On the relation between detectability and strict dissipativity for nonlinear discrete time systems

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Workshop
Elgersburg

Outline

1. Introduction
2. Nonlinear Detectability
3. Strict Dissipativity
4. Relation between detectability and strict dissipativity
   - When does detectability imply strict dissipativity?
   - When does strict dissipativity imply detectability?

Setup

Consider the discrete time nonlinear system

$$x(k+1) = f\big(x(k), u(k)\big)$$
$$x(0) = x_0$$

with $f : X \times U \to X$ and $x_0 \in X$ where X and U are vector spaces.

$$\min_{u \in \mathbb{U}^K} J_K(x_0, u) \text{ with}$$

$$J_K(x_0, u) = \sum_{k=0}^{K-1} \ell\big(x(k), u(k)\big)$$

$$\text{s.t. } x(k) \in \mathbb{X}, u(k) \in \mathbb{U} \; \forall \, k = 0, \dots, K-1$$

with $K \in \mathbb{N}$ fixed, constraint sets $\mathbb{X} \subseteq X$, $\mathbb{U} \subseteq U$ and $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ continuous.

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Standing assumption: $(0,0) \in \mathbb{X} \times \mathbb{U}$, $f(0,0) = 0$ and $\ell(0,0) = 0$.

Description: Model Predictive Control

MPC is a method in which an optimal control problem on infinite horizon

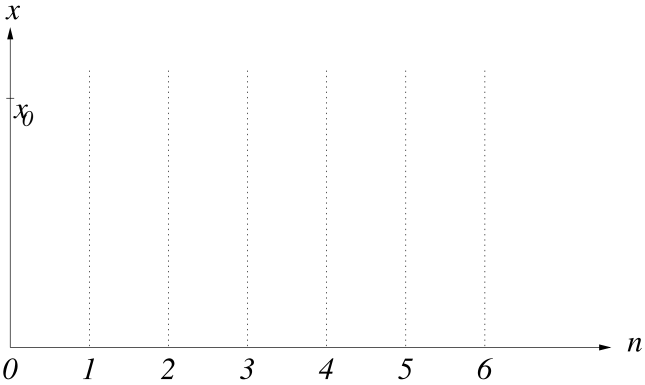
$$\textit{minimize } J_\infty(x_0, u) = \sum_{k=0}^{\infty} \ell\big(x(k), u(k)\big)$$

is approximated by the iterative solution of finite horizon problems

$$\textit{minimize } J_K(x_0, u) = \sum_{k=0}^{K-1} \ell\big(x(k), u(k)\big)$$

with fixed $K \in \mathbb{N}$ and $x(k+1) = f\big(x(k), u(k)\big)$.

MPC from the trajectory point of view



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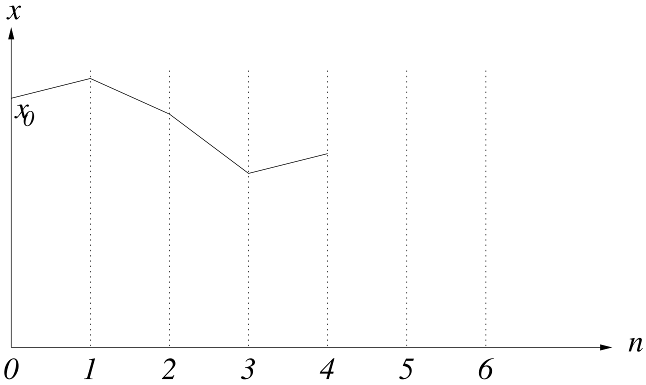

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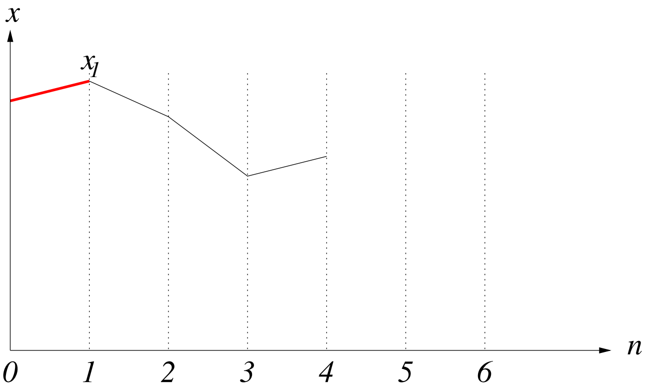

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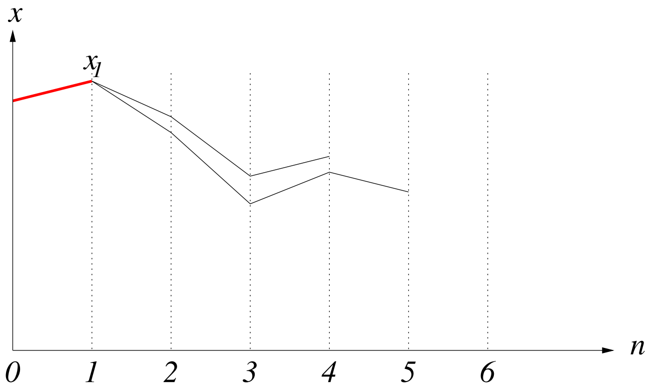

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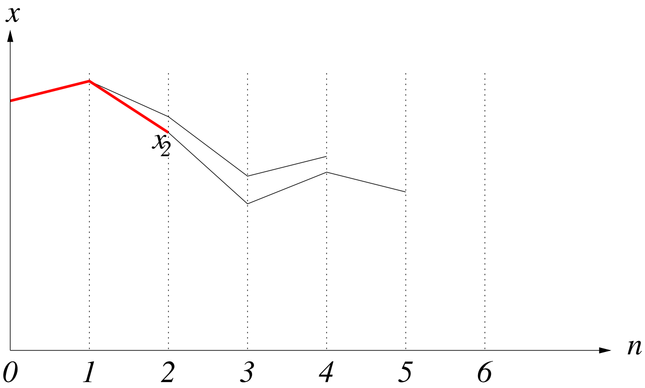

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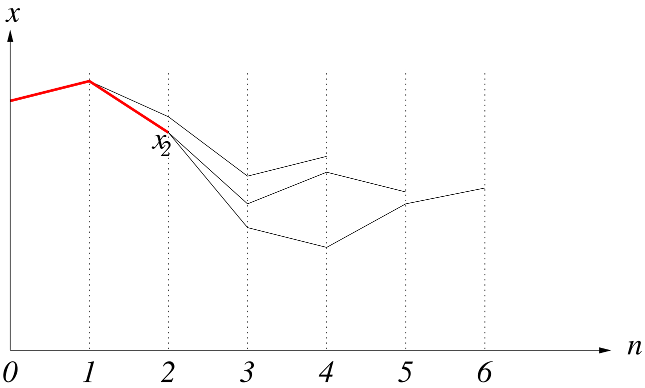

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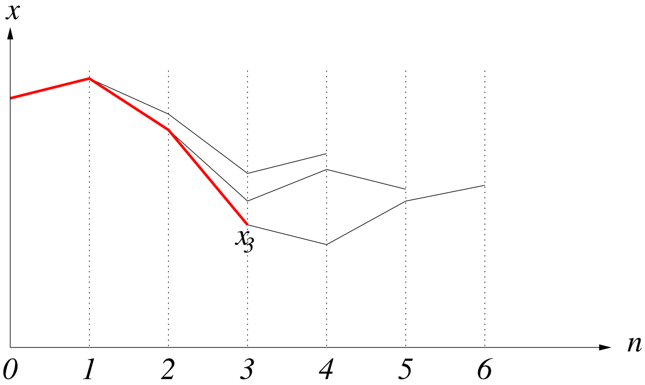

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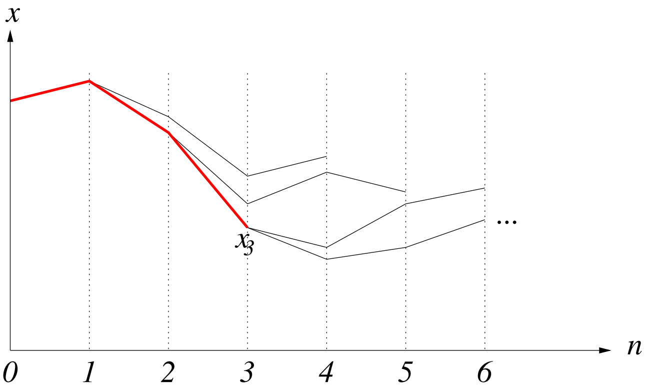

