Analysis and numerical solution of differential-algebraic equations with delay

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Mathematics for key technologies

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Outline

1. Introduction
2. Analysis of DDAEs
3. Linear DAE theory
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6. The method of steps
7. Generalized method of steps
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Delay differential-algebraic equations (DDAE) of the form

\[ F(t, x(t), \dot{x}(t), x(t - \tau), u(t)) = 0, \]

in a time interval \([0, t_f)\), with state \(x : [-\tau, t_f) \rightarrow \mathbb{C}^n\), control \(u : [-\tau, t_f) \rightarrow \mathbb{C}^m\), delay time \(\tau\).

Here \(\dot{x}\) denotes the time derivative of the vector valued function \(x\).
Linearizing around non-stationary reference solution yields linear Delay Differential-Algebraic Equation (DDAE)

\[
E(t)\dot{x}(t) = A(t)x(t) + D(t)x(t - \tau) + B(t)u(t) + r(t),
\]

\(E, A, D : [-\tau, t_f) \to \mathbb{C}^{n,n}, B : [-\tau, t_f) \to \mathbb{C}^{n,m}, r : [-\tau, t_f) \to \mathbb{C}^n.\)

For uniqueness one needs initial (or boundary) functions

\[
\phi : [-\tau, 0] \to \mathbb{C}^n, \ x|_{[-\tau,0]} = \phi.
\]

For simplicity, we assume that \(t_f = \ell\tau,\) so that the time interval is \(I := [0, \ell\tau),\) for an integer \(\ell \in \mathbb{N}.\)

In the following we only discuss the forward problem, no control. More delays, nonlinear version, \(\tau = \tau(x, t),\) and many more extensions possible.
DAEs provide a unified framework for the analysis, simulation and control of (coupled) dynamical systems (continuous and discrete time).

- Automatic modeling (modelica, dymola, simscape, ...) leads to DAEs. *(Constraints at interfaces).*
- Conservation laws lead to DAEs. *(Conservation of mass, energy, momentum).*
- Coupling of solvers leads to DAEs *(discrete time).*
- Control problems are DAEs *(behavior).*
Most real world problems have (typically small) delay and are constrained.

- electrical circuits,
- stick-slip friction,
- brain models,
- real time feedback control,
- juggling and balancing,
- . . . .
Model/software based control of automatic transmission.
Project with Daimler AG
Half-toroid model
Mathematical tasks

- Development of control/optimization methods for coupled system.
- Model reduction to make control/optimization feasible.
- Real time control software for transmission on board computer. Small delay
- Our tasks, build a controller.

Ultimate goals: Decrease fuel consumption, save money on production, improve switching.
Classical control engineering approach

- Build a prototype.
- Measure the input/output behavior.
- Build a (linear) model of the input/output model.
- Construct a controller.
- Apply it in the physical system.

This took a while, was very expensive, but worked reasonably well. But prototypes are expensive. Model based approach failed: Very different space and time scales; delays due to different time scales.

Pressure control of hydraulic system.
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DDAE

\[ E(t)\dot{x}(t) = A(t)x(t) + D(t)x(t - \tau) + r(t), \]

In intervals \([(k - 1)\tau, k\tau]\) we just have a DAE

\[ E(t)\dot{x}(t) = A(t)x(t) + f(t), \]

with

\[ f(t) = D(t)x(t - \tau) + r(t) \]

But to solve this DAE we need solvability and consistency of initial conditions.
Definition

1. A function \( x : [-\tau, t_f) \rightarrow \mathbb{C}^n \) is called a *piecewise differentiable solution* of the DDAE, if it is continuous, piecewise continuously differentiable and satisfies the DDAE a.e..

2. An initial function \( \phi \) is called *consistent* if the ivp for the DDAE has at least one piecewise differentiable solution.

3. The DDAE is called *solvable* if it has at least one piecewise differentiable solution. It is called *regular* if in addition, for every consistent initial function, the solution of the initial value problem is unique.

We could consider weak solutions in \( H_1 \). Delay PDAEs master thesis Christoph Zimmer 2015, but here \( C^0 \) and piecewise \( C^1 \).
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Derivative arrays

For numerical methods and for the design of controllers, we use derivative arrays (Campbell 1989). We assume that derivatives of original functions are available or can be obtained via computer algebra or automatic differentiation.

