Pfadverfolgung –
Von Normalformen zu prädiktiven Reglern

Timm Faulwasser
Laboratoire d’Automatique, Ecole Polytechnique Fédérale de Lausanne
Institut für Angewandte Informatik, Karlsruher Institut für Technologie

Joint work with:
Tobias Weber, Janine Matschek, Juan Pablo Zometa, Rolf Findeisen (OvGU MD)
Sven Lorenz, Johann Dauer (DLR Braunschweig)

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Stabilization, Trajectory Tracking and Path Following?

\[ \sum: \quad \dot{x} = f(x, u) \]

\[ u = k(x) \]

- **Reference set-point** \( x_s \in \mathbb{R}^n \)
- **Control error**
  \[ e(t) = x(t) - x_s \]
  \[ \dot{e}(t) = f(x, u) \]

**Set-point Stabilization**

- time invariant problem
Stabilization, Trajectory Tracking and Path Following?

**Trajectory Tracking**

- Reference trajectory $t \mapsto r(t) \in \mathbb{R}^n$
- Tracking error $e(t) = x(t) - r(t)$
  
  \[ \dot{e}(t) = f(x, u) - \dot{r}(t) \]

  time varying problem
  performance limits?

**Set-point Stabilization**

- Reference set-point $x_s \in \mathbb{R}^n$
- Control error $e(t) = x(t) - x_s$
  
  \[ \dot{e}(t) = f(x, u) \]

  time invariant problem


Stabilization, Trajectory Tracking and Path Following?

**Trajectory Tracking**
- Reference trajectory: \( t \mapsto r(t) \in \mathbb{R}^n \)
- Tracking error:
  \[
  e(t) = x(t) - r(t) \\
  \dot{e}(t) = f(x, u) - \dot{r}(t)
  \]

**Path Following**
- Reference path: \( \theta \mapsto p(\theta) \in \mathbb{R}^n \)
- Path-following error:
  \[
  e(t) = x(t) - p(\theta(t)) \\
  \dot{e}(t) = f(x, u) - \nabla p \cdot g(\theta, v)
  \]

Applications? Controller design?
Path Following

Typical Applications
- Autonomous vehicles
- Machine tools
- Motion problems
- ...

Problem definition? Controller design?
Outline

Path Following
- Set-point stabilization, trajectory tracking and path following
- Problem analysis: tailored normal forms
- Path followability results
- Examples
- Feedforward path following and closed-loop path following

Model Predictive Path Following Control
- Convergence properties
- MPFC for a robotic manipulator
- Feedforward path following for an autonomous helicopter

Outlook and Summary
Output Path-following Problem

- Dynamic system
\[ \Sigma : \begin{cases} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i, \\ y &= h(x) \end{cases}, \quad x(0) = x_0 \]

- Output path \( \subset \) regular 1d curve
\[ \mathcal{P} = \{ y \in \mathbb{R}^m | \theta \in \mathbb{R} \mapsto p(\theta) \} \]
\[ p(\theta) \in \mathcal{C}^k \]

Control task
- Convergence to path
\[ \lim_{t \to \infty} \| y(t) - p(\theta(t)) \| = 0 \]
- Convergence on path
\[ \lim_{t \to \infty} \theta(t) = 0, \quad \dot{\theta}(t) \geq 0 \]
- Satisfaction of constraints
\[ \forall t \geq 0 : x(t, x_0 | u(\cdot)) \in \mathcal{X}, \quad u(t) \in \mathcal{U} \]

Questions
- Problem structure \( \rightarrow \) Suitable formulation?
- Constraints \( \rightarrow \) Controller design?
Analysis of Path-following Problems

Path Followability

Given a system $\Sigma$ and a path $\mathcal{P}$, under which conditions is it possible to follow $\mathcal{P}$ exactly? That is, does an input $u : [0, \infty) \rightarrow \mathbb{R}^m$ exist, such that $h(x(t)) - p(\theta(t)) = 0$ while $\theta \rightarrow 0$?

- $\mathcal{I}_\mathcal{P} = \text{path manifold}$

How to characterize $\mathcal{I}_\mathcal{P}$?

