Robust stability of interconnected time-varying linear systems

Dedicated to Achim Ilchmann on the occasion of his 60th birtday

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10. Elgersburg-Workshop 7.-11. 2. 2016

Work in progress
based on past joint work with A. Ilchmann and A.J. Pritchard

Outline

1. Previous work: Robust stability of single time-varying linear systems
   - Problem formulation and basic setting
   - I/O-operator associated with (A, B, C)
   - Riccati equations and the quadratic stability problem
   - Extension to dynamic (causal) perturbations

2. Uncertain interconnected systems: Problem formulation
   - Introduction
   - Couplings of structure E
   - Composite system equation
   - Example: Interconnection of four subsystems
   - Stability radius for time-varying perturbations of structure E
   - Problems

3. Results
   - Transformation of perturbations Δ to block-diagonal form
   - Estimate for the time-varying stability radius of structure E
   - Special interconnection structures
   - Joint Lyapunov functions

Modeling uncertainty

$$\mathbf{\Sigma}: \quad \dot{x}(t) = A(t)x(t), \quad t \geq 0, \qquad \text{where } A(\cdot) \in L^{\infty}_{loc}(\mathbb{R}_+; \mathbb{K}^{n\times n}),\ \mathbb{K} = \mathbb{R}, \mathbb{C}$$

Assumption: $\mathbf{\Sigma}$ uniformly exponentially stable (u.e.s.)
Def.: $\mathbf{\Sigma}$ u.e.s. $\quad \Leftrightarrow \quad \exists M, \omega > 0 : \|\Phi(t,s)\| \leq Me^{-\omega(t-s)},\ t \geq s \geq 0$

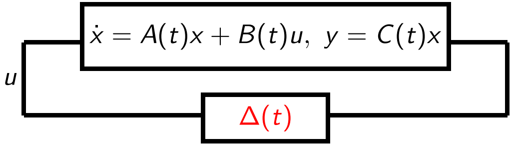
Perturbed system (time-varying parameter perturbations)

$$\mathbf{\Sigma}_\Delta : \quad \dot{x}(t) = [A(t) + B(t)\Delta(t)C(t)]x(t), \quad t \geq 0,$$

$B(\cdot) \in L^{\infty}(\mathbb{R}_+; \mathbb{K}^{n\times \ell}),\ C(\cdot) \in L^{\infty}(\mathbb{R}_+; \mathbb{K}^{q\times n})$ (perturbation structure and scaling)

$\Delta(\cdot) \in L^{\infty}(\mathbb{R}_+; \mathbb{K}^{\ell\times q})$ unknown $\qquad \mathbf{\Sigma}_\Delta$ "uncertain" system

Feedback interpretation of $\mathbf{\Sigma}_\Delta$: $\Delta(\cdot)$ memoryless time-varying feedback operator



$y\,, \quad u(t) = \Delta(t)y(t)$

Uncertainty modeled by coupling the output of (A, B, C) to its input via an unknown feedback operator $\Delta(t)$

uncertain coupling $\longleftrightarrow$ perturbation

Stability radius of time-varying linear systems

Problem: Which bounds on $\|\Delta\|_{L^\infty} := \sup_{t\geq 0} \|\Delta(t)\|$ guarantee that $\mathbf{\Sigma}_\Delta$ is u.e.s.?

Def.: $r_{\mathbb{K}}(A, B, C) = \inf\{\|\Delta(\cdot)\|_{L^\infty}; \Delta(\cdot) \in L^\infty(\mathbb{R}_+; \mathbb{K}^{\ell\times q}),\ \mathbf{\Sigma}_\Delta \textit{not u.e.s.}\}$
stability radius

Equivalent stability criteria

Prop.: For $\mathbf{\Sigma} : \dot{x}(t) = A(t)x(t)$ with transition matrix $\Phi(t, s)$ equivalent:

(i) $\mathbf{\Sigma}$ is uniformly exponentially stable.

(ii) $\Phi(t, s)$ is uniformly exponentially bounded (i.e. $\|\Phi(t, s)\| \leq Me^{\beta(t-s)}$, $t \geq s \geq 0$ for some $\beta \in \mathbb{R}$), and there exists a constant $c > 0$ such that

$$\int_{t_0}^{\infty} \|\Phi(t, t_0)\|^2 dt \leq c\,, \quad t_0 \geq 0\,.$$

(iii) There exist constants $c_1, c_2 > 0$ s.t. for all solutions $x(t; t_0, x^0) := \Phi(t, t_0)x^0$,

$$\|x(t; t_0, x^0)\|_{\mathbb{K}^n} \leq c_1\|x^0\|_{\mathbb{K}^n}, \quad t \geq t_0 \geq 0,$$

$$\|x(\cdot; t_0, x^0)\|_{L^2(t_0,\infty;\mathbb{K}^n)} \leq c_2\|x^0\|_{\mathbb{K}^n}\,, \quad t_0 \geq 0\,.$$

For (i) $\Leftrightarrow$ (ii), see
Daleckii and Krein (1974), Stability of Solutions of Diff. Equations in Banach Spaces

I/O-operator associated with $(A(\cdot),B(\cdot),C(\cdot))$ on interval $[t_0,\infty)\subset\mathbb{R}_+$

$$\mathbb{L}_{t_0} : u(\cdot) \mapsto y(\cdot) \quad \text{where } y(t) = \int_{t_0}^{t} C(t)\Phi(t,s)B(s)u(s)ds,\ \ t \geq t_0$$

Lemma: (i) $\mathbb{L}_{t_0}$ is a bounded linear operator from $L^2(t_0,\infty;\mathbb{K}^\ell)$ to $L^2(t_0,\infty;\mathbb{K}^q)$.