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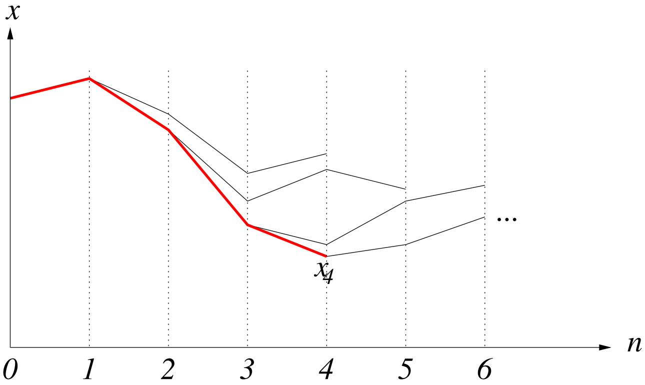

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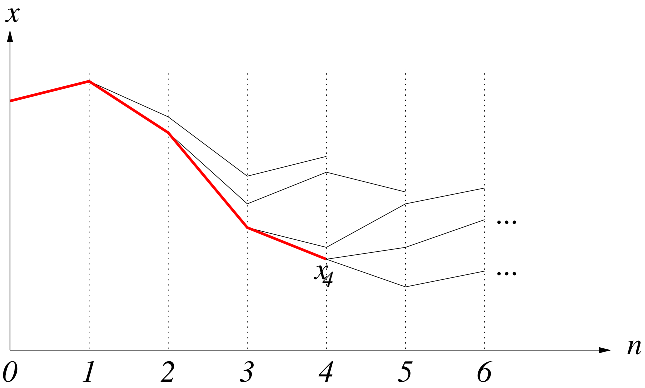

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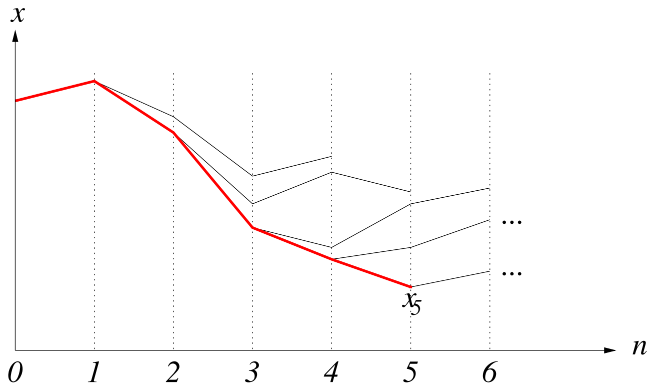

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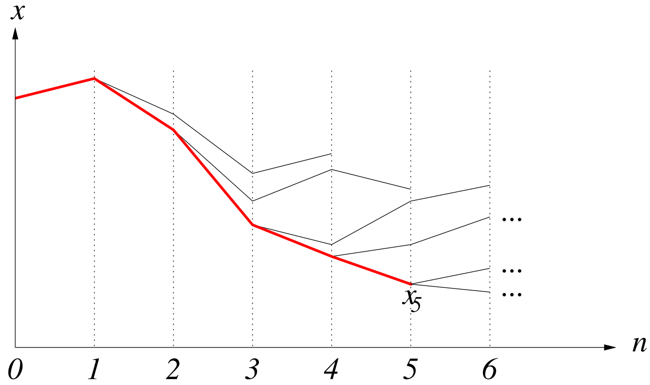

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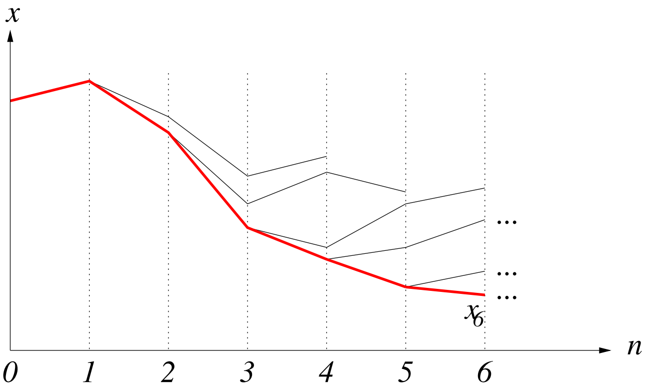

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Stability results

Available stability results for MPC:

- strict dissipativity: practical asymptotic stability [Grüne, Stieler '14]
- detectability: practical asymptotic stability [Grimm, Messina, Tuna, Teel '05]
- detectability + additional conditions: (non-practical) asymptotic stability [Grimm, Messina, Tuna, Teel '05]

Notation

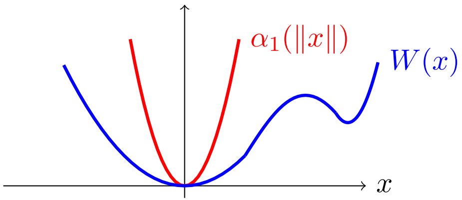
- The class $\mathcal{G}$ is the set of all continuous and nondecreasing functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$.
- The class $\mathcal{K}_\infty$ is the set of all strictly increasing and unbounded functions $\alpha \in \mathcal{G}$.

Nonlinear Detectability [Grimm, Messina, Tuna, Teel '05]

The system is detectable from ℓ with respect to $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$ if there exists a continuous function $W : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that

$$0 \leq W(x) \leq \alpha_1(\|x\|)$$
$$W\big(f(x,u)\big) - W(x) \leq -\alpha_2(\|x\|) + \alpha_3\big(\ell(x,u)\big)$$

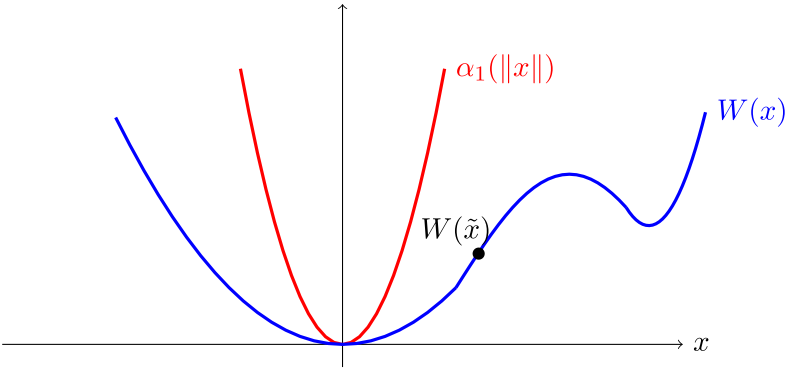
for all $(x,u) \in \mathbb{X} \times \mathbb{U}$ with $f(x,u) \in \mathbb{X}$.



Example 1: $\ell(\tilde{x},\tilde{u}) = 0$:

$$W\big(f(\tilde{x},\tilde{u})\big) - W(\tilde{x}) \leq -\alpha_2(\|\tilde{x}\|) + \alpha_3\big(\ell(\tilde{x},\tilde{u})\big)$$

$$\overset{\ell(\tilde{x},\tilde{u})=0}{\iff} \quad W\big(f(\tilde{x},\tilde{u})\big) - W(\tilde{x}) \leq -\alpha_2(\|\tilde{x}\|) + 0$$



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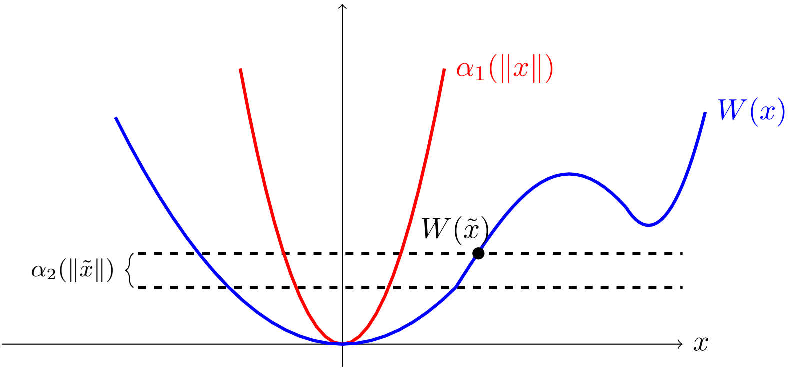

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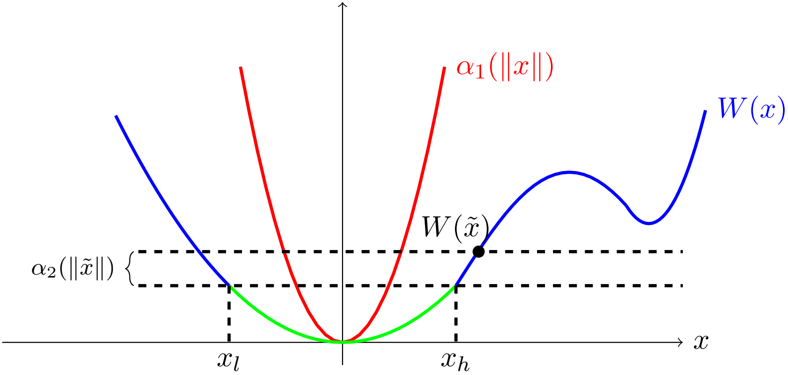