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \\
0 = h(x) - p(\theta)
\]
Vector Relative Degree (I)

\[
\Sigma : \begin{cases} 
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, & x(0) = x_0 \\
y = h(x)
\end{cases}
\]

**Definition:** System \( \Sigma \) is said to have a vector relative degree \( r = (r_1, \ldots, r_m) \) at \( x_0 \) if

- \( L_{g_j} L_f^k h_i(x) = 0 \iff \frac{\partial}{\partial u_j} \left( \frac{d^{k+1} y_i}{dt^{k+1}} \right) = 0 \)
  
  for all \( 1 \leq j \leq m \), for all \( k < r_i - 1 \), for all \( 1 \leq i \leq m \), and for all \( x \in \mathcal{N}(x_0) \).

- the matrix

\[
A(x) = 
\begin{pmatrix}
L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\
L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\
\vdots & & \vdots \\
L_{g_1} L_f^{r_m-1} h_1(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x)
\end{pmatrix}
\]

is nonsingular at \( x = x_0 \).

[Isidori `95; Nijmeijer & van der Schaft `90]
Vector Relative Degree (II)

**Proposition:** If

\[
\Sigma : \begin{cases} 
    \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x(0) = x_0 \\
    y &= h(x)
\end{cases}
\]

has vector relative degree \( r = (r_1, \ldots, r_m) \) at \( x_0 \) then there exists a local diffeomorphism \( \Phi : \mathbb{R}^n \to \mathbb{R}^{\sum_{i=1}^{m} r_i} \times \mathbb{R}^{n-\sum_{i=1}^{m} r_i} \), \((\xi, \eta) = \Phi(x)\) such that \( \Sigma \) can be written as

\[
\begin{align*}
\dot{\xi}_i &= \begin{pmatrix} 0 & I^{r_i-1} \\ 0 & 0 \end{pmatrix} \xi_i + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad i = 1, \ldots, m \\
\dot{\eta} &= \beta(\xi, \eta, u)
\end{align*}
\]

\((\text{BINF})\)

\[
\xi = \left( y_1, \dot{y}_1, \ddot{y}_1, \ldots, \frac{d^{r_1-1} y_1}{dt^{r_1-1}}, \ldots, y_m, \ldots, \frac{d^{r_m-1} y_m}{dt^{r_m-1}} \right) \in \mathbb{R}^{\sum_{i=1}^{m} r_i}, \quad \eta \in \mathbb{R}^{n-\sum_{i=1}^{m} r_i}
\]

[Isidori '95; Nijmeijer & van der Schaft '90]
Vector Relative Degree (III)

Nonlinear part of BINF can be written as

\[
\begin{pmatrix}
\dot{\xi}_1, r_1 \\
\vdots \\
\dot{\xi}_m, r_m
\end{pmatrix}
= A(\xi, \eta)u + b(\xi, \eta)
\]

\[\dot{\eta} = \beta(\xi, \eta, u)\]

Zero dynamics

\[
\begin{pmatrix}
one \\
\vdots \\
0
\end{pmatrix}
= A(0, \eta_0)u_0 + b(0, \eta_0) \quad \Rightarrow \quad u_0 = -A(0, \eta_0)^{-1}b(0, \eta_0)
\]

\[\dot{\eta}_0 = \beta(0, \eta_0, u_0)\]

Vector relative degree at \(x_0\) \quad \Rightarrow \quad \Sigma \text{ is input-output feedback linearizable (static)}

[Isidori `95; Nijmeijer & van der Schaft `90]
Analysis of Path-following Problems

State space $\mathbb{R}^n$

Output space $\mathbb{R}^{m<n}$

$\mathcal{I}_p = \text{path manifold}$

$\mathcal{I}_p$ = path manifold

Timing law

\[
z = (\theta, \dot{\theta}, \ldots, \theta^{(\hat{r}+1)})^T
\]

\[
\dot{z} = \begin{pmatrix} 0 & \hat{r}^{-1} \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} v,
\]

\[
\theta = z_1
\]

Augmented system

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x_i) u_i
\]

\[
\hat{\theta} = \tilde{h}(x) E v \quad p(\theta)
\]

\[
e = h(x) - p(z_1)
\]

\[
\theta = z_1
\]
Transverse Normal Form

**Idea:** map to suitable coordinates

\[ \xi = \left( e_1, \dot{e}_1, \ldots, e_1^{(r_1-1)}, e_2, \ldots, e_m^{(r_m-1)} \right)^T \]

Augmented system

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\
\dot{z} &= \tilde{I}z + Ev \\
e &= h(x) - p(\theta) \\
\theta &= z_1
\end{align*}
\]