(ii) $t_0 \mapsto \|\mathbb{L}_{t_0}\|$ is monotonically decreasing.

(iii) If $A(\cdot), B(\cdot), C(\cdot)$ periodic with common period, then $t_0 \mapsto \|\mathbb{L}_{t_0}\|$ constant.

Prop.: For all $t_0 \geq 0$, $\quad \|\mathbb{L}_{t_0}\|^{-1} \leq r_{\mathbb{C}}(A,B,C) \leq r_{\mathbb{R}}(A,B,C)$.

Remark: Let (A,B,C) time-invariant. Then $\|\mathbb{L}_{t_0}\|^{-1} = r_{\mathbb{C}}(A,B,C)$ for all $t_0 \geq 0$.
There exists a joint "Lyapunov function" for all $\mathbf{\Sigma}_\Delta$, $\|\Delta\| < r_{\mathbb{C}}(A,B,C)$.
If the data (A,B,C) are real ($\mathbb{K}=\mathbb{R}$), in general, $r_{\mathbb{C}}(A,B,C) < r_{\mathbb{R}}(A,B,C)$.

Question: $\lim_{t_0\to\infty} \|\mathbb{L}_{t_0}\|^{-1} = r_{\mathbb{C}}(A,B,C)$ in the time-varying case??

Counterexample: Can construct scalar periodic system $(A,B,C) = (a(\cdot),1,1)$ for which $r_{\mathbb{C}}(A,B,C) = 1$ and $\|\mathbb{L}_{t_0}\|^{-1} = \|\mathbb{L}_0\|^{-1} < 1$.

Quadratic stability problem: Does there exist a joint Lyapunov for all time-varying $\mathbf{\Sigma}_\Delta$, $\Delta \in L^\infty(\mathbb{R}_+;\mathbb{K}^{\ell\times q})$, $\|\Delta\|_{L^\infty} < r_{\mathbb{K}}(A,B,C)$?

Riccati equations and the quadratic stability problem

Suppose the time-varying $\mathbf{\Sigma} = (A, B, C)$ is uniformly exponentially stable (u.e.s).

Parametrized differential Riccati equation $(DRE)_\rho$, $\rho \geq 0$:

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) + \rho^2 C^*(t)C(t) + P(t)B(t)B^*(t)P(t) = 0, \quad t \geq t_0$$

Thm.(HIP 1989): $(DRE)_\rho$ has a (unique) bounded stabilizing Hermitian solution $P(\cdot) = P_\rho(\cdot)$ on $[t_0, \infty)$ iff $\rho < \|\mathbb{L}_{t_0}\|^{-1}$.

This thm. characterizes $\|\mathbb{L}_{t_0}\|^{-1}$ – but not $r_{\mathbb{C}}(A, B, C)$ – in terms of $(DRE)_\rho$.

Thm.(HIP 1989): Suppose $t_0 \geq 0$ and $\rho < \|\mathbb{L}_{t_0}\|^{-1}$. Then:

(i) $V(t, x) = \langle x, P_\rho(t)x \rangle$ is a joint “Lyapunov function” for all $\mathbf{\Sigma}_\Delta$, $\|\Delta\| \leq \rho$.

(ii) If $(y, t) \to \Delta(y, t)$ is continuously differentiable in $y \in \mathbb{K}^q$, measurable in $t \in [t_0, \infty)$ and satisfies $\sup_{t \geq t_0} \|\Delta(y, t)\|_{\mathbb{K}^\ell} \leq \rho \|y\|_{\mathbb{K}^q}$ then the nonlinear system

$$\mathbf{\Sigma}_\Delta : \quad \dot{x}(t) = A(t)x(t) + B(t)\Delta(C(t)x(t), t), \quad t \geq t_0$$

is uniformly exponentially stable in the sense that the trajectories of $\mathbf{\Sigma}_\Delta$ satisfy $\|x(t)\|_{\mathbb{K}^n} \leq M e^{-\omega(t - t_0')} \|x(t_0')\|_{\mathbb{K}^n}$, $t \geq t_0' \geq t_0$ for some constants $M, \omega > 0$.

Extension of perturbation class from linear $\Delta(t) : \mathbb{K}^q \to \mathbb{K}^\ell$ to nonlinear $\Delta(\cdot, t) : \mathbb{K}^q \to \mathbb{K}^\ell$.

