Kleinste-Quadrate Kollokation für Differential-Algebraische Gleichungen

Least-Squares Collocation for Differential-Algebraic Equations

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Literature:

▼ R. Lamour, R. März, and C. Tischendorf: *Differential-Algebraic Equationss: A Projector Based Analysis*. DAE-F Springer 2013.

▼ M. Hanke, R. März, C. Tischendorf, E. Weinmüller, and S. Wurm: *Least-squares collocation for linear higher-index differential-algebraic equations.* J. Comput. Appl. Math. 317, 403-431, 2017.

▼ M. Hanke, R. März, and C. Tischendorf: *Least-squares collocation for higher-index linear differential-algebraic equations: Estimating the instability threshold.* Mathematics of Computation, accepted July 2018.

▼ M. Hanke and R. März: *A least-squares collocation method for nonlinear higher-index differential-algebraic equations.* In preparation. NUMDIFF-15, Halle 2018

▼ M. Hanke and R. März: *Questions concerning differential-algebraic operators: Toward a reliable direct numerical treatment of differential-algebraic equations.* In preparation. NUMDIFF-15, Halle 2018

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1. Polynomial collocation: Basic approach
2. Overdetermined least-squares collocation
3. Higher-index DAEs: Numerical experiments
4. Hitherto existing mathematical background
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Outline

Basic approach

$$f((Dx)'(t),x(t),t)=0,\ t\in[a,b],\quad g(x(a),x(b))=0, \tag{1}$$
$$D=[I\ 0]\in\mathbb{R}^{k\times m},\quad \operatorname{rank} D=k$$

Given a mesh

$$\pi: a=t_0<t_1<\cdots<t_n=b,\quad h_j=t_j-t_{j-1},$$

with maximal stepsize $h=h_\pi>0$, and additional points $t_{ji}=t_{j-1}+\sigma_i h_j$,

$$0\le\sigma_1<\sigma_2<\cdots<\sigma_M\le 1,$$

a number $N\ge 1$, and the $(nNm+k)$-dimensional ansatz space

$$X_\pi=\{p:[a,b]\to\mathbb{R}^m : Dp\in\mathcal{C}([a,b],\mathbb{R}^k):$$
$$p_s|_{[t_{j-1}\,t_j)}\in\mathfrak{P}_N,\ s=1,\ldots,k,\ p_s|_{[t_{j-1}\,t_j)}\in\mathfrak{P}_{N-1},\ s=k+1,\ldots,m,$$
$$j=1,\ldots,n\},$$

we seek $x_\pi \in X_\pi$ such that

$$f(Dx_\pi)'(t_{ji}), x_\pi(t_{ji}), t_{ji}) = 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, n, \qquad (2)$$

$$g(x_\pi(t_0), x_\pi(t_n)) = 0. \qquad (3)$$

In general: $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^d$, with $d :=$ dynamical degree of freedom, $\boxed{0 \leq d \leq k \leq m}$.

Number of unknowns: $nNm + k$, number of equations: $nMm + d$

- Standard or classical collocation: $M = N$, add $k - d$ extra conditions.
- Overdetermined least-squares collocation: $M \geq N + 1$, solve (2),(3) in a least-squares sense.
  x_π is a least-squares solution!

Standard polynomial collocation for ODEs

$$x'(t) + \varphi(x(t), t) = 0, \ t \in [a, b], \quad g(x(a), x(b)) = 0 \tag{4}$$

Here: $D = I$, $k = m$, and $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$.

Definition (AMR 1988, p.218)

A collocation solution for (4) is a continuous, piecewise polynomial function x_π which reduces to a polynomial of degree at most N on each subinterval (which means $x_\pi \in X_\pi$) and satisfies

$$\begin{aligned} x'_\pi(t_{ji}) + \varphi(x_\pi(t_{ji}), t_{ji}) = 0, \ i = 1, \ldots, M = N, j = 1, \ldots, n, \\ g(x_\pi(t_0), x_\pi(t_n)) = 0. \end{aligned}$$

Theorem (≈ AMR 1988, p.226)

If $x_* \in \mathcal{C}^1([a,b],\mathbb{R}^m)$ *is an isolated solution of* (4), *then there are positive constants* ρ *and* h_* *such that the following hold for all meshes with* $h \leq h_*$.