Transverse normal form

\[
\begin{align*}
\dot{\xi} &= \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \xi_i + \begin{pmatrix} 0 \\ \ddots \end{pmatrix} \\
\eta_1 &= \beta(\xi, \eta_1, \eta_2, u, v) \\
\eta_2 &= \tilde{I}\eta_2 + Ev \\
e &= C\xi \\
\theta &= \eta_{2,1}
\end{align*}
\]

**Lemma (Existence of TNF).**

If system \( \Sigma \) has a well-defined vector relative degree

\[ r = (r_1, \ldots, r_m)^T, \sum_{i=1}^{m} r_i \leq n \text{ at } x_0 \]

and \( \hat{r} = \max\{r_i\} \),

then the augmented system can be mapped into a transverse normal form.

[ Nielsen & Maggiore `08; Banaszuk & Hauser `95; F. & Findeisen `16 ]
Transverse Normal Form

Idea: map to suitable coordinates

\[ \xi = \left( e_1, \dot{e}_1, \ldots, e_1^{(r_1-1)}, e_2, \ldots, e_m^{(r_m-1)} \right)^T \]

Augmented system

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\
\dot{z} &= \tilde{I}z + Ev \\
e &= h(x) - p(\theta) \\
\theta &= z_1
\end{align*}
\]

Transverse normal form

\[ (\xi, \eta) = \Phi(x, z) \]

\[ \Phi^{-1}(\xi, \eta) = (x, z) \]

\[
\begin{align*}
\dot{\xi}_i &= \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \xi_i + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\
\eta_1 &= \beta(\xi, \eta_1, \eta_2, u, v) \\
\eta_2 &= \tilde{I} \eta_2 + Ev \\
e &= C\xi \\
\theta &= \eta_2, 1
\end{align*}
\]

Example? Path followability?
Example – Fully Actuated Robot

\[ P = \left\{ y \in \mathbb{R}^2 \mid \theta \mapsto \begin{pmatrix} p_1(\theta) \\ p_2(\theta) \end{pmatrix} \right\} \]

\[ x_1 = (q_1, q_2)^T; x_2 = (\dot{q}_1, \dot{q}_2)^T \]

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
x_2 \\
B^{-1}(x_1) \left( u - C(x_1, x_2)x_2 - g(x_1) \right)
\end{pmatrix}
\]

\[ y = x_1 \]

**Augmented system**

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
B^{-1}(x_1) \left( u - C(x_1, x_2)x_2 - g(x_1) \right)
\end{pmatrix}
\]

\[ z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v 
\]

\[ e = x_1 - p(z_1) \]

\[ \theta = z_1 \]
Example – Fully Actuated Robot

Augmented system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = 
\begin{pmatrix}
x_2 \\
B^{-1}(u - Cx_2 - g)
\end{pmatrix}, \quad 
\dot{z} = 
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} z + 
\begin{pmatrix}
0 \\
1
\end{pmatrix} v
\]

\[e = x_1 - p(z_1)\]
\[\theta = z_1\]

\[\Phi : \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\eta
\end{pmatrix} = 
\begin{pmatrix}
x_1 - p(z_1) \\
x_2 - \frac{\partial p}{\partial z_1} z_2 \\
z
\end{pmatrix}\]

Transverse normal form

\[
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\eta}
\end{pmatrix} = 
\begin{pmatrix}
\xi_2 \\
\alpha(\xi, \eta, u, v) \\
0
\end{pmatrix}
\]

\[
\dot{\eta} = 
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \eta + 
\begin{pmatrix}
0 \\
1
\end{pmatrix} v
\]

\[e = \xi_1\]
\[\theta = \eta_1\]

\[\xi = 0\]

\[0 = \alpha(0, \eta, u_P, v_P)\]

\[\dot{\eta} = 
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \eta + 
\begin{pmatrix}
0 \\
1
\end{pmatrix} v_P\]

\[e = 0\]
\[\theta = \eta_1\]
Path Followability and Transverse Normal Forms

Augmented system
\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \]
\[ \dot{z} = \tilde{I}z + Ev \]
\[ e = h(x) - p(\theta) \]
\[ \theta = z_1 \]