Dynamic (causal) perturbations

Def.: A (linear) operator $\Delta : L^2(\mathbb{R}_+;\mathbb{K}^q) \to L^2(\mathbb{R}_+;\mathbb{K}^\ell)$, $u \mapsto \Delta(u)$ is said to be causal if for all $w(\cdot), v(\cdot) \in L^2(\mathbb{R}_+;\mathbb{K}^q)$

$$w(\cdot)|[0,t] = v(\cdot)|[0,t] \Rightarrow \Delta(w)|[0,t] = \Delta(v)|[0,t]$$

Perturbed system (with memory)

$$\mathbf{\Sigma}_\Delta : \quad \dot{x}(t) = A(t)x(t) + B(t)\Delta(C(\cdot)x(\cdot))(t), \quad t \geq 0,$$

Fact: For every $(t_0, x^0) \in \mathbb{R}_+ \times \mathbb{K}^n$ there exists a unique continuous mild solution $x_\Delta(\cdot) = x_\Delta(\cdot; t_0, x^0) : [t_0, \infty) \to \mathbb{K}^n$ satisfying

$$x_\Delta(t) = \Phi(t,t_0)x^0 + \int_{t_0}^t \Phi(t,s)B(s)\Delta([Cx_\Delta]_{t_0})(s)ds\,, \quad t \geq t_0$$

where $[Cx_\Delta]_{t_0}$ is the trivial extension of $Cx_\Delta : [t_0,\infty) \to \mathbb{K}^n$ to $\mathbb{R}_+$.

Def.: $\mathbf{\Sigma}_\Delta$ L^2-stable if there exists a constant $c_\Delta > 0$ such that for all $t_0 \geq 0$

$$\|x_\Delta(t;t_0,x^0)\|_{\mathbb{K}^n} \leq c_\Delta \|x^0\|_{\mathbb{K}^n},\ t \geq t_0; \quad \|x_\Delta(\cdot;t_0,x^0)\|_{L^2} \leq c_\Delta \|x^0\|_{\mathbb{K}^n}\,.$$

Def.: Stability radius of (A, B, C) w.r.t. linear dynamic (causal) perturbations

$$r_{\mathbb{K},dyn}(A,B,C) = \inf\{\|\Delta(\cdot)\|;\ \Delta : L^2(\mathbb{R}_+;\mathbb{K}^q) \to L^2(\mathbb{R}_+;\mathbb{K}^\ell) \text{ causal}, \mathbf{\Sigma}_\Delta \text{not } L^2\text{-stable}\}$$

Theorem (B. Jacob 1998): (i) $r_{\mathbb{C},dyn}(A,B,C) = \lim_{t_0\to\infty} \|\mathbb{L}_{t_0}\|^{-1}$.
(ii) If $\mathbb{K} = \mathbb{R}$, then $r_{\mathbb{C},dyn}(A,B,C) = r_{\mathbb{R},dyn}(A,B,C) = \lim_{t_0\to\infty} \|\mathbb{L}_{t_0}\|^{-1}$.

Question: Does equality hold, if we restrict the perturbation class to causal operators Δ which are i/o-operators of time-varying linear systems?

Uncertain interconnected systems: Introduction

An interconnected or composite system consists of subsystems $\Sigma_1, \ldots \Sigma_N$ which are interacting.

Composite system models emerge in two ways:
They either represent a real composite system assembled by interconnecting a finite number of physical systems or they are obtained by analytical decomposition of a complex or large scale system:

- decomposition from a physical point of view
- decomposition from a mathematical point of view (e.g. for numerical purposes)

Purpose: obtain physical insight, reduce dimensions.

Typical examples: electrical and traffic networks, power systems, multispecies communities.

Uncertainties of the composite system arise because of uncertainties in the subsystems or uncertainties in the couplings.

Central problem: If the nominal subsystems are exponentially stable, under which bounds on the couplings can we assert stability of the interconnected system?

Objects: We consider time-varying uncertainties of arbitrary but fixed structure of time-varying composite systems. We want sharp bounds.

Subsystems and interconnections

For the mathematical description of a composite system we need to specify the subsystems and the couplings between them.

Given N subsystems:

$$\boldsymbol{\Sigma}_i = (A_i, B_i, C_i): \quad \begin{array}{ll} \dot{x}_i(t) & = A_i(t)x_i(t) + B_i(t)u_i(t) \\ y_i(t) & = C_i(t)x_i(t) \end{array}, \quad i = 1, \dots, N.$$

The interconnection structure between these subsystems is described by a structure matrix $E = (e_{ij}) \in \{0, 1\}^{N \times N}$ where

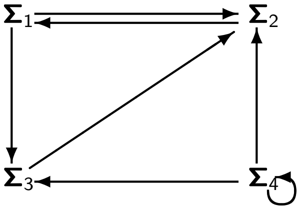
- $e_{ij} = 1$ if output of $\boldsymbol{\Sigma}_j$ is coupled to input of $\boldsymbol{\Sigma}_i$
- $e_{ij} = 0$ output of $\boldsymbol{\Sigma}_j$ is not coupled to input of $\boldsymbol{\Sigma}_i$.

The directed graph $\Gamma(E)$ associated with E has the vertices $\boldsymbol{\Sigma}_1, \dots \boldsymbol{\Sigma}_N$ and a directed arc from $\boldsymbol{\Sigma}_j$ to $\boldsymbol{\Sigma}_i$ iff $e_{ij} = 1$. "from column index to row index"

$\Gamma(E)$ describes the coupling structure of the interconnected system.