Linear case: We put \( E(t) \dot{x} = A(t)x + f(t) \) and its derivatives up to order \( \mu \) into a large DAE

\[
M_k(t) \dot{z}_k = N_k(t) z_k + g_k(t), \quad k \in \mathbb{N}_0
\]

for \( z_k = [x^T, \dot{x}^T, \ldots, x^{(k)}^T] \).

\[
M_2 = \begin{bmatrix}
    E & 0 & 0 \\
    \dot{E} - A & E & 0 \\
    -2\dot{A} + \ddot{E} & 2\dot{E} - A & E
\end{bmatrix}, \quad N_2 = \begin{bmatrix}
    A & 0 & 0 \\
    \dot{A} & 0 & 0 \\
    \ddot{A} & 0 & 0
\end{bmatrix}, \quad z_2 = \begin{bmatrix}
    x \\
    \dot{x} \\
    \ddot{x}
\end{bmatrix}.
\]
Theorem (Kunkel/M. 1996)

Under some constant rank assumptions, for every linear DAE there exist integers $\mu$, $a$, $d$ and $v$ such that:

1. $\text{corank } M_{\mu + 1}(t) - \text{corank } M_{\mu}(t) = v$.

2. $\text{rank } M_{\mu}(t) = (\mu + 1)m - a - v$ on $\mathbb{I}$, and there exists a smooth matrix function $Z_{2,3}$ (left nullspace of $M_{\mu}$) with $Z_{2,3}^T M_{\mu}(t) = 0$.

3. The projection $Z_{2,3}$ can be partitioned into two parts: $Z_2$ (left nullspace of $(M_{\mu}, N_{\mu})$) so that the first block column $\hat{A}_2$ of $Z_2^* N_{\mu}(t)$ has full rank $a$ and $Z_3^* N_{\mu}(t) = 0$. Let $T_2$ be a smooth matrix function such that $\hat{A}_2 T_2 = 0$, (right nullspace of $\hat{A}_2$).

4. $\text{rank } E(t) T_2 = d = \ell - a - v$ and there exists a smooth matrix function $Z_1$ of size $(n, d)$ with $\text{rank } \hat{E}_1 = d$, where $\hat{E}_1 = Z_1^T E$.

The minimal $\mu$ is called the strangeness-index.
Reduced problem

\( Z_{2,3}^T \) operates on the derivative array

\[
M_\mu(t) \dot{z}_\mu = N_\mu(t) z_\mu + g_\mu(t),
\]

and picks out the algebraic and the redundant part.

\( Z_3^T \) operates on the original system

\[
E(t) \dot{x} = A(t)x + f(t),
\]

and picks out the dynamic part.

Note: No change of variables
We obtain a numerically computable condensed form

\[
\begin{align*}
\dot{E}_1(t)\dot{x} & = \dot{A}_1(t)x + \dot{f}_1(t), \quad d_\mu \text{ equations} \\
0 & = \dot{A}_2(t)x + \dot{f}_2(t), \quad a_\mu \text{ equations} \\
0 & = \dot{f}_3(t), \quad v_\mu \text{ equations}
\end{align*}
\]

where \( \dot{A}_1 = Z_1^T A, \dot{f}_1 = Z_1^T f, \) and \( \dot{f}_2 = Z_2^T g_\mu, \dot{f}_3 = Z_3^T g_\mu. \) This system is strangeness-free/index one.
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To understand the regularization process for DDAEs, we rewrite the DDAE as

$$E(t)\dot{x}(t) = A(t)x(t) + T(t)\lambda(t) + f(t),$$

for all $t \in I$, together with an initial condition

$$x(0) = x^0,$$

which comes from the initial function.

The function parameter $\lambda$ is called \textit{consistent} if the DDAE ivp is solvable (not necessarily uniquely solvable).
The linear DAE is called **retarded**, **neutral**, or **advanced** if the minimum smoothness requirement for a consistent function parameter $\lambda$ is that $\lambda \in C^0_p(I, \mathbb{C}^p)$, $\lambda \in C^1_p(I, \mathbb{C}^p)$, or $\lambda \in C^k_p(I, \mathbb{C}^p)$ for some $k \geq 2$. 
Definition

A linear DDAE is called *causal* if for a consistent initial function the solution \( x(t) \) of the corresponding ivp at the current time \( t \) depends only on the inhomogeneity \( f \) at current and past time points.