Transverse normal form
\[ (\xi, \eta) = \Phi(x, z) \]
\[ \Phi^{-1}(\xi, \eta) = (x, z) \]
\[ \dot{\xi}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xi_i + \begin{pmatrix} 0 \\ \cdots \\ \alpha_i(\xi, \eta, u, v) \end{pmatrix} \]
\[ \eta_1 = \beta(\xi, \eta_1, \eta_2, u, v) \]
\[ \eta_2 = \tilde{I}\eta_2 + Ev \]
\[ e = C\xi \]
\[ \theta = \eta_{2,1} \]

Theorem (Suff. conditions for path followability). If system \( \Sigma \) has a well-defined vector relative degree \( r = (r_1, \ldots, r_m)^T \), \( \sum_{i=1}^{m} r_i \leq n \) at \( x_0 \), and \( \Phi(x_0, z_0) = (0, \eta_0)^T \), then \( \mathcal{P} \) is locally exactly followable by \( \Sigma \) such that \( \dot{\theta} > 0 \) holds. [Faulwasser '13]

Necessary conditions? Constraints? Feedback design?
Path Followability in the Presence of Constraints

**Theorem** (Suff. conditions for constrained path followability).

If

- $\Sigma$ has vector relative degree $r = (r_1, \ldots, r_m)^T$, $\sum_{i=1}^{m} r_i = n$,
- the coordinate change $\Phi^{-1} : (\xi, \eta) \mapsto (x, z)$ is continuous, and
- $\mathcal{P}$ is steady-state consistent (+ technical assumptions),

then $\mathcal{P}$ is **exactly followable** by $\Sigma$ such that $\dot{\theta} > 0$ holds.

---

**Steady state consistent path** = each point on $\mathcal{P}$ corresponds to an admissible steady state.

\[
\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^{n+m}
\]

\[
\Phi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m
\]

\[
\tilde{h} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m
\]

\[
\mathcal{S} = \{(x, u) \in \mathbb{R}^{n+m} \mid 0 = f(x, u)\}
\]

**Feedback control?**
Feedforward Path Following

\[ u_P = -A^{-1}(0, \eta)b(0, \eta) \]
\[ \dot{\eta}_1 = \beta(0, \eta, u_P, v_P) \]
\[ \dot{\eta}_2 = \dot{\eta} + Ev_P \]

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \]

Feedforward path following:

- Feedforward control \( u_P \) and reference motion \( p(\theta(t)) \) generated in open loop.
- Feedback (high-gain) to stabilize reference \( p(\theta(t)) \).
- Path following is reformulated as trajectory-tracking problem.
- Verscheure et al.'09; Grizzle et al. '06; ...
Closed-loop Path Following

\[ u_P = -A^{-1}(0, \eta)b(0, \eta) \]
\[ \dot{\eta}_1 = \beta(0, \eta, u_P, v_P) \]
\[ \dot{\eta}_2 = \dot{\eta} + Ev_P \]

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \]

Closed-loop path following:

- **Feedforward control** \( u_P \) and reference motion \( p(\theta(t)) \) generated in closed loop.
- **Feedback** (high-gain) to stabilize reference \( p(\theta(t)) \).
- **Direct error feedback on path evolution** \( \rightarrow \) dynamic feedback.
- **Challenging MIMO design problem**: Aguiar ’06; Skjetne ’04; F. & Hackl ’14; ...
Outline

Path Following
- Set-point stabilization, trajectory tracking and path following
- Problem analysis: tailored normal forms
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- Examples
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Model Predictive Path Following Control
- Convergence properties
- MPFC for a robotic manipulator
- Feedforward path following for an autonomous helicopter

Outlook and Summary
Principle of Predictive Control

Model Predictive Control (MPC) = repeated optimal control

1. State observation $\hat{x}(t_k)$ at $t_k$

2. Solve

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(t), u(t))d\tau + E(x(t_k + T))$$

\[ \forall \tau \in [t_k, t_k + T] : \dot{x} = f(x, u), \quad x(t_k) = \hat{x}(t_k) \]

\[ x(\tau) \in X, \quad u(\tau) \in U \]

\[ x(t_k + T) \in E \]

3. Apply $u^*(\tau)$ for $\tau \in [t_k, t_k+1)$

Advantages:
- constraints
- MIMO systems
- optimization of transients
- dist. implementation

NMPC for path-following problems?
Principle of Predictive Path Following

Idea
• Prediction based on augmented dynamics.
• Costs penalize path-following error (and its time derivative).