Example:

$$E = (e_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Couplings of structure E (neglect dependence on t)

Given N subsystems $\mathbf{\Sigma}_i = (A_i, B_i, C_i)$ and $E = (e_{ij}) \in \{0,1\}^{N \times N}$

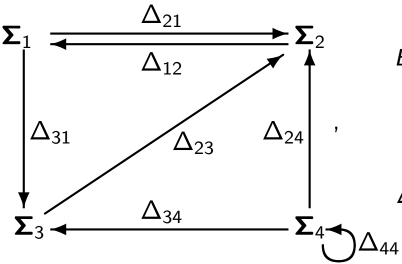
Def.: A coupling matrix of structure E is a block matrix $\Delta = (\Delta_{ij})_{i,j \in \underline{N}}$ with blocks $\Delta_{ij} \in \mathbb{K}^{\ell_i \times q_j}$ s.t. $\Delta_{ij} = 0$ if $e_{ij} = 0$.

The vector space of coupling matrices of structure E:

$$\mathbf{\Delta}_{\mathbb{K}}(E) := \{\Delta = (\Delta_{ij});\ \Delta_{ij} \in \mathbb{K}^{\ell_i \times q_j},\ \Delta_{ij} = 0 \text{ if } e_{ij} = 0\}.$$

Subsystem $\mathbf{\Sigma}_j$ is coupled to $\mathbf{\Sigma}_i$ via the uncertain coupling matrix Δ_{ij} if and only if $e_{ij} = 1$.

Example:



,

$$E = (e_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & \Delta_{12} & 0 & 0 \\ \Delta_{21} & 0 & \Delta_{23} & \Delta_{24} \\ \Delta_{31} & 0 & 0 & \Delta_{34} \\ 0 & 0 & 0 & \Delta_{44} \end{bmatrix} \in \mathbf{\Delta}_{\mathbb{K}}(E)$$

Composite system equation

Given N subsystems $\dot{x}_i = A_i(t)x_i + B_i(t)u_i,\ \ y_i = C_i(t)x_i,\quad E = (e_{ij}) \in \{0,1\}^{N\times N}$

Time-varying couplings of structure E:

$u_i = \sum_{j=1}^N \Delta_{ij}(t)y_j = \sum_{j=1}^N \Delta_{ij}(t)C_j(t)x_j,\ i \in \underline{N},$ where $\Delta(t) = (\Delta_{ij}(t)) \in \mathbf{\Delta}_{\mathbb{K}}(E)$.

Coupled system equations:

$$\dot{x}_i(t) = A_i(t)x_i(t) + B_i(t)\sum_{j=1}^N \Delta_{ij}(t)C_j(t)x_j(t), \quad i \in \underline{N}.$$

Coupled system equation in vector form: (without the argument t)

$$\Sigma_\Delta : \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \underbrace{(A + B\Delta C)}_{A(\Delta)} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \begin{matrix} A = \mathrm{diag}(A_1, \ldots, A_N), \\ B = \mathrm{diag}(B_1, \ldots, B_N), \\ C = \mathrm{diag}(C_1, \ldots, C_N), \end{matrix}$$

i-th pertubed system equation:

$$\dot{x}_i(t) = \underbrace{[A_i(t) + B_i(t)\Delta_{ii}(t)C_i(t)]}_{\text{uncertain } i\text{-th subsystem}} x_i(t) + \sum_{j \in \underline{N}\setminus\{i\}} \underbrace{B_i(t)\Delta_{ij}(t)C_j(t)x_j(t)}_{\text{uncertain influence of } \mathbf{\Sigma}_j \text{ on } \mathbf{\Sigma}_i}$$

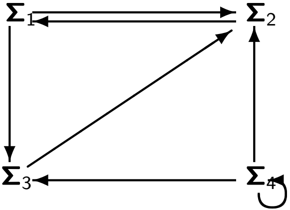
Note: Some of the subsystems may represent exosystems, i.e. dynamic models of exogeneous disturbance signals. uncertain couplings $\longleftrightarrow$ perturbations

Example: Interconnection of four subsystems

$$\mathbf{\Sigma}_i : \quad \dot{x}_i(t) = A_i(t)x_i(t) + B_i(t)u_i(t), \quad y_i(t) = C_i(t)x_i(t), \qquad i = 1, \ldots, 4.$$

Interconnection structure:

$$E = (e_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$\Gamma(E)$

Uncertain couplings of structure E:

$u_i(t) = \sum_{j=1}^{4} \Delta_{ij}(t)\, y_j(t), \; i = 1, \ldots, 4,$ where $\Delta(t) = (\Delta_{ij}(t)) \in \mathbf{\Delta}_{\mathbb{K}}(E)$ unknown

yield
$$\dot{x}_i(t) = A_i(t)x_i(t) + B_i(t) \sum_{j=1}^{4} \Delta_{ij}(t) C_j(t) x_j(t), \qquad i = 1, \ldots, 4.$$

Coupled system: (omitting argument t)

$$\mathbf{\Sigma}_\Delta : \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_1 & B_1\Delta_{12}C_2 & 0 & 0 \\ B_2\Delta_{21}C_1 & A_2 & B_2\Delta_{23}C_3 & B_2\Delta_{24}C_4 \\ B_3\Delta_{31}C_1 & 0 & A_3 & B_3\Delta_{34}C_4 \\ 0 & 0 & 0 & A_4 + B_4\Delta_{44}C_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Largest $r > 0$ s.t. $\mathbf{\Sigma}_\Delta$ is u.e.s. for all couplings Δ of structure E with $\|\Delta(t)\| < r$?