Both DDEs and DAEs are causal, DDAEs are not always causal. For example, the scalar equation

\[
0 \cdot \dot{x}(t) = 0 \cdot x(t) + x(t - \tau) - f(t), \quad \text{for all } t \in (0, \infty),
\]

is noncausal, since the unique solution \( x(t) = f(t + \tau) \) depends on \( f \) at the future time point \( t + \tau \).
Causality

- Noncausality problem comes from delays in the algebraic equations.
- Are they realistic?
- Yes, e.g. in the optimality conditions for optimal control of DDAEs and in feedback control based on anticipated states.
- Noncausality requires regularization, either by reformulation or by boundary conditions
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Theorem

Consider a causal linear DDAE and assume that the strangeness-index $\mu$ is well-defined for the pair of functions $(E, A)$. Then the DDAE has the same solution set as the DDAE

$$
\begin{bmatrix}
\hat{E}_1(t) \\
0 \\
0
\end{bmatrix} \dot{x}(t) = 
\begin{bmatrix}
\hat{A}_1(t) \\
\hat{A}_2(t)
\end{bmatrix} x(t) + 
\begin{bmatrix}
\hat{D}_{0,1}(t) \\
\hat{D}_{0,2}(t)
\end{bmatrix} x(t - \tau) 
+ \sum_{i=1}^{\mu} 
\begin{bmatrix}
0 \\
\hat{D}_{i,2}(t)
\end{bmatrix} x^{(i)}(t - \tau) + 
\begin{bmatrix}
\hat{f}_1(t) \\
\hat{f}_2(t) \\
\hat{f}_3(t)
\end{bmatrix},
$$

\[ d \]

\[ a \]

\[ v \]

where $[\hat{E}_1^T \hat{A}_2^T]^T$ has pointwise full row rank.
Solvability

Corollary

Assume that all requirements of previous Theorem are satisfied.

1. **The DDAE is solvable if and only if either** \( v = 0 \) \( \) or \( \hat{f}_3(t) = 0 \), for all \( t \geq 0 \).

2. **An initial function** \( \phi \) **is consistent if and only if** \( \phi \) **is sufficiently smooth and satisfies**

\[
0 = \hat{A}_2(0)\phi(0) + \sum_{i=0}^{\mu} \hat{B}_{i,3}(0)\phi^{(i)}(-\tau) + \hat{f}_2(0).
\]

3. **The DDAE is regular if and only if in addition,** \( \begin{bmatrix} \hat{E}^T_1 & \hat{A}_2^T \end{bmatrix}^T \) **is pointwise invertible, i.e.,** \( d + a = n \).

A square DDAE is causal if and only if the associated DAE is regular.
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Introducing sequences of matrix and vector valued functions $E_i$, $A_i$, $B_i$, $f_i$, $x_i$, for each $i \in \mathbb{N}$, on the time interval $[0, \tau]$ via

\[
E_i(t) := E(t + (i - 1)\tau), \quad A_i(t) := A(t + (i - 1)\tau),
\]

\[
D_i(t) := D(t + (i - 1)\tau), \quad f_i(t) := f(t + (i - 1)\tau),
\]

\[
x_i(t) := x(t + (i - 1)\tau), \quad x_0(t) := \phi(t - \tau),
\]

one can rewrite the DDAE ivp as the sequence of DAEs

\[
E_i(t) \dot{x}_i(t) = A_i(t)x_i(t) + D_i(t)x_{i-1}(t) + f_i(t),
\]

for $t \in (0, \tau)$, and for $i = 1, 2, \ldots, \ell$, with initial conditions

\[
x_i(0) = x_{i-1}(\tau).
\]

So if $x_{i-1}$ has been computed then we can compute $x_i$. This works well for DDEs but has problems for DDAEs.
Problems with DDAEs

- For method of steps we need that the ivp has a unique solution $x_i$ for any sufficiently smooth function $D_i(t)x_{i-1}(t) + f_i(t)$ and for any consistent $x_i(0)$.
- If this holds then we can just apply a DAE solver, but it does not work for non-causal DDAEs.
- Inherited from the theory of DAEs, hidden structures may exist in DDAEs and in fact, even though it looks like of retarded type, the DDAE can possess an underlying DDE of neutral or advanced type.
- Furthermore, the order can increase.
Consider the DDAE

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \dot{x}(t) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} x(t) + \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix} x(t - \tau) + \begin{bmatrix}
1 \\
-t
\end{bmatrix},
\]

for all \( t \in (0, \infty) \) and an initial function \( x(t) = \phi(t) \) for \( t \in [-\tau, 0] \). For \( t \in (0, \tau) \), inserting \( x(t - \tau) = \phi(t - \tau) \) yields a system, which does not uniquely determine the second component of \( x(t) \). But, the DDAE at \( t + \tau \) gives rise to the system

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \dot{x}(t + \tau) = \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix} x(t) + \begin{bmatrix}
1 \\
-t - \tau
\end{bmatrix},
\]

which contains the algebraic constraint \( 0 = \begin{bmatrix}
1 & 1
\end{bmatrix} x(t) - t - \tau \).
To regularize DDAEs we need two different operators:

- The *shift (forward) operator* that maps the equation into the equation
  \[ E(t + \tau) \dot{x}(t + \tau) = A(t + \tau)x(t + \tau) + B(t + \tau)x(t) + f(t + \tau), \]
  provided that the point \( t \) satisfies \( t < t_f - \tau \).