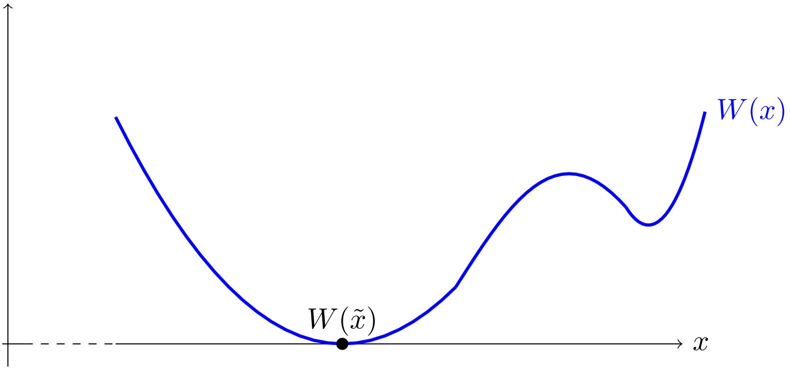
$$\Rightarrow f(\tilde{x}, \tilde{u}) \overset{!}{\in} [x_l, x_h]$$

Example 2: $\tilde{x} \neq 0$ and $W(\tilde{x}) = 0$:

$$W\big(f(\tilde{x},\tilde{u})\big) - W(\tilde{x}) \leq -\alpha_2(\|\tilde{x}\|) + \alpha_3\big(\ell(\tilde{x},\tilde{u})\big)$$

$$\overset{W(\tilde{x})=0}{\iff} W\big(f(\tilde{x},\tilde{u})\big) - 0 \leq -\alpha_2(\|\tilde{x}\|) + \alpha_3\big(\ell(\tilde{x},\tilde{u})\big)$$

$$\iff \ell(\tilde{x},\tilde{u}) \geq \alpha_3^{-1}\Big(\alpha_2(\|x\|) + W\big(f(\tilde{x},\tilde{u})\big)\Big) \overset{\tilde{x}\neq 0}{>} 0$$



Input-Output-to-State Stability

A system is input-output to state stable (IOSS) if, for some $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$,

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma_1(\|u(t)\|_\infty) + \gamma_2\big(\|h(x(t))\|_\infty\big)$$

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Nonlinear detectability is a generalization of IOSS:

$$\ell(x, u) = \|h(x)\| + \|u\|.$$

Strict Dissipativity [Willems '72]

The system is called strictly dissipative with respect to the supply rate ℓ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and a function $\rho \in \mathcal{K}_\infty$ such that

$$\ell(x, u) + \lambda(x) - \lambda\big(f(x, u)\big) \geq \rho(\|x\|)$$

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Physical interpretation:

- $\lambda(x)$: energy stored in the system
- $\ell(x,u)$: energy supplied to the system
- a certain amount of energy depending on $\|x\|$ must be dissipated

Turnpike

Informal definition: An optimal trajectory stays near an equilibrium x_e most of the time.

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Theorem (Grüne '13)

Under an appropriate reachability assumption, strict dissipativity with supply rate $s(x,u) = \ell(x,u) - \ell(x_e,u_e)$ implies the turnpike property.

Example: a macroeconomic model

Consider a classical 1d macroeconomic model

[Brock/Mirman '72]

Minimize the finite horizon objective with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $f(x,u) = u$

on $\mathbb{X} = \mathbb{U} = [0, 10]$

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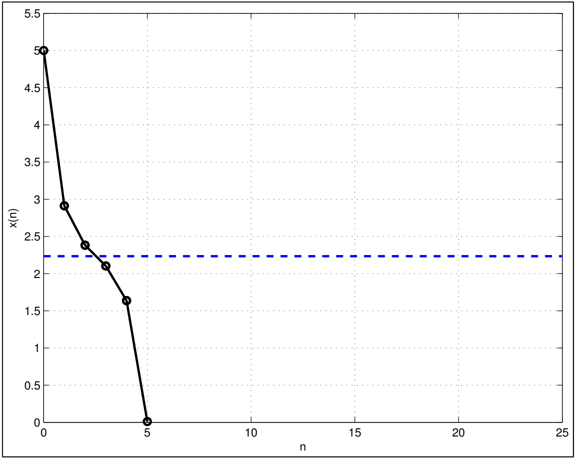
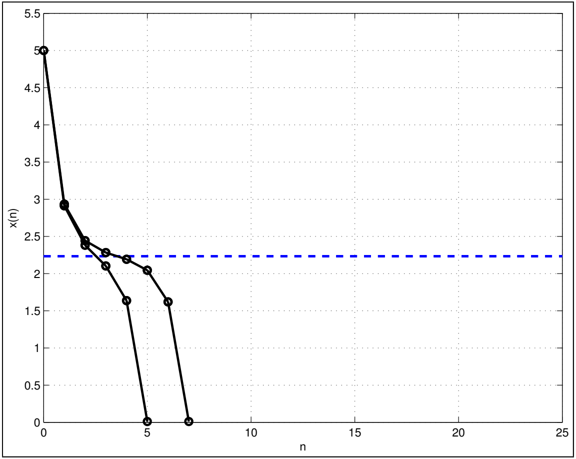
One may thus expect that finite horizon optimal trajectories also stay for a long time near that equilibrium.

Example: optimal trajectories



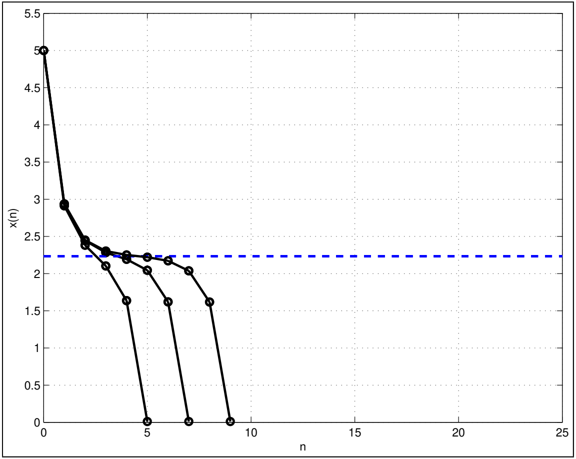
Optimal trajectory for $N = 5$

Example: optimal trajectories



Optimal trajectories for $N = 5, \dots, 7$

Example: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 9$

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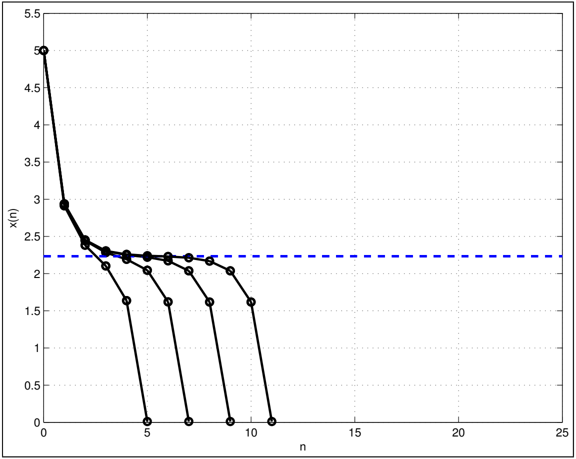
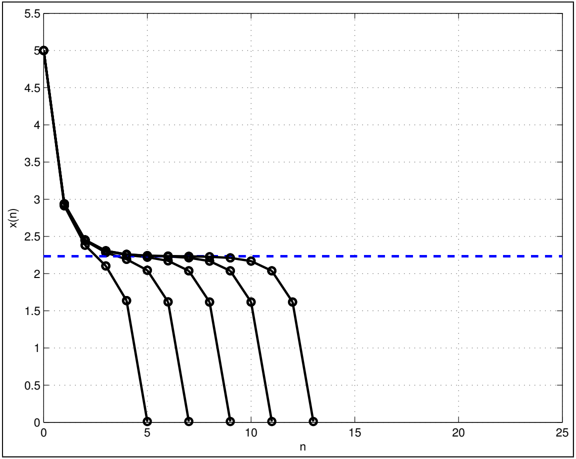


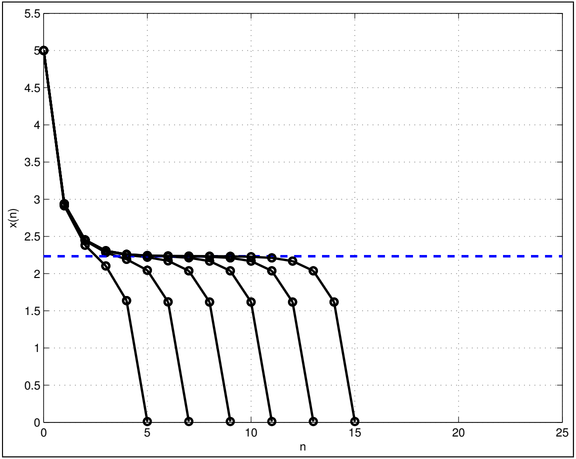
Optimal trajectories for $N = 5, \ldots, 11$