Stability radius for time-varying perturbations of structure E

Given N time-varying subsystems $\mathbf{\Sigma}_i = (A_i, B_i, C_i)$ u.e.s., $E = (e_{ij}) \in \{0,1\}^{N\times N}$.
$A := \operatorname{diag}(A_1, \ldots, A_N)$, $B := \operatorname{diag}(B_1, \ldots, B_N)$, $C := \operatorname{diag}(C_1, \ldots, C_N)$.

Provide $\mathbf{\Delta}_{\mathbb{K}}(E) = \{\Delta = (\Delta_{ij}); \Delta_{ij} \in \mathbb{K}^{\ell_i \times q_j}, \Delta_{ij} = 0 \text{ if } e_{ij} = 0\}$ with norm
$\|\Delta\|_{\mathbf{\Delta}_{\mathbb{K}}(E)} := \max_{i\in\underline{N}} \sigma_{\max}([\Delta_{i1}, \ldots, \Delta_{iN}]) = \max_{i\in\underline{N}} \left[\lambda_{\max}(\sum_{j\in\underline{N}} \Delta_{ij}\Delta_{ij}^*)\right]^{1/2}$.

Def.: $\mathbf{\Delta}_{\mathbb{K}}^{tv}(E)$ normed space of time-varying perturbations of structure E

$$\Delta(\cdot) : \mathbb{R}_+ \to \mathbf{\Delta}_{\mathbb{K}}(E) \text{ bounded measurable}$$

$$\|\Delta(\cdot)\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}} := \sup_{t\geq 0} \|\Delta(t)\|_{\mathbf{\Delta}_{\mathbb{K}}(E)} = \sup_{t\geq 0} \max_{i\in\underline{N}} \left[\lambda_{\max}\Big(\sum_{j\in\underline{N}} \Delta_{ij}(t)\Delta_{ij}(t)^*\Big)\right]^{1/2}$$

Perturbed system equation:

$$\mathbf{\Sigma}_\Delta \quad \dot{x} = [A(t) + B(t)\Delta(t)C(t)]x, \quad \Delta(\cdot) \in \mathbf{\Delta}_{\mathbb{K}}^{tv} := \mathbf{\Delta}_{\mathbb{K}}^{tv}(E).$$

Def.: Stability radius of (A, B, C) w.r.t. time-varying perturbations of structure E

$$r_{\mathbb{K}} = r_{\mathbb{K}}(A, B, C, E) := \inf\{\|\Delta(\cdot)\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}}; \Delta(\cdot) \in \mathbf{\Delta}_{\mathbb{K}}^{tv}(E) \text{ and } \mathbf{\Sigma}_\Delta \text{ not u.e.s.}\}.$$

Problems:

Problem 1: For which $r > 0$ does $\Delta \in \mathbf{\Delta}_{\mathbb{K}}^{tv}(E)$, $\|\Delta\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}} \leq r$ imply the uniform exponential stability of $\mathbf{\Sigma}_{\Delta}$? The maximal r with this property is the stability radius $r_{\mathbb{K}}(A, B, C, E)$.

Problem 2: For which $r > 0$ does there exist a joint quadratic Liapunov function guaranteeing uniform exponential stability for all the systems

$$\mathbf{\Sigma}_{\Delta}: \quad \dot{x} = [A(t) + B(t)\Delta(t)C(t)]x, \quad \Delta \in \mathbf{\Delta}_{\mathbb{K}}^{tv}(E), \|\Delta\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}} \leq r.$$

(In this case nonlinear and dynamic perturbations of norm $\leq r$ will not destroy uniform exponential stability.)

Transformation of Δ to block-diagonal form

Given N u.e. stable tv systems $\mathbf{\Sigma}_i = (A_i, B_i, C_i)$, $\quad E = (e_{ij}) \in \{0,1\}^{N\times N}$.

Associated i/o-operators: $\mathbb{L}^i_{t_0} : L^2(t_0,\infty;\mathbb{K}^{\ell_i}) \to L^2(t_0,\infty;\mathbb{K}^{q_i})$ defined by:

$$\mathbb{L}^i_{t_0} : u_i(\cdot) \mapsto y_i(\cdot) \quad \text{where } y_i(t) = \int_{t_0}^t C_i(t)\Phi_i(t,s)B_i(s)u_i(s)ds,\ \ t \geq t_0$$

$$A = \underset{n\times n}{\mathrm{diag}(A_1,\dots,A_N)},\ B = \underset{n\times \ell}{\mathrm{diag}(B_1,\dots,B_N)},\ C = \underset{q\times n}{\mathrm{diag}(C_1,\dots,C_N)} \quad \text{(omitting } t\text{)}$$

Diagonalization trick:

With $\Delta(\cdot) = (\Delta_{ij}(\cdot)) \in \mathbf{\Delta}^{tv}_{\mathbb{K}}(E)$ associate tv block-diagonal matrix of size $\ell \times Nq$:

$\tilde{\Delta} = \mathrm{diag}(\Delta^1,\dots,\Delta^N)$, $\ \Delta^i = [\Delta_{i1},\dots,\Delta_{iN}]$ block rows of Δ = diagonal blocks of $\tilde{\Delta}$!