(a) *There is a unique collocation solution* x_π *in a tube of radius* ρ *around* x_*

(b) x_π *can be obtained by Newton's method, which converges quadratically provided the initial iterate is sufficiently close to* x_*.

(c) $|x_\pi(t_j) - x_*(t_j)| = O(h^p), \quad j = 1, \dots, n,$

$\|x_\pi - x_*\|_\infty \leq c\, h^N.$

Standard polynomial collocation for regular index-1 DAEs

$$f((Dx)'(t), x(t), t) = 0,\ t \in [a, b], \quad g(x(a), x(b)) = 0,$$
$$D = [I\ 0] \in \mathbb{R}^{k\times m}, \quad \operatorname{rank} D = k < m, \quad d = k$$

Theorem (≈ LMW 2015, p. 241)

If $x_* \in \mathcal{C}^1_D([a,b], \mathbb{R}^m)$ is an isolated solution of (1), then there are positive constants ρ and h_* such that the following hold for all meshes with $h \leq h_*$.

(a) *There is a unique collocation solution x_π in a tube of radius ρ around x_**

(b) *x_π can be obtained by Newton's method, which converges quadratically provided the initial iterate is sufficiently close to x_*.*

(c) $|(Dx_\pi)(t_j) - (Dx_*)(t_j)| = O(h^p), \quad j = 1, \ldots, n,$

$\|x_\pi - x_*\|_\infty \leq c\, h^N.$

Outline

Introduce $\mathcal{C}_\pi = \mathcal{C}_\pi([a,b],\mathbb{R}^m)$ and the restriction operator $R_{\pi,M} : \mathcal{C}_\pi \to \mathcal{C}_\pi$ by

$$R_{\pi,M} w|_{[t_{j-1},t_j)} \in \mathfrak{P}^m_{M-1},$$
$$(R_{\pi,M} w)(t_{ji}) = w(t_{ji}), \quad i = 1,\cdots,M,\ j = 1,\cdots,n,$$

We also assign to $w \in \mathcal{C}_\pi([a,b],\mathbb{R}^m)$ the vector $W \in \mathbb{R}^{mMn}$,

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} \in \mathbb{R}^{mMn}, \quad W_j = \left(\frac{h_j}{M}\right)^{\frac{1}{2}} \begin{bmatrix} w(t_{j1}) \\ \vdots \\ w(t_{jM}) \end{bmatrix} \in \mathbb{R}^{mM},$$

which yields

$$\boxed{\|R_{\pi,M} w\|^2_{L^2} = W^T \mathcal{L} W, \quad w \in \mathcal{C}_\pi([a,b],\mathbb{R}^m)}, \tag{5}$$

with a positive definite, symmetric matrix $\mathcal{L}$ the entries of which are independent of π. There are further constants $\kappa_l, \kappa_u > 0$ such that

$$\boxed{\kappa_l\, |W|^2 \le W^T \mathcal{L} W \le \kappa_u\, |W|^2, \quad W \in \mathbb{R}^{mMn}.} \tag{6}$$

The overdetermined least-squares collocation means now that we seek an element $x_\pi \in X_\pi$ minimizing the functional

$$\psi_{\pi,M}(x) = \|R_{\pi,M}\underbrace{f((Dx)'(\cdot),x(\cdot),\cdot)}_{=w}\|^2_{L^2} + |g(x(t_0),x(t_n))|^2$$
$$= W^T \mathcal{L}\, W + |g(x(t_0),x(t_n))|^2, \quad x \in X_\pi$$

$$x_\pi \in \mathrm{argmin}\{W^T\mathcal{L}\,W + |g(x(t_0),x(t_n))|^2 \colon\; x \in X_\pi\}$$