Example: optimal trajectories



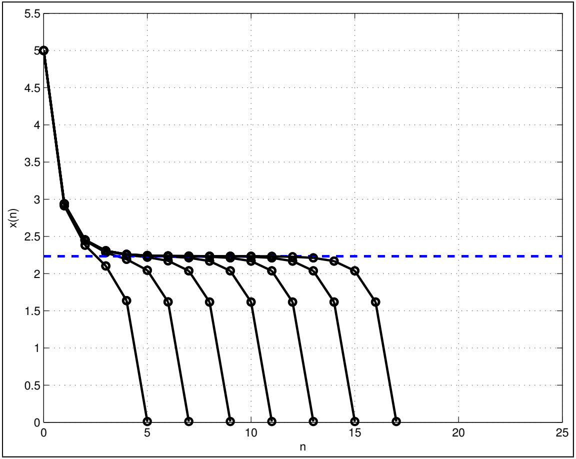
Optimal trajectories for $N = 5, \ldots, 13$

Example: optimal trajectories



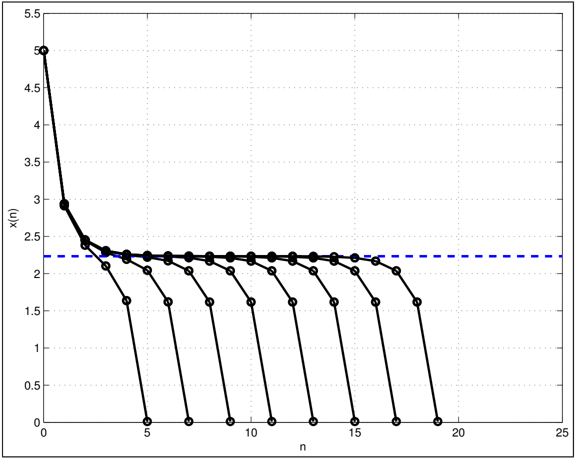
Optimal trajectories for $N = 5, \dots, 15$

Example: optimal trajectories



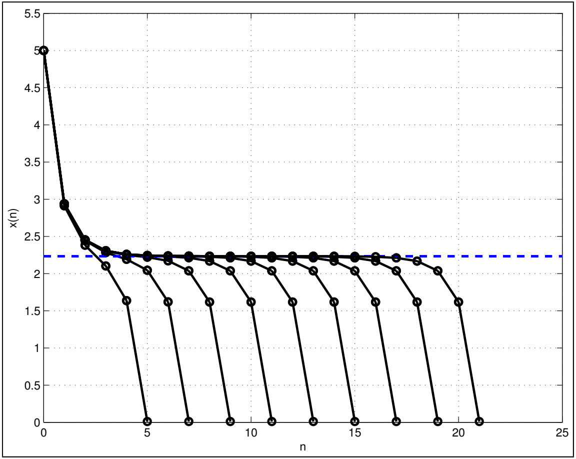
Optimal trajectories for $N = 5, \ldots, 17$

Example: optimal trajectories



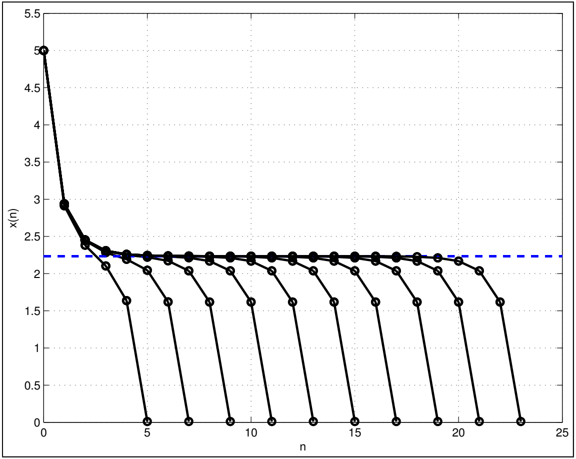
Optimal trajectories for $N = 5, \ldots, 19$

Example: optimal trajectories



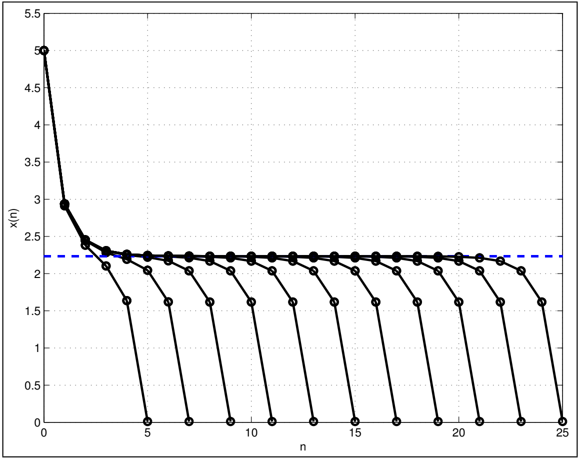
Optimal trajectories for $N = 5, \ldots, 21$

Example: optimal trajectories



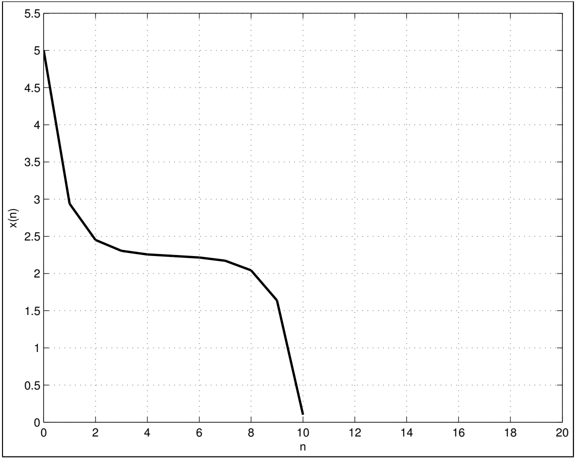
Optimal trajectories for $N = 5, \dots, 23$

Example: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 25$

MPC for the Example with $N = 10$



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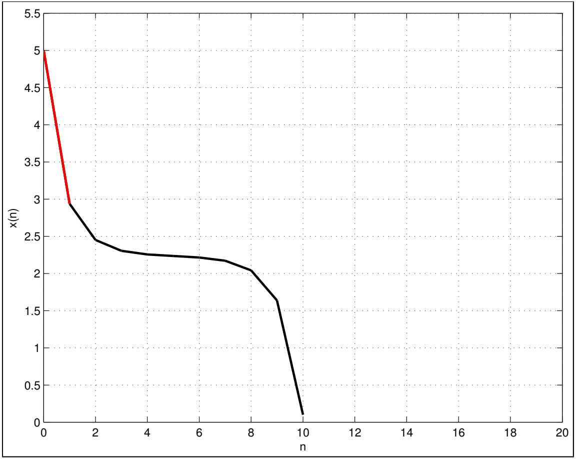

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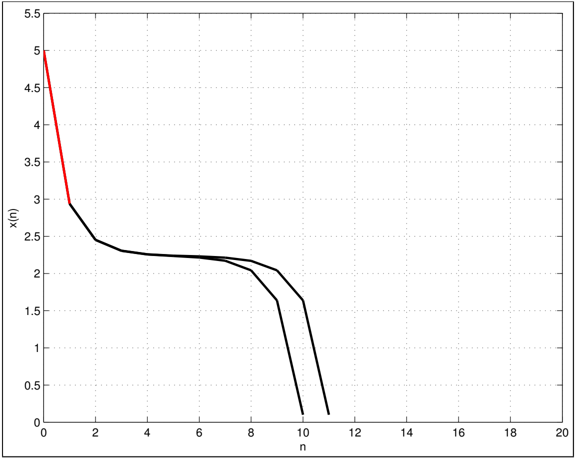

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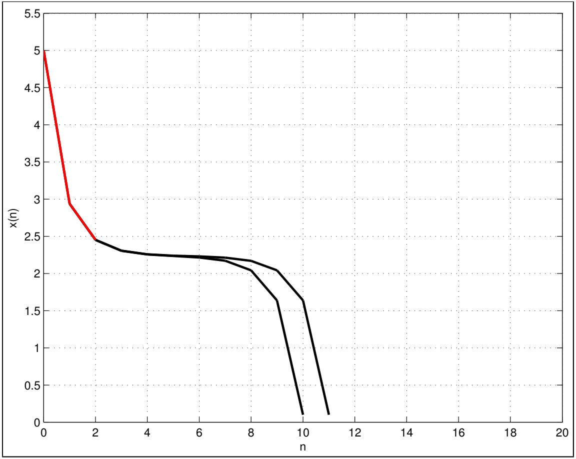