Spectral norm of $\tilde{\Delta}(t) : \mathbb{K}^{Nq} \to \mathbb{K}^{\ell}$: $\quad \|\tilde{\Delta}(t)\| = \max_{i\in\underline{N}} \|\Delta^i(t)\| = \|\Delta(t)\|_{\mathbf{\Delta}_{\mathbb{K}}(E)}$

Hence

$$\|\tilde{\Delta}(\cdot)\|_{L_\infty} = \sup_{t\geq 0}\|\tilde{\Delta}(t)\| = \sup_{t\geq 0} max_{i\in\underline{N}}\|\Delta^i(t)\| = \sup_{t\geq 0}\|\Delta(t)\|_{\mathbf{\Delta}_{\mathbb{K}}(E)} = \|\Delta(\cdot)\|_{\mathbf{\Delta}^{tv}_{\mathbb{K}}}$$

If $\tilde{C} = \begin{bmatrix} \tilde{C}^1 \\ \vdots \\ \tilde{C}^N \end{bmatrix}$, $\ \tilde{C}^i = \mathrm{diag}\,(e_{i1}C_1,\dots,e_{iN}C_N)$, then $B(t)\Delta(t)C(t) = B(t)\tilde{\Delta}(t)\tilde{C}(t)$.

Blow up of dimensions: $\tilde{C}(t) \in \mathbb{K}^{Nq\times n}$, $\tilde{\Delta}(t) \in \mathbb{K}^{\ell\times Nq}$. $\quad$ What is the advantage?

Estimate for the time-varying stability radius of structure E

$\tilde\Delta(\cdot) = \mathrm{diag}(\Delta^1(\cdot), ..., \Delta^N(\cdot)), \;\; \Delta^i(\cdot) = [\Delta_{i1}(\cdot), ..., \Delta_{iN}(\cdot)], \;\; \|\tilde\Delta(\cdot)\| = \|\Delta(\cdot)\|_{\mathbf{\Delta}^{tv}_{\mathbb{K}}}$

For $\gamma = (\gamma_1, ..., \gamma_N) > 0$ let $L_\gamma = \mathrm{diag}(\gamma_1^{-1} I_{\ell_1}, ..., \gamma_N^{-1} I_{\ell_N})$, $R_\gamma = \mathrm{diag}(\gamma_1 I_q, ..., \gamma_N I_q)$, where $q = q_1 + \cdots + q_N$. Then $L_\gamma \tilde\Delta(t) R_\gamma = \tilde\Delta(t)$ and
$B(t)\Delta(t)C(t) = B(t)\tilde\Delta(t)\tilde C(t) = B(t)\, L_\gamma \tilde\Delta(t) R_\gamma\, \tilde C(t) = B_\gamma(t)\tilde\Delta(t)\tilde C_\gamma(t)$.

Perturbed system equation:

$$\mathbf{\Sigma}_\Delta : \dot x = [A(t) + B(t)\Delta(t)C(t)]x = [A(t) + B_\gamma(t)\tilde\Delta(t)\tilde C_\gamma(t)]x, \quad \Delta(\cdot) \in \mathbf{\Delta}^{tv}_{\mathbb{K}}(E).$$

Theorem: Let $\mathbb{L}_{t_0,\gamma} : L^2(t_0,\infty;\mathbb{K}^\ell) \to L^2(t_0,\infty;\mathbb{K}^{Nq})$ i/o-operator of $(A, B_\gamma, \tilde C_\gamma)$. Then

$$\|\mathbb{L}_{t_0,\gamma}\|^2 = \max_{j\in\underline{N}} \frac{\|\mathbb{L}^j_{t_0}\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2, \qquad t_0 \ge 0,\ \gamma > 0.$$

If $\|\Delta(\cdot)\|_{\mathbf{\Delta}^{tv}_{\mathbb{K}}} < \|\mathbb{L}_{t_0,\gamma}\|^{-1}$ then $\mathbf{\Sigma}_\Delta$ is uniformly exponentially stable. In particular,

$$r_{\mathbb{K}}(A,B,C,E) = \inf\{\|\Delta(\cdot)\|_{\mathbf{\Delta}^{tv}_{\mathbb{K}}};\, \Delta(\cdot) \in \mathbf{\Delta}^{tv}_{\mathbb{K}}(E) \text{ and } \mathbf{\Sigma}_\Delta \text{ not u.e.s.}\}$$
$$\ge \left[\max_{j\in\underline{N}} \frac{\|\mathbb{L}^j_{t_0}\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2\right]^{-1/2}, \quad \gamma = (\gamma_1, ..., \gamma_N) > 0.$$