Optimal Control Problem

\[
\min_{\bar{u}(\cdot), \bar{v}(\cdot)} \int_{t_k}^{t_k+T} F (\bar{e}, \dot{\bar{e}}, \bar{\theta}, \bar{u}, \bar{v}) \, d\tau + E (\bar{x}, \bar{z}) \bigg|_{t_k+T}
\]

\[
\dot{\bar{x}} = f(\bar{x}) + \sum_{i=1}^{m} g_i(\bar{x})\bar{u}_i, \quad \bar{x}(t_k) = x(t_k)
\]

\[
\dot{\bar{z}} = \tilde{I}\bar{z} + \dot{\bar{e}} + E\bar{v}, \quad \bar{z}(t_k) = \bar{z}(t_k, \bar{z}(t_{k-1}))
\]

\[
\bar{e} = h(\bar{x}) - r(\bar{\theta})
\]

\[
\bar{\theta} = \bar{z}_1
\]

\[
\bar{x} \in \mathcal{X}, \quad \bar{u} \in \mathcal{U}
\]

\[
\bar{z} \in \mathcal{Z}, \quad \bar{v} \in \mathcal{V}
\]

\[
(\bar{x}, \bar{z}) \bigg|_{t_k+T} \in \mathcal{E}
\]

Stability/convergence? Example?
Application to a Robotic Manipulator

Benchmark Problem

• Robot KUKA LBR IV $\rightarrow$ up to 7 joints.
• Make the robot write on a blackboard.
• Allow interaction between user and robot.
Implementation of Predictive Path-Following on KUKA LWR IV

- Dynamics of the robot
  - State space representation
    \[
    \begin{pmatrix}
    \dot{x}_1 \\
    \dot{x}_2
    \end{pmatrix} = \begin{pmatrix}
    B^{-1}(x_1) \\
    x_2
    \end{pmatrix}(u - C(x_1, x_2)x_2 - g(x_1))
    \]
  - Path parameter dynamics
    \[
    \dot{z} = \begin{pmatrix}
    0 & 1 \\
    0 & 0
    \end{pmatrix} z + \begin{pmatrix}
    0 \\
    1
    \end{pmatrix} v, \quad \theta = z_1
    \]
  - Error outputs
    \[
    e = h(x_1) - p(z_1)
    \]
    \[
    \theta = z_1
    \]
- Cost function, terminal penalty
  \[
  F(e, \dot{e}, \theta, \dot{\theta}, u, v) = \left\| \begin{pmatrix}
  e \\
  \dot{e}
  \end{pmatrix} \right\|_{Q_e}^2 + \left\| \begin{pmatrix}
  \theta - \theta_T \\
  \dot{\theta} - \dot{\theta}_{\text{ref}}
  \end{pmatrix} \right\|_{Q_\theta}^2 + \left\| \begin{pmatrix}
  u \\
  v
  \end{pmatrix} \right\|_R^2, \quad E(x, z) = 0
  \]
- Prediction horizon \( T = 50 - 200\, ms \), sampling time \( \delta = 1 - 5\, ms \)
- Optimization solved with ACADO Toolkit

**Results?**
Application to a Robotic Manipulator

Joint work with: Juan P. Zometa, Janine Matschek, Tobias Weber, Rolf Findeisen
(all OvG Magdeburg)

[Faulwasser et al. ‘15]
Application to a Robotic Manipulator

Joint work with: Juan P. Zometa, Janine Matschek, Tobias Weber, Rolf Findeisen (all OvG Magdeburg)

[Faulwasser et al. '15]
Application to a Robotic Manipulator
Application to a Robotic Manipulator
Lyapunov and back-stepping approaches to path following

- ...

Differential geometric reformulation

- ...

Feedforward path following

Predictive control for path following (theory)

- **Faulwasser & Findeisen (2008).** Nonlinear model predictive path-following control. *Proc. of Workshop Future Directions of Nonlinear Model Predictive Control, Pavia, Italy.*


- **Faulwasser (2013).** Optimization-based Solutions to Constrained Trajectory-tracking and Path-following Problems. *Shaker, Aachen, Germany.*

Predictive control for path following (applications)


Summary and Conclusion

Summary

• Path following: combines trajectory generation and tracking
• Typical applications of path following: autonomous vehicles & robots, machine tools, ...
• Tailored Byrnes-Isidori Normal Form is the key step for analysis
• MPC can be used to tackle path-following problems

Open questions

• Combination of path following and force control?
• Model uncertainty and repetitive motions?
• Implication of exo-systems with inputs for regulator problems?
• ...

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