- The *differentiation operator* that maps the DDAE into the equation
  \[ \frac{d}{dt} (E(t) \dot{x}(t) - A(t)x(t)) = \frac{d}{dt} (B(t)x(t - \tau) + f(t)). \]

- In general the two operators do not commute, since the derivatives of the functions \( E, A, B, x, f \) may exist at the point \( t + \tau \) but not at the point \( t \), or vice versa.

- Finding an optimal way to combine the differentiation and shift operator is still an open problem.
Problems with DDAEs

DDAEs can have hidden algebraic constraints and higher order differential equations.

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t)
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t-1) \\
x_2(t-1) \\
x_3(t-1)
\end{bmatrix}
+ 
\begin{bmatrix}
-t \\
-1 - e^{t-1}
\end{bmatrix}.
\]

Hidden second order equation:

\[
0 = \ddot{x}_1(t-1) - e^{t-1}.
\]

gives a consistency condition for \( \phi \) in \([0, 1]\) and shifting forward, we obtain a hidden second order differential equation.

We cannot select all the differential equations for the dynamics from the original DDAE.
Generalized method of steps

Compute the solution $x$ of the DDAE ivp in $[(i - 1)\tau, i\tau]$, $1 \leq i \leq \ell$, or equivalently, the function $x_i$, provided that $x_{i-1}, \ldots, x_0$ are already known. The sequence of DAEs

$$E_j(t)\dot{x}_j(t) = A_j(t)x_j(t) + D_j(t)x_{j-1}(t) + f_j(t), \quad j = 1, \ldots, i - 1,$$

contains redundant equations, which do not contribute to the determination of $x_i$, but the solvability of $x_i$ is governed by the sequence of DAEs

$$E_{i+j}(t)\dot{x}_{i+j}(t) = A_{i+j}(t)x_{i+j}(t) + D_{i+j}(t)x_{i+j-1}(t) + f_{i+j}(t), \quad j = 0, \ldots, \ell - i.$$

The sequence of DAEs may have finitely many equations ($\ell < \infty$) or infinitely many equations ($\ell = \infty$).
**Definition**

For a fixed $i \leq \ell$, consider the sequence of DAEs in the method of steps. The minimum integer $k \geq 0$ such that the so-called *shift-inflated system*

$$E_{i+j}(t)\dot{x}_{i+j}(t) = A_{i+j}(t)x_{i+j}(t) + B_{i+j}(t)x_{i+j-1}(t) + f_{i+j}(t), \quad j = 0, \ldots, k,$$

has a unique solution $x_i$, provided $x_{i-1}$ is given and assumed that the initial vector $x_i(0) = x_{i-1}(\tau)$ is consistent, is called the *shift index* with respect to $i$, and denoted by $\kappa(i)$. 

Theorem

Suppose that the DDAE is not of advanced type and consider the shift inflated system. If the strangeness indices for the associated DAEs are well-defined, then there exists a unique shift index $\kappa(i)$ with respect to $i$.

Shifting and differentiating gives a strangeness-free DDAE for which numerical methods can be directly employed in the method of steps.
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Numerical Procedure

Generalized method of steps for DDAEs of retarded or neutral type.

- Form double-inflated array by first differentiating and then shifting.
- Filter out pointwise the regular, strangeness-free formulation from the double-inflated system.
- Perform solver on this system.
Consider the DDAE ivp

\[
\begin{bmatrix}
0 & 0 \\
1 & -t
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix}
= \begin{bmatrix}
1 & -t \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t-\tau) \\
x_2(t-\tau)
\end{bmatrix}
+ \begin{bmatrix}
f_1(t) \\
f_2(t)
\end{bmatrix},
\]

for \( t \in [0, \infty) \), \( \tau = 1 \), with \( \phi(t) := \begin{bmatrix} e^{t/10} \\ t \end{bmatrix} \), for \( t \in [-\tau, 0] \).

The analytic solution is

\[
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
= \begin{bmatrix} e^{t/10} \\ t \end{bmatrix}.
\]

DDAE is noncausal, has hidden differential equation and \( \kappa = 1 \).

Regularized strangeness-free DDAE is

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
1 & -t
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1(t-\tau) \\
x_2(t-\tau)
\end{bmatrix}
+ \begin{bmatrix}
-\dot{f}_1(t+\tau) - f_2(t+\tau) \\
-f_1(t)
\end{bmatrix}.
\]
RADAU 2a solver RADAR5 fails for original system but successfully handles the regularized DDAE computed automatically by the reformulation.

Figure: Numerical solution and absolute error.
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Thank you very much for your attention.