Optimal scaling

$\mathbf{\Sigma}_\Delta$ is uniformly exponentially stable if $\Delta(\cdot) \in \mathbf{\Delta}_{\mathbb{K}}^{tv}(E)$ and

$$\|\Delta(\cdot)\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}} < \rho_\gamma = \left[\max_{j\in\underline{N}} \frac{\|\mathbb{L}_{t_0}^j\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2\right]^{-1/2} = \|\mathbb{L}_{t_0,\gamma}\|^{-1}, \quad \gamma = (\gamma_1, \dots, \gamma_N) > 0.$$

Prop.: Let $D_{t_0} = \operatorname{diag}(\|\mathbb{L}_{t_0}^1\|, \dots, \|\mathbb{L}_{t_0}^N)\|)$. Then

$$\sup_{\gamma>0} \rho_\gamma = \left[\inf_{\gamma>0} \max_{j\in\underline{N}} \frac{\|\mathbb{L}_{t_0}^j\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2\right]^{-1/2} = \varrho(ED_{t_0}^2)^{-1/2} = \varrho(D_{t_0}^2 E)^{-1/2}.$$

spectral radius

Cor.:
$r_{\mathbb{R}}(A,B,C,E) \geq r_{\mathbb{C}}(A,B,C,E) \geq \lim_{t_0\to\infty} \varrho(ED_{t_0}^2)^{-1/2} = \sup_{t_0\geq 0} \varrho(ED_{t_0}^2)^{-1/2}$.

Remark: For time-invariant (A,B,C) and time-varying perturbations of structure E equality holds: $r_{\mathbb{C}}(A,B,C,E) = \varrho(ED_{t_0}^2)^{-1/2}$ for any $t_0 \geq 0$.

Conjecture: Equality holds if dynamical interconnections of structure E are admitted.

Question: Under which conditions $\exists$ optimal scaling vector $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_N) > 0$?

Prop.: $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_N) > 0$ is optimal, i.e. $\rho_{\hat{\gamma}} = \sup_{\gamma>0} \rho_\gamma$, if and only if $\hat{\gamma}^2 := (\hat{\gamma}_1{}^2, \dots, \hat{\gamma}_N{}^2)$ is a left eigenvector of $ED_{t_0}^2$ that corresponds to $\varrho(ED_{t_0}^2)$.
$\hat{\gamma}$ exists if $ED_{t_0}^2$ is irreducible ($\Leftrightarrow$ interconnection graph $\Gamma(E)$ strongly connected).

Optimal scaling

$\mathbf{\Sigma}_\Delta$ is uniformly exponentially stable if $\Delta(\cdot) \in \mathbf{\Delta}_{\mathbb{K}}^{tv}(E)$ and

$$\|\Delta(\cdot)\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}} < \rho_\gamma = \left[\max_{j\in\underline{N}} \frac{\|\mathbb{L}_{t_0}^j\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2\right]^{-1/2} = \|\mathbb{L}_{t_0,\gamma}\|^{-1}, \quad \gamma = (\gamma_1, \dots, \gamma_N) > 0.$$

Prop.: Let $D_{t_0} = \operatorname{diag}(\|\mathbb{L}_{t_0}^1\|, \dots, \|\mathbb{L}_{t_0}^N)\|)$. Then

$$\sup_{\gamma>0} \rho_\gamma = \left[\inf_{\gamma>0} \max_{j\in\underline{N}} \frac{\|\mathbb{L}_{t_0}^j\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2\right]^{-1/2} = \varrho(ED_{t_0}^2)^{-1/2} = \varrho(D_{t_0}^2 E)^{-1/2}.$$

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$\hat{\gamma}$ exists if $ED_{t_0}^2$ is irreducible ($\Leftrightarrow$ interconnection graph $\Gamma(E)$ strongly connected).

Special interconnection structures

1. $E = (e_{ij})$ strictly upper triangular, i.e. $e_{ij} = 0$ if $1 \le i \le j \le N$.
No subsystem $\mathbf{\Sigma}_j$ uncertain (perturbed), $\mathbf{\Sigma}_j$ acts on $\mathbf{\Sigma}_i$ only if $i < j$.
Then $ED^2_{t_0}$ nilpotent and $\varrho(ED^2_{t_0}) = 0$, hence $r_{\mathbb{K}}(A,B,C,E) = \infty$.

2. $E = I_N$. No interaction between different subsystems, but all subsystems uncertain.
Then $ED^2_{t_0} = \mathrm{diag}(\|\mathbb{L}^1_{t_0}\|^2, \dots, \|\mathbb{L}^N_{t_0}\|^2)$ and $\varrho(ED^2_{t_0}) = \max_{j \in \underline{N}} \|\mathbb{L}^j_{t_0}\|^2$ and so
$r_{\mathbb{K}}(A,B,C,E) \ge \lim_{t_0 \to \infty} \varrho(ED^2_{t_0})^{-1/2} = \lim_{t_0 \to \infty} \min_{j \in \underline{N}} \|\mathbb{L}^j_{t_0}\|^{-1}$.