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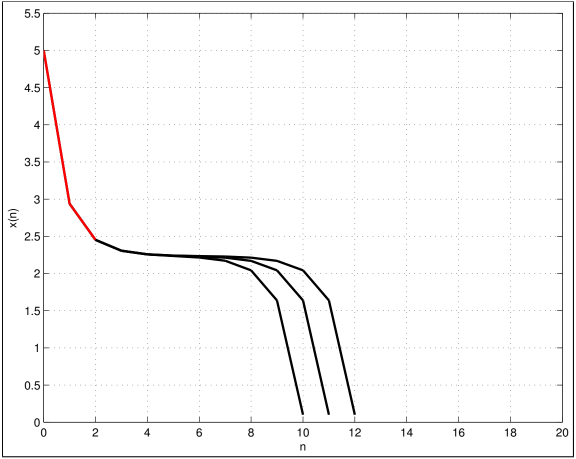

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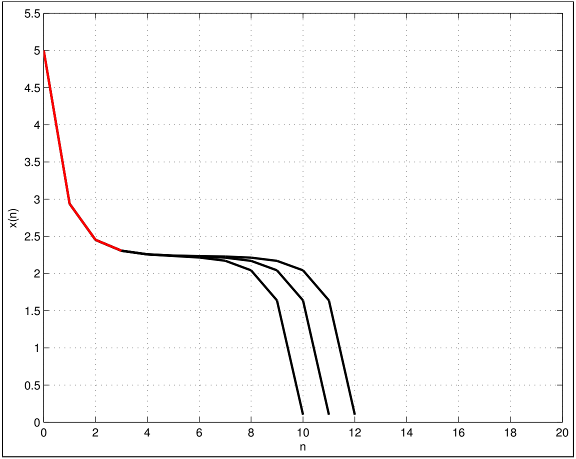

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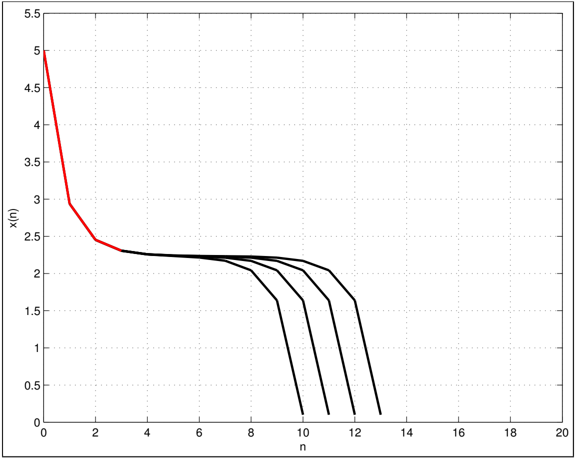

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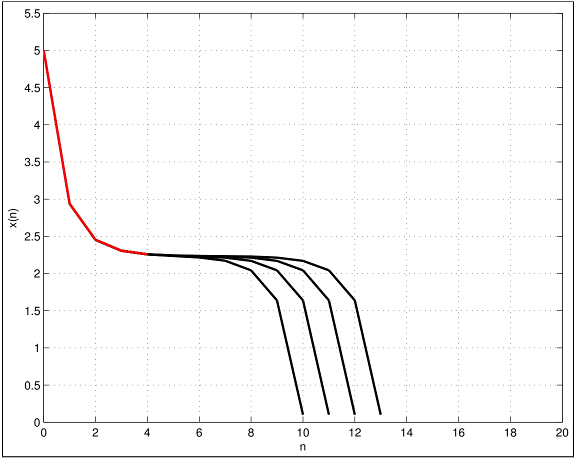

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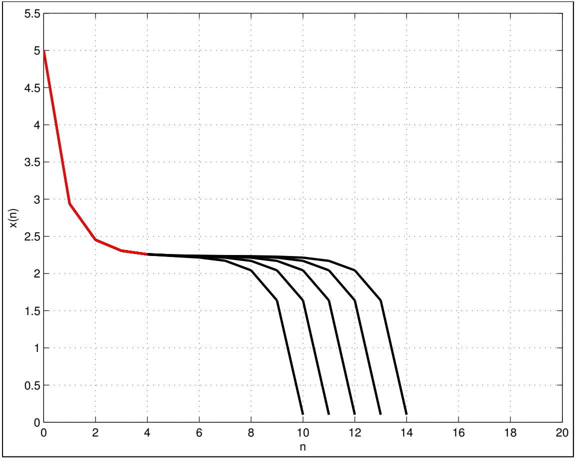

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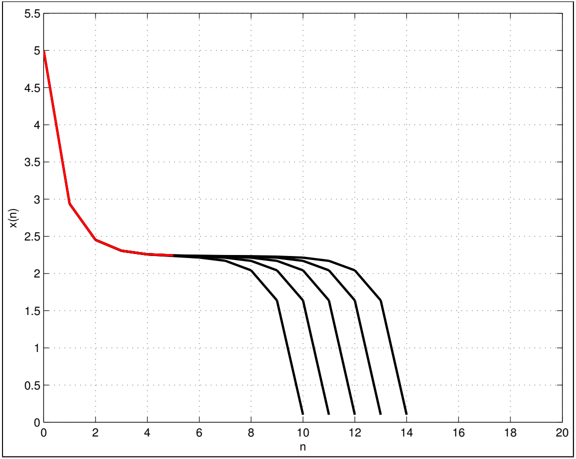

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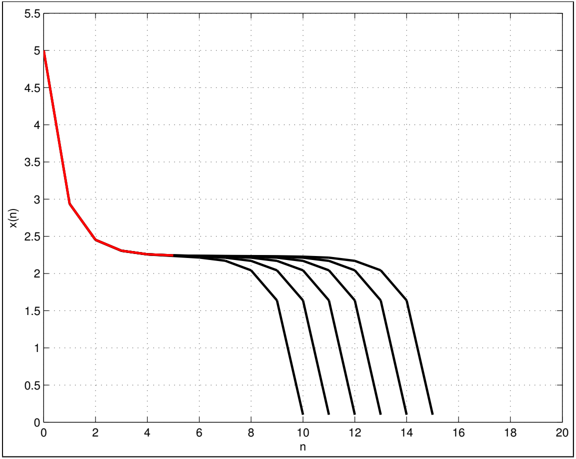

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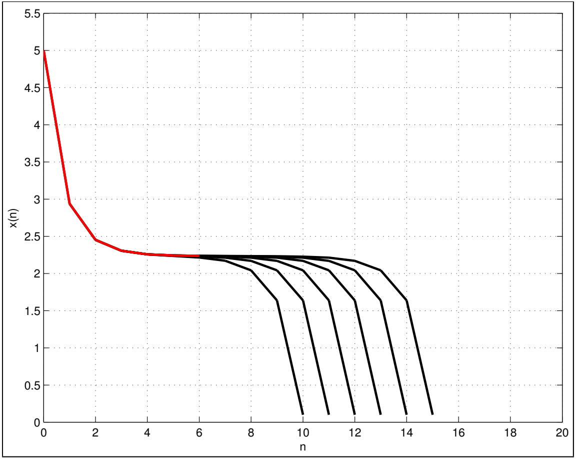

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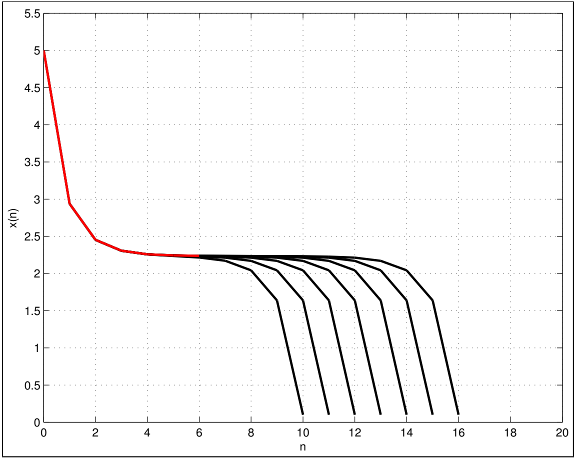
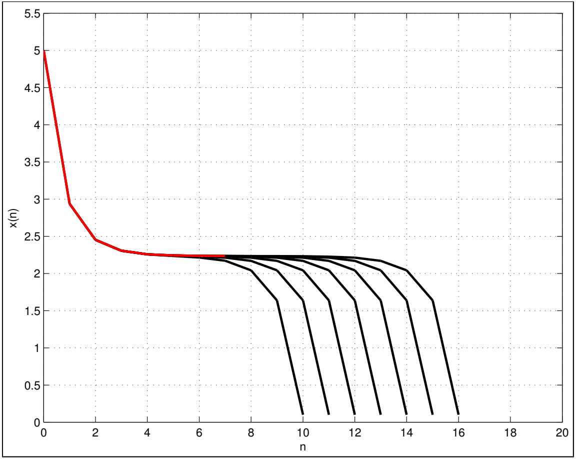


