^* X x_{j_2} - x_{j_1}^* X x_{j_1} \\
&= \sum_{k=j_1}^{j_2-1} \sigma(x_k^* X x_k) - x_k^* X x_k, \quad \sigma x_k = x_{k+1} \\
&= \sum_{k=j_1}^{j_2-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \begin{bmatrix} A^* X A - X & A^* X B \\ B^* X A & B^* X B \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}
\end{aligned}
$$

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& - x_0^* X x_0 \\
&= \sum_{k=0}^{\infty} \sigma(x_k^* X x_k) - x_k^* X x_k, \quad \sigma x_k = x_{k+1} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \begin{bmatrix} A^*XA - X & A^*XB \\ B^*XA & B^*XB \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}
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&= \sum_{k=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \left(\begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} - \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \right) \begin{bmatrix} x_j \\ u_j \end{bmatrix} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} - \mathcal{J}^\sigma(x, u) = \|Kx + Lu\|_{\ell^2}^2 - \mathcal{J}^\sigma(x, u)
\end{aligned}
$$

Application to Infinite Horizon Optimal Control

- Suppose, optimal (x, u) fulfills $\|Kx + Lu\|_{\ell^2}^2 = 0$.
- Using the deflating subspace it holds that

$$\begin{bmatrix} 0 & \sigma I_n - A & -B \\ \sigma A^* - I_n & -Q & -S \\ \sigma B^* & -S^* & -R \end{bmatrix} \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}$$

$$= Z \begin{bmatrix} \sigma I_n - A & -B \\ K & L \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} \quad , \quad Z \in \mathbb{K}^{2n+m \times n+q}.$$

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- Thus, $\mu_j := -Xx_j$ solves the boundary value problem

$$\begin{bmatrix} 0 & I_n & 0 \\ A^* & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} \sigma \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ I_n & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix},$$

$$x_0 = x^0, \qquad \lim_{j\to\infty} \mu_j = 0.$$

Palindromic Matrix Pencils

Reformulation of the BVP [Schröder 2008] with

$$m_j := \mu_j - \mu_{j+1}$$

yields

$$\begin{bmatrix} 0 & I_n & 0 \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \sigma \begin{bmatrix} m \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ I_n & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} m \\ x \\ u \end{bmatrix},$$

$$x_0 = x^0, \qquad \sum_{j=0}^{\infty} m_j = \mu_j$$

with corresponding palindromic matrix pencil

$$z\mathcal{A}^* - \mathcal{A} = z \begin{bmatrix} 0 & I_n & 0 \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ I_n & Q & S \\ 0 & S^* & R \end{bmatrix} \in \mathbb{K}^{2n+m \times 2n+m}[z].$$

Deflating Subspace for Palindromic Matrix Pencils

- Find deflating subspace Y of

$$z\mathcal{A}^* - \mathcal{A} = z \begin{bmatrix} 0 & I_n & 0 \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ I_n & Q & S \\ 0 & S^* & R \end{bmatrix}$$

given a solution of

$$\begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix},$$

i. e., $(z\mathcal{A}^* - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A})$ for $\tilde{E}, \tilde{A} \in \mathbb{K}^{n+q \times n+m}$.

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- $Y = \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}$ does not work due to the variable transformation.

Deflating Subspace for Palindromic Matrix Pencils

- From $\mu_j = -Xx_j = X \begin{bmatrix} -I_n & 0 \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}$ it follows that

$$m_j = \mu_j - \mu_{j+1} = X(x_{j+1} - x_j) = X \begin{bmatrix} A - I_n & B \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}.$$

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- Leads to the deflating subspace

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} X(A - I_n) & XB \\ I_n & 0 \\ 0 & I_m \end{bmatrix}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ (I_n - A^*)X & K^* \\ -B^*X & L^* \end{bmatrix},$$

and

$$z\tilde{E} - \tilde{A} = \begin{bmatrix} zI_n - A & -B \\ (z-1)K & (z-1)L \end{bmatrix}.$$

Deflating Subspace for Palindromic Matrix Pencils

Theorem

Let $(I_n, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ *and consider the palindromic pencil* $z\mathcal{A}^* - \mathcal{A}$. *Further, let* $q = \mathrm{rk}_{\mathbb{K}(z)}\,\Phi(z)$ *and assume that* $\mathrm{rk}\,[\,I_n - A \quad -B\,] = n$. *Then the following are equivalent:*

- *There exists a solution of the (discrete time) Lur'e equation.*
- *The KYP inequality has a solution and there exist* $Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}, Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix}, \tilde{E}, \tilde{A}$, *such that* $(z\mathcal{E} - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A})$ *and*
  - *the matrix* $\begin{bmatrix} I_n - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$ *has full row rank* n*;*
  - *the space* $\mathcal{Y} = \mathrm{im}\,Y$ *is the largest space such that* $Y^*(\mathcal{A}^* - \mathcal{A})Y = 0$.

Deflating Subspace for IDEs

- In the IDE case find deflating subspace Y of

$$z\mathcal{A}^* - \mathcal{A} = z \begin{bmatrix} 0 & E & 0 \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ E^* & Q & S \\ 0 & S^* & R \end{bmatrix}$$

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- Part of the Solution:

$$Y = \begin{bmatrix} X(A-E) + G_1 & XB + G_2 \\ V_1^\Sigma & V_2^\Sigma \end{bmatrix},$$

where im $\begin{bmatrix} G_1 & G_2 \end{bmatrix} \subseteq \ker E^*$, and $\begin{bmatrix} V_1^\Sigma & V_2^\Sigma \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{V}^\Sigma$.

Concluding Remarks

- Conclusions
  - We have seen the discrete time counterpart of the KYP Lemma for DAEs.
  - We have seen the discrete time counterpart of Lur'e equations and corresponding deflating subspaces for (explicit) difference equations.
  - The results on Lur'e equations for DAEs can also be adapted to Lur'e equations on IDEs.

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- What was not shown
  - The spectral structure of palindromic matrix pencils can be characterized w.r.t. the solvability of the KYP inequality.
  - Certain discretizations display relations between continuous time results and their discrete time counterpart.

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  - Certain discretizations display relations between continuous time results and their discrete time counterpart.

Thank you for your attention!

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