Outline

Example (1: Linear time-varying index-3 DAE, $m = 3, k = 2, d = 0$)

The DAE

$$\begin{aligned} x_2'(t) + x_1(t) &= g_1(t), \\ t\eta x_2'(t) + x_3'(t) + (\eta+1)x_2(t) &= g_2(t), \\ t\eta x_2(t) + x_3(t) &= g_3(t), \quad t \in [0,1], \end{aligned}$$

can be cast into the proper form

$$\begin{bmatrix} 1 & 0 \\ t\eta & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x\right)'(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\eta & 0 \\ 0 & t\eta & 1 \end{bmatrix} x(t) = q(t), \quad t \in [0,1].$$

There is exactly one solution to each sufficiently smooth q. Set $\eta = -2$ and determine q_1, q_2, q_3 sucht that the solution becomes

$$x_{*\,1}(t) = e^{-t}\sin t, \quad x_{*\,2}(t) = e^{-2t}\sin t, \quad x_{*\,3}(t) = e^{-t}\cos t.$$

Example (1)

The most sensible component concerning numerical computations is x_1.
Uniformly distributed points $0 < \sigma_1 < \ldots < \sigma_M < 1$, uniform mesh.

Table: Collocation results, $\eta = -2$, $N = 3$

n	Standard: $\|x_{*1} - x_{\pi 1}\|_\infty$	Least-squares: $\|x_{*1} - x_{\pi 1}\|_\infty$, $M = 7$
20	3.74e+006	3.26e–4
40	9.84e+016	7.52e–5
80	3.51e+038	1.81e–5
160	2.04e+082	4.42e–6
320	2.98e+170	1.11e–6

Table: Componentwise maximal error, polynomial degree $N = 3$, $M = 7$

n	Standard collocation			Least-squares collocation		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
20	5.56e+006	3.03e+004	5.99e+004	2.09e-4	1.10e-06	2.18e-06
40	1.55e+017	4.23e+014	8.41e+014	5.03e-5	1.31e-07	2.65e-07
80	5.70e+038	7.76e+035	1.55e+036	1.23e-5	1.60e-08	3.20e-08
160	3.36e+082	2.29e+079	4.57e+079	3.06e-6	1.98e-09	4.00e-09
320	4.93e+170	1.68e+167	3.35e+167	7.68e-7	2.50e-10	5.00e-10

Example (1)

Table: Error $(\|e_1\|_{L^2}^2 + \|e_2\|_{H^1}^2 + \|e_3\|_{H^1}^2)^{\frac{1}{2}}$ of the collocation solution for $N = 3$

	$M = 2N + 1$ uniform points		$M = N + 1$ Gaussian points	
n	error	order	error	order
10	6.31e-4		6.46e-4	
20	1.44e-4	2.1	1.45e-4	2.2
40	3.47e-5	2.1	3.47e-5	2.1
80	8.53e-6	2.0	8.53e-6	2.0
160	2.12e-6	2.0	2.12e-6	2.0
320	5.27e-7	2.0	5.27e-7	2.0

Example (1)

$e = x_* - x_\pi$

Table: Error $(\|e_1\|^2_{L^2} + \|e_2\|^2_{H^1} + \|e_3\|^2_{H^1})^{\frac{1}{2}}$ of the collocation solution for $N = 1$

	$M = 3$ uniform points		$M = 2$ Gaussian points	
n	error	order	error	order
10	5.65e-1		5.65e-1	
20	3.93e-1	0.5	3.93e-1	0.5
40	2.49e-1	0.6	2.49e-1	0.7
80	1.85e-1	0.4	1.85e-1	0.4
160	1.42e-1	0.4	1.42e-1	0.4
320	1.12e-1	0.3	1.12e-1	0.3

Example (2: Mathematical pendulum in first-order formulation)

This problem has been used in many publications for demonstrating properties of algorithms for the solution of differentail-algebraic systems. We apply the first-order formulation ($m = 5$) of

$$\begin{aligned} x''(t) &= -x(t)\lambda(t), \\ y''(t) &= -y(t)\lambda(t) - g, \\ 0 &= x(t)^2 + y(t)^2 - L^2, \quad t \in [0,1], \end{aligned}$$

$g = 16$, $L = \sqrt{8}$. This problem has index 3 and dynamical degree $d = 2$.
Initial conditions: $y(0) = 2$ and $y'(0) = 0$.
Choose n equidistant grid points and $M = N + 1$ uniformly distributed collocation points.