MPC for the Example with $N = 10$



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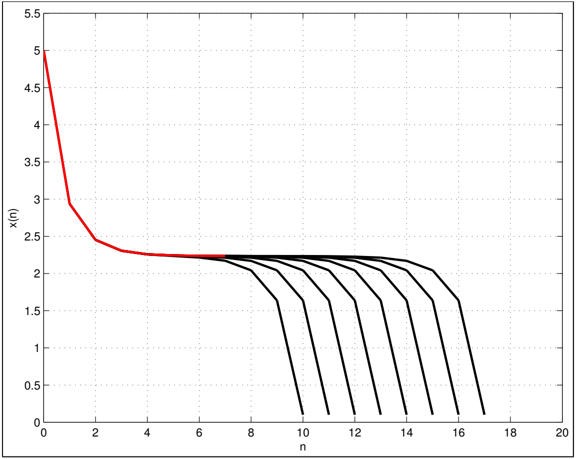

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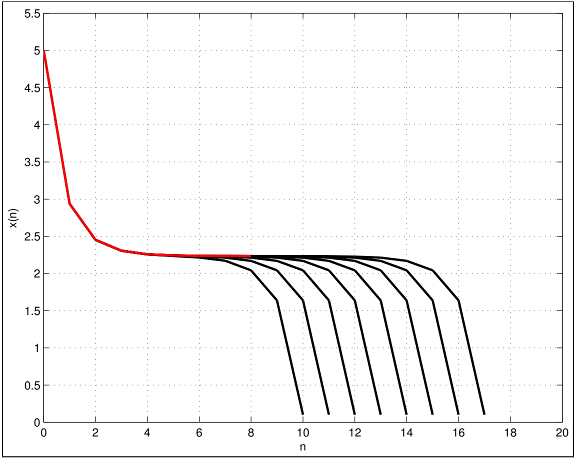

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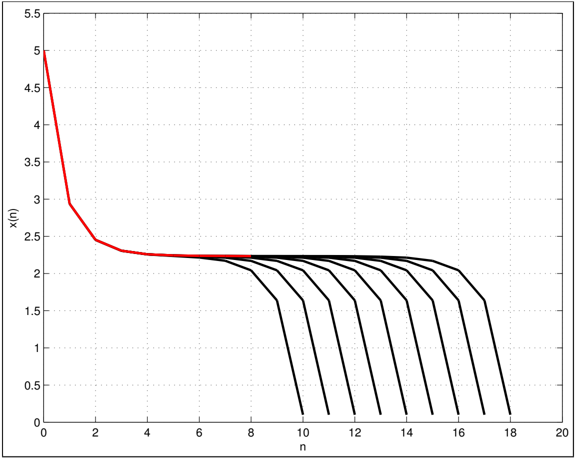

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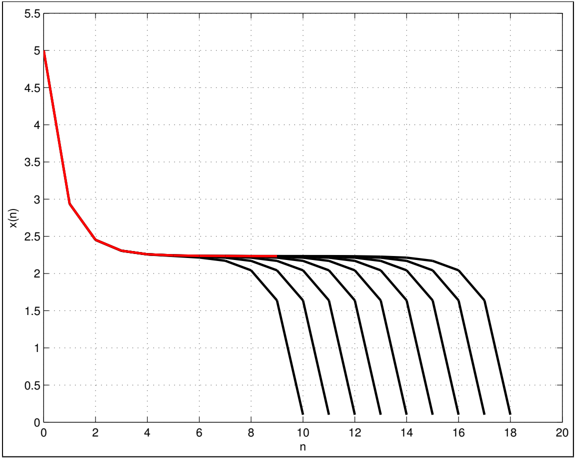

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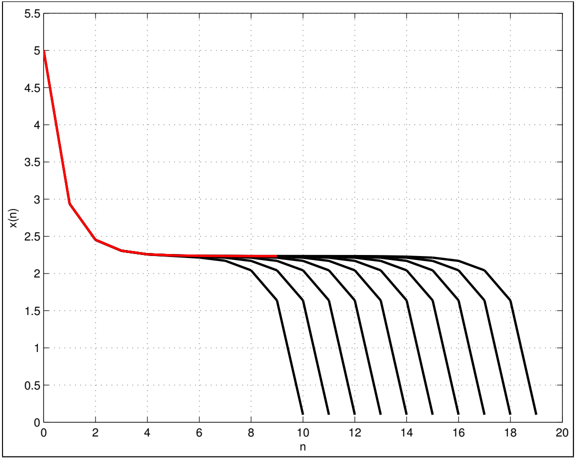

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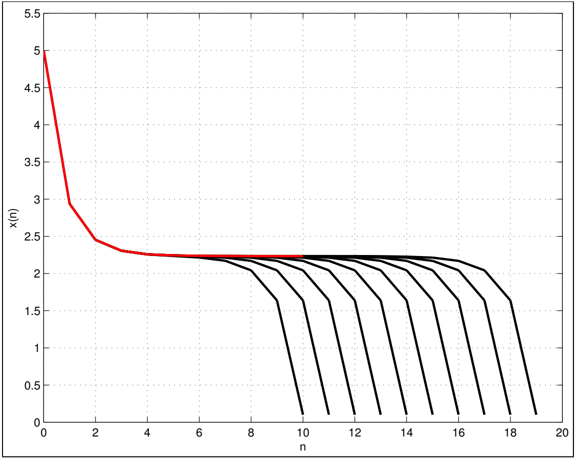

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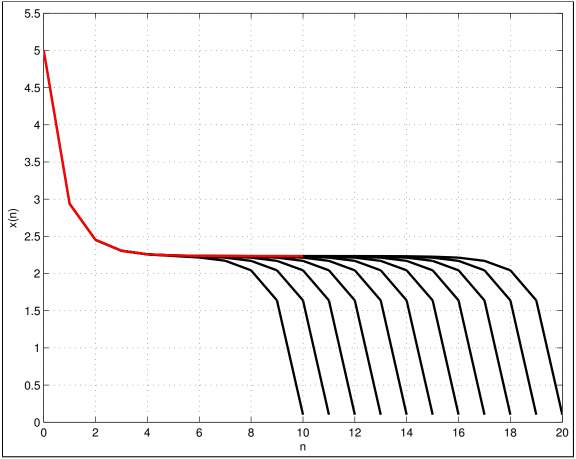

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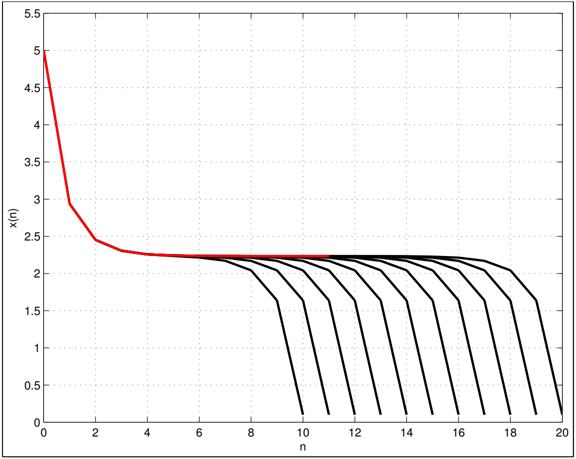

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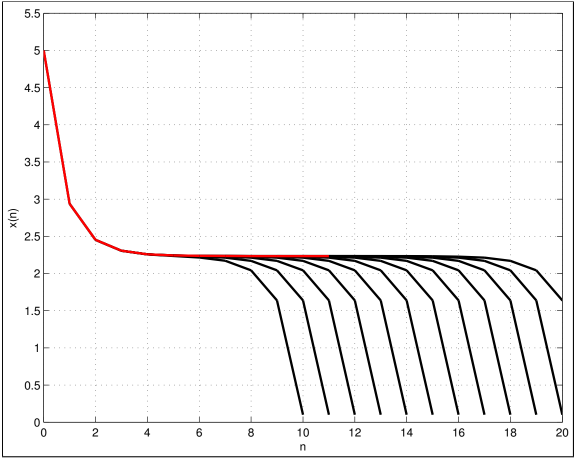

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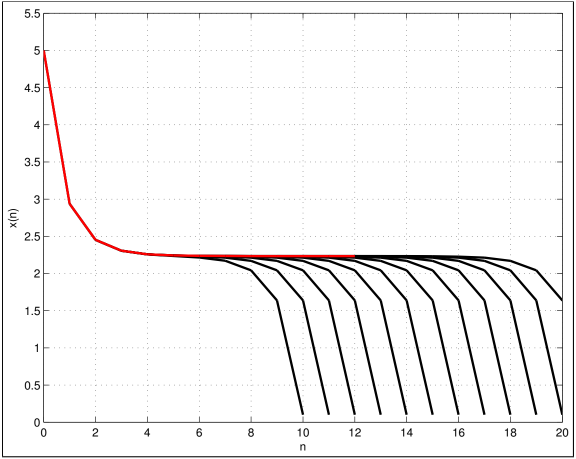