3. E cyclic:

$$E = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad ED^2_{t_0} = \begin{bmatrix} 0 & \|\mathbb{L}^2_{t_0}\|^2 & 0 & \cdots & 0 \\ 0 & 0 & \|\mathbb{L}^3_{t_0}\|^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \|\mathbb{L}^N_{t_0}\|^2 \\ \|\mathbb{L}^1_{t_0}\|^2 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

No subsystem perturbed, but $\mathbf{\Sigma}_j$ acts on $\mathbf{\Sigma}_i$ if $i > j$ and $\mathbf{\Sigma}_1$ acts on $\mathbf{\Sigma}_N$.

$$r_{\mathbb{K}}(A, B, C, E) \ge \varrho(ED^2_{t_0})^{-1/2} = \sqrt[N]{\|\mathbb{L}^1_{t_0}\|^2 \cdots \|\mathbb{L}^N_{t_0}\|^2}^{\,-1/2} = \sqrt[N]{\|\mathbb{L}^1_{t_0}\| \cdots \|\mathbb{L}^N_{t_0}\|}^{\,-1}$$

4. $E = [\text{ones}]$, i.e. all $e_{ij} = 1$. All subsystems perturbed, all couplings uncertain.
Then $r_{\mathbb{K}}(A, B, C, E) \ge \varrho(ED^2_{t_0})^{-1/2} = \left[\sum_{j=1}^N \|\mathbb{L}^j_{t_0}\|^2\right]^{-1/2}$.

Scaled Riccati equations and joint Lyapunov functions

N time-varying systems $\mathbf{\Sigma}_i = (A_i, B_i, C_i)$, $i \in \underline{N}$ u.e.s., $E = (e_{ij}) \in \{0,1\}^{N\times N}$.

Let $A = \operatorname{diag}(A_1, \dots, A_N)$ and for any scaling vector $\gamma = (\gamma_1, \dots, \gamma_N) > 0$ define

$$B_\gamma = BL_\gamma = \operatorname{diag}(\gamma_1^{-1}B_1, \dots, \gamma_N^{-1}B_N),\ \tilde{C}_\gamma = R_\gamma \tilde{C} = \begin{bmatrix} \gamma_1 \tilde{C}^1 \\ \vdots \\ \gamma_N \tilde{C}^N \end{bmatrix},\ \tilde{C}^i = \operatorname{diag}(e_{i1}C_1, \dots, e_{iN}C_N)$$

Scaled parametrized differential Riccati equation $(DRE)_{\rho,\gamma}$, $\gamma > 0$, $\rho \geq 0$:

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) + \rho^2 \tilde{C}_\gamma^*(t)\tilde{C}_\gamma(t) + P(t)B_\gamma(t)B_\gamma^*(t)P(t) = 0, \quad t \geq t_0$$

Thm: (i) $(DRE)_{\rho,\gamma}$ has a bounded stabilizing Hermitian solution $P(\cdot) = P_{\rho,\gamma}(\cdot)$ on $[t_0, \infty)$ iff $\rho < \rho_\gamma := \|\mathbb{L}_{t_0,\gamma}\|^{-1} = \left[\max_{j\in\underline{N}} \frac{\|\mathbb{L}_{t_0}^j\|^2}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_{ij}^2\right]^{-1/2}$.

(ii) If $\rho < \varrho(ED_{t_0}^2)^{-1/2} = \sup_{\gamma>0} \rho_\gamma$, choose scaling $\gamma > 0$ such that $\rho < \rho_\gamma$. Then $V_{\rho,\gamma}(x) = \langle x, P_{\rho,\gamma}x\rangle$ is a joint quadratic "Liapunov function" guaranteeing uniform exponential stability for all the interconnected systems

$$\mathbf{\Sigma}_\Delta : \quad \dot{x} = [A(t) + B(t)\Delta(t)C(t)]x, \ \ t \geq t_0, \quad \Delta(\cdot) \in \mathbf{\Delta}_{\mathbb{K}}^{tv}(E), \|\Delta\|_{\mathbf{\Delta}_{\mathbb{K}}^{tv}} \leq \rho.$$

$P_{\rho,\gamma} \succ 0$ if E has no zero column and the pairs (A_j, C_j), $j \in \underline{N}$ are uniformly observable.

Summary

Results:

- Using a scaling method we have found a lower bound for $r_{\mathbb{K}}(A, B, C, E)$ which may be sharp if dynamic interconnections instead of time-varying memoryless interconnections of structure E are considered.
- This bound $\varrho(ED_{t_0}^2)^{-1/2}$ can easily be computed if the norms of the i/o-operators of the time-varying linear subsystems are known.
- For all interconnected systems $\mathbf{\Sigma}_\Delta$ where the norm of $\Delta(\cdot)$ is below the bound $\varrho(ED_{t_0}^2)^{-1/2}$ we have constructed a joint quadratic Lyapunov function by means of a scaled parametrized Riccati equation. This Lyapunov function can be used to deal with time-varying nonlinear and dynamic interconnections.

To do:

- Prove that the lower bound $\varrho(ED_{t_0}^2)^{-1/2}$ of $r(A, B, C, E)$ is sharp if arbitrary causal linear perturbation operators of structure E (instead of time-varying memoryless perturbations) are considered.
- Restrict the perturbation class from arbitrary causal linear operators to i/o-operators of time-varying linear systems.

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