Example (2)

Table: Errror in $L^2(0,1)$ for $N = 3$ for the pendulum example.

n	x	x'	y	y'	λ
10	4.78e-02	1.25e-01	1.98e-02	1.05e-01	6.19e-01
20	6.58e-03	1.88e-02	2.71e-03	2.03e-02	3.32e-01
40	8.97e-04	3.18e-03	3.69e-04	4.51e-03	1.72e-01
80	1.20e-04	6.35e-04	4.93e-05	1.08e-03	8.67e-02
160	1.53e-05	1.44e-04	6.30e-06	2.64e-04	4.34e-02
320	1.90e-06	3.46e-05	7.82e-07	6.49e-05	2.14e-02
640	3.07e-08	1.08e-06	1.27e-08	2.03e-06	1.34e-03

Example (2)

Table: Order estimate for $N = 3$ for the pendulum example.

n	x	x'	y	y'	λ
10	2.9	2.7	2.9	2.4	0.9
20	2.9	2.6	2.9	2.2	0.9
40	2.9	2.3	2.9	2.1	1.0
80	3.0	2.1	3.0	2.0	1.0
160	3.0	2.1	3.0	2.0	1.0
320	5.9	5.0	5.9	5.0	4.0

Example (2)

Table: Errror in $L^2(0,1)$ for $N = 5$ for the pendulum example.

n	x	x'	y	y'	λ
10	2.49e-04	9.79e-04	1.12e-04	1.88e-03	3.61e-02
20	4.92e-05	1.29e-04	2.01e-05	1.42e-04	4.91e-03
40	1.53e-06	4.60e-06	6.27e-07	7.00e-06	6.18e-04
80	3.68e-08	1.85e-07	1.51e-08	3.98e-07	7.74e-05
160	1.09e-09	1.05e-08	4.47e-10	2.48e-08	9.81e-06
320	2.49e-14	5.35e-14	3.98e-14	2.21e-14	5.00e-12

Example (2)

Table: Order estimate for $N = 5$ for the pendulum example.

n	x	x'	y	y'	λ
10	2.3	2.9	2.5	3.7	2.9
20	5.0	4.8	5.0	4.3	3.0
40	5.4	4.6	5.4	4.1	3.0
80	5.1	4.1	5.1	4.0	3.0
160	15.4	17.6	13.5	20.1	20.9

Example (3: S.L.Campbell, E. Moore, 1995)

$$
\begin{aligned}
x_1' - x_4 &= 0,\\
x_2' - x_5 &= 0,\\
x_3' - x_6 &= 0,\\
x_4' - x_6 \cos t + x_3 \sin t + x_5 - 2x_1(1 - r(x_1^2 + x_2^2)^{-\frac{1}{2}})x_7 &= 0,\\
x_5' - x_6 \sin t - x_3 \cos t - x_4 - 2x_2(1 - r(x_1^2 + x_2^2)^{-\frac{1}{2}})x_7 &= 0,\\
x_6' + x_3 - 2x_3x_7 &= 0,\\
x_1^2 + x_2^2 + x_3^2 - 2r(x_1^2 + x_2^2)^{\frac{1}{2}} + r^2 - \rho^2 &= 0, \;\; t \in [0, 5],
\end{aligned}
$$

in which $r > \rho$ is supposed and the numerical experiments are carried out for $\rho = 5$ and $r = 10$. We apply the same parameters. The problem has index 3.

Here we use n equidistant grid points and $M = N + 1$ Gaussian collocation points.