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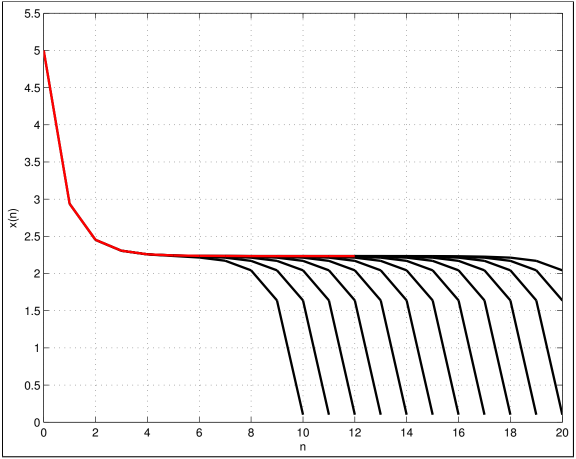

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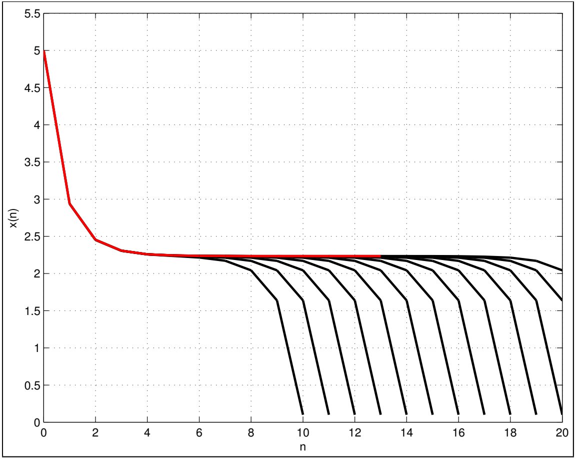

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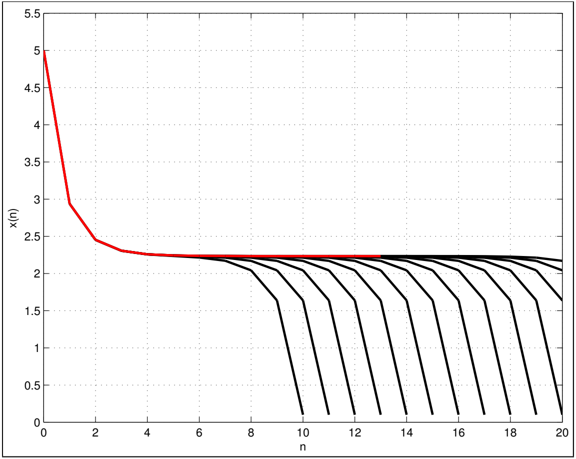

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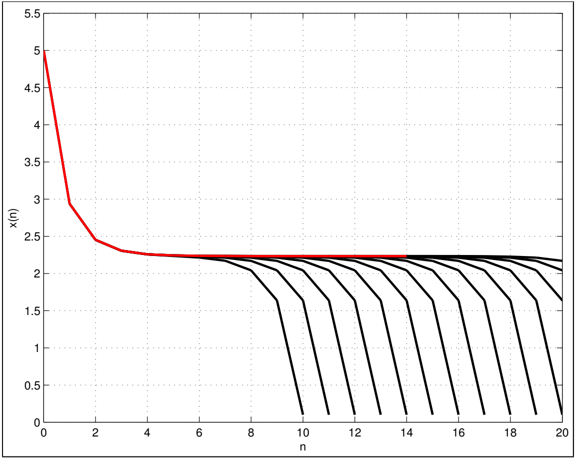

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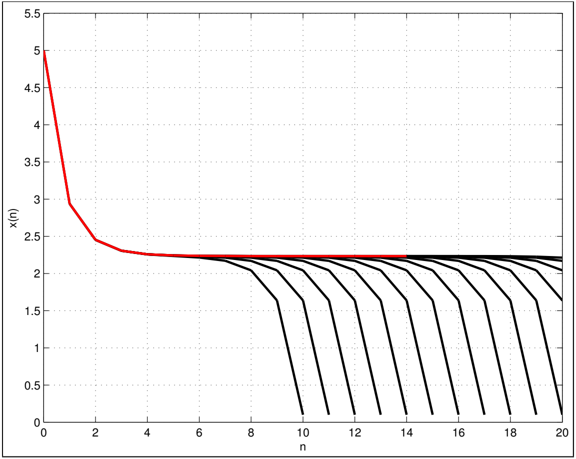

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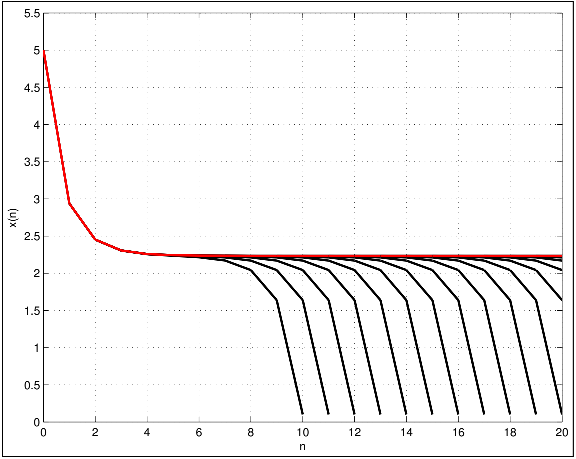

Comparison of Detectability and Dissipativity

Similarity of both definitions:

$$0 \leq W(x) \leq \alpha_1(\|x\|)$$
$$W\big(f(x,u)\big) - W(x) \leq -\alpha_2(\|x\|) + \alpha_3\big(\ell(x,u)\big)$$

$$\lambda\big(f(x,u)\big) - \lambda(x) \leq -\rho(\|x\|) + \ell(x,u)$$

with

- $W : \mathbb{X} \to \mathbb{R}_{\geq 0}$, $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below
- $\alpha_2, \rho \in \mathcal{K}_\infty$

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α_3 linear function

Proposition

If the system is detectable from ℓ with respect to some $(\alpha_1, \alpha_2, c \cdot \mathrm{Id}) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$ with $c \in \mathbb{R}_{>0}$ constant, then the system is strictly dissipative with respect to the supply rate ℓ.

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Proof:

$$
\begin{aligned}
& W\big(f(x,u)\big) - W(x) \leq -\alpha_2(\|x\|) + c \cdot \ell(x,u) \\
\overset{c>0}{\Longleftrightarrow}\ & \frac{1}{c} W\big(f(x,u)\big) - \frac{1}{c} W(x) \leq -\frac{1}{c}\alpha_2(\|x\|) + \ell(x,u) \\
\Longleftrightarrow\ & \lambda\big(f(x,u)\big) - \lambda(x) \leq -\rho(\|x\|) + \ell(x,u)
\end{aligned}
$$

with $\lambda(x) := \frac{W(x)}{c}$ and $\rho(r) := \frac{\alpha_2(r)}{c}$.

α_3 bounded by an affine function

Theorem

Let the system be detectable from ℓ with respect to some $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$. If α_3 is bounded from above by an affine function $\gamma : \mathbb{R} \to \mathbb{R}$, then the system is strictly dissipative with respect to the supply rate ℓ.

Scaling functions

Lemma (Grimm, Messina, Tuna, Teel '05)

Let the system be detectable from ℓ *with respect to some* $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$. *Let* $p \in \mathcal{K}_\infty$ *be such that* $q(s) := (dp/ds)(s)$ *is well defined, continuous and nondecreasing. Then*

$$
\begin{aligned}
&(p \circ W \circ f)(x,u) - (p \circ W)(x) \\
&\leq -(q \circ \frac{\alpha_2}{4})(\|x\|) \cdot (\frac{\alpha_2}{4})(\|x\|) \\
&+ 2q\Big((\alpha_3 \circ \ell)(x,u) + (\alpha_1 \circ \alpha_2^{-1} \circ 2\alpha_3 \circ \ell)(x,u)\Big) \cdot (\alpha_3 \circ \ell)(x,u)
\end{aligned}
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for all $(x,u) \in \mathbb{X} \times \mathbb{U}$.

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&+ 2q\Big((\alpha_3 \circ \ell)(x,u) + (\alpha_1 \circ \alpha_2^{-1} \circ 2\alpha_3 \circ \ell)(x,u)\Big) \cdot (\alpha_3 \circ \ell)(x,u)
\end{aligned}
$$

for all $(x,u) \in \mathbb{X} \times \mathbb{U}$.

 Relation between detectability and strict dissipativity

Idea of Proof: α_3 bounded by an affine function

q from the Lemma can be chosen such that $q(\cdot) \leq c_q$ for a constant $c_q \in \mathbb{R}_{\geq 0}$ and

$$2q\Big((\alpha_3 \circ \ell)(x,u) + (\alpha_1 \circ \alpha_2^{-1} \circ 2\alpha_3 \circ \ell)(x,u)\Big) \cdot (\alpha_3 \circ \ell)(x,u)$$
$$\leq \begin{cases} \ell(x,u) \cdot 1 & \text{, \textit{if} } \big(\alpha_3 \circ \ell\big)(x,u) \leq 1 \\ 2c_q \cdot \big(\alpha_3 \circ \ell\big)(x,u) & \text{, \textit{otherwise}} \end{cases}$$
$$\overset{\alpha_3 \leq \gamma}{\leq} c \cdot \ell(x,u)$$

holds.

Idea of Proof: α_3 bounded by an affine function

Applying the Lemma:

$$(p \circ W \circ f)(x,u) - (p \circ W)(x) \leq -\,(q \circ \frac{\alpha_2}{4})(\|x\|) \cdot (\frac{\alpha_2}{4})(\|x\|) + c \cdot \ell(x,u)$$

$\Rightarrow$ strict dissipative with

$$\lambda(x) = \frac{1}{c}(p \circ W)(x)$$
$$\rho(r) = \frac{1}{c}(q \circ \frac{\alpha_2}{4})(r) \cdot (\frac{\alpha_2}{4})(r)$$

Relation between detectability and strict dissipativity

Bounded stage cost

Corollary

If ℓ is bounded from above on $\mathbb{X} \times \mathbb{U}$ then detectability from ℓ implies strict dissipativity with respect to the supply rate ℓ.

Example: α_3 not bounded by an affine function

The system

$$f(x,u) = \begin{cases} \max\{\frac{x}{2}, x-1\} + e^{\|u\|} - 1, & x \geq 0 \\ \min\{\frac{x}{2}, x+1\} + e^{\|u\|} - 1, & x < 0 \end{cases}$$

$$l(x,u) = \big(\|x\| + 1\big)\|u\|$$

with $\mathbb{X} = \mathbb{R}$ and $\mathbb{U} = \mathbb{R}$ is detectable with

$$W(x) = x^2$$

$$\alpha_1(r) = r^2$$

$$\alpha_2(r) = \min\{r, \frac{3}{4}r^2\}$$

$$\alpha_3(r) = 2(e^r - 1) + (e^r - 1)^2$$

Example: α_3 not bounded by an affine function

Assume the system is strictly dissipative for some $\lambda, \rho \in \mathcal{K}_\infty$.
This implies for $x = 0$:

$$f(x,u) = e^{\|u\|} - 1,\ \ell(x,u) = \|u\|$$
$$\lambda(e^{\|u\|} - 1) \leq \|u\| + \lambda(0)$$
$$\Rightarrow \lambda(x) \leq \ln(x+1) + \lambda(0)\ \forall x > 0$$

and for $x \geq 2$ and $u = 0$:

$$f(x,u) = x - 1,\ \ell(x,u) = 0$$
$$\lambda(x) - \lambda(x-1) \geq \rho(\|x\|) \geq \rho(2)$$
$$\Rightarrow \lambda(x) \geq \rho(2) + \lambda(x-1)$$
$$\Rightarrow \lambda(x) \geq (x-1)\rho(2) + \lambda(1)$$

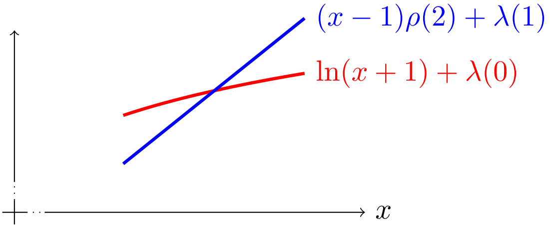
Example: α_3 not bounded by an affine function

So for $x \geq 2$ and $u = 0$:

$$(x-1)\rho(2) + \lambda(1) \leq \lambda(x) \leq \ln(x+1) + \lambda(0)$$

but for x sufficiently large:

$$(x-1)\rho(2) + \lambda(1) > \ln(x+1) + \lambda(0)$$



 Relation between detectability and strict dissipativity

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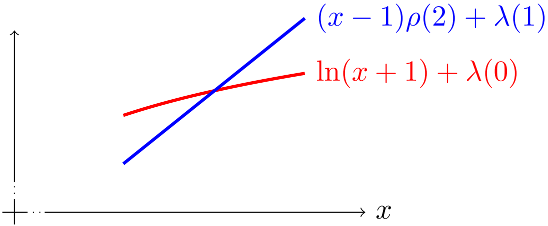

$\Rightarrow$ The system is not strictly dissipative.

When does strict dissipativity imply detectability?

Theorem

Let the system be strictly dissipative with a continuous storage function λ satisfying $\lambda(0) \leq \lambda(x)$ for all $x \in \mathbb{X}$. Then the system is detectable with

$$W(x) := \lambda(x) - \lambda(0)$$
$$\alpha_2 := \rho$$
$$\alpha_3 := \mathrm{Id}$$

where ρ is taken from the definition of strict dissipativity and $\alpha_1 \in \mathcal{G}$ suitably chosen.

Proof of the Theorem

Proof

- $W(x) = \lambda(x) - \lambda(0) \geq 0$
- $W(0) = 0 \ \Rightarrow \exists \alpha_1 \in \mathcal{G} : W(x) \leq \alpha_1(\|x\|)$

$$
\begin{aligned}
& W\big(f(x,u)\big) - W(x) \\
& = \Big(\lambda\big(f(x,u)\big) - \lambda(0)\Big) - \Big(\lambda(x) - \lambda(0)\Big) \\
& = \lambda\big(f(x,u)\big) - \lambda(x) \\
& \overset{str.diss.}{\leq} \quad -\rho(\|x\|) + \ell(x,u) \\
& = -\alpha_2(\|x\|) + \alpha_3\big(\ell(x,u)\big)
\end{aligned}
$$

$\Rightarrow$ *The system is detectable from ℓ with respect to $(\alpha_1, \alpha_2, \alpha_3)$.*

Example: λ not minimized at 0

The system

$$f(x,u) = 2x + u$$
$$l(x,u) = u^2$$

with $\mathbb{X} = [-2,2]$ and $\mathbb{U} = [-3,3]$ is strictly dissipative with $\lambda(x) = -\frac{x^2}{2}$. So λ is maximized in 0.

Example: λ not minimized at 0

Assume the system is detectable from ℓ with respect to some $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$.
$\Rightarrow \exists W : \mathbb{X} \to \mathbb{R}_{\geq 0} : \forall x \in [-1, 1] \setminus \{0\}$ and $u = 0$:

$$\begin{aligned} W\big(f(x,u)\big) - W(x) &= W(2x) - W(x) \\ &\leq -\alpha_2(\|x\|) + \alpha_3(0) \\ &< 0 \end{aligned}$$

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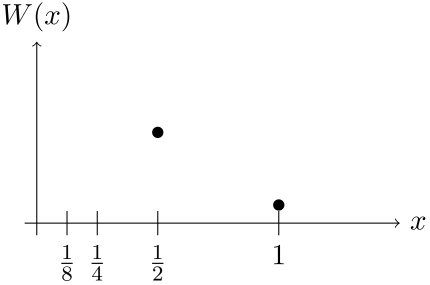

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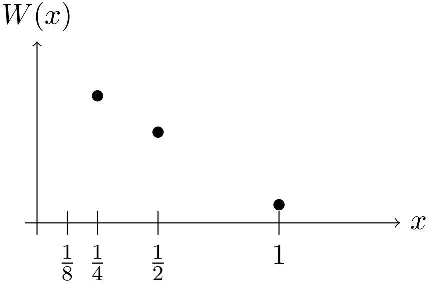

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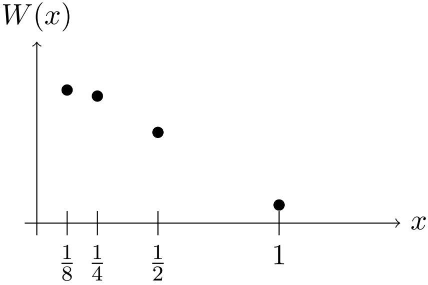

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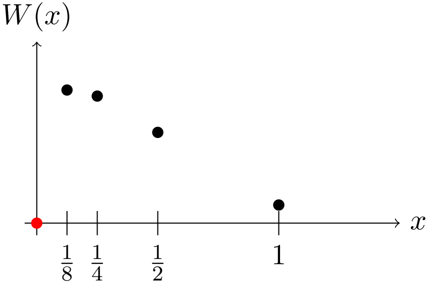

Example: λ not minimized at 0

$\Rightarrow \forall x \in [-1,1] \setminus \{0\}$:

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This contradicts continuity of W in $x = 0$.
$\Rightarrow$ The system is not detectable.

Relation between detectability and strict dissipativity

Conclusion

Summary:

- detectable with α_3 bounded by an affine function
  $\Rightarrow$ strict dissipative
- detectable with ℓ bounded on $\mathbb{X} \times \mathbb{U}$
  $\Rightarrow$ strict dissipative
- strict dissipative with continuous storage function λ and $\lambda(0) \leq \lambda(x)$ for all $x \in \mathbb{X}$
  $\Rightarrow$ detectable