Example (3)

The solution considered in the reference is

$$
\begin{aligned}
x_{*1}(t) &= (\rho\cos(2\pi - t) + r)\cos t = (\rho\cos t + r)\cos t,\\
x_{*2}(t) &= (\rho\cos(2\pi - t) + r)\sin t = (\rho\cos t + r)\sin t,\\
x_{*3}(t) &= \rho\sin(2\pi - t) = -\rho\sin t,\\
x_{*4}(t) &= -(\rho\cos(2\pi - t) + r)\sin t + \rho\sin(2\pi - t)\cos t,\\
x_{*5}(t) &= (\rho\cos(2\pi - t) + r)\cos t + \rho\sin(2\pi - t)\sin t,\\
x_{*6}(t) &= -\rho\cos(2\pi - t),\\
x_{*7}(t) &= 0.
\end{aligned}
$$

Example (3)

Table: Errors for the components x_i. $N = 5$, $M = N + 1$

	$n = 50$	$n = 100$	$n = 200$
$\|x_{\pi,1} - x_{*1}\|_\infty$	3.56e-10	5.70e-12	6.53e-12
$\|x_{\pi,2} - x_{*2}\|_\infty$	3.75e-10	7.34e-12	7.74e-12
$\|x_{\pi,3} - x_{*3}\|_\infty$	2.46e-10	6.33e-12	7.77e-12
$\|x_{\pi,4} - x_{*4}\|_\infty$	2.50e-08	1.43e-09	2.04e-09
$\|x_{\pi,5} - x_{*5}\|_\infty$	2.16e-08	1.24e-09	1.76e-09
$\|x_{\pi,6} - x_{*6}\|_\infty$	3.86e-08	2.17e-09	3.04e-09
$\|x_{\pi,7} - x_{*7}\|_\infty$	1.16e-06	1.36e-07	1.63e-07

Comparable with the results in the reference but much less cost!

Example (3)

Table: Errors in $H^1_D((0,5),\mathbb{R}^7)$

n	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
10	3.32e+1	4.53e+0	3.82e-1	7.02e-2	1.47e-3
20	3.32e+1	7.51e-1	1.02e-1	1.26e-2	1.24e-4
40	3.32e+1	3.03e-1	3.14e-2	2.52e-3	1.30e-5
80	3.32e+1	1.80e-1	1.22e-2	5.45e-4	1.54e-6
160	3.32e+1	1.17e-1	5.67e-3	1.25e-4	1.20e-6
320	3.32e+1	7.95e-2	2.73e-3	1.25e-4	1.20e-6

Example (3)

Table: Order estimation. The row "theory" contains the expected orders.

n	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
20	0.0	2.6	1.9	2.5	3.6
40	0.0	1.3	1.7	2.3	3.3
80	0.0	0.7	1.4	2.2	3.1
160	0.0	0.6	1.1	2.1	0.6
theory		(0)	(1)	(2)	3

Example (4: Index-4 Jordan DAE)

$$-\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x'(t) + x(t) = q(t), \quad t \in [0,1],$$

$$x_*(t) = \begin{bmatrix} e^{-t} \sin t \\ e^{-2t} \sin t \\ e^{-t} \cos t \\ e^{-2t} \cos t \end{bmatrix}$$

Example (4: Index-4 Jordan DAE)

Table: Numerical estimates for the convergence order.

N	Theory	$M = 2N + 1$ L^2	$M = 2N + 1$ $\mathbb{R}$	$M = N + 1$ L^2	$M = N + 1$ $\mathbb{R}$
1u	0	0.1	0.0	0.0	0.0
1g	0	0.1	0.0	0.0	0.0
2u	0	0.4	0.4	0.4	0.4
2g	0	0.4	0.4	0.4	0.3
3u	0	1.1	1.1	1.1	1.1
3g	0	1.1	1.1	1.1	1.1
4u	1	2.1	2.1	2.1	2.1
4g	1	2.1	2.1	2.1	2.1
5u	2	2.6	2.7	3.2	3.1
5g	2	2.9	2.9	3.1	3.1
6u	3	3.5	3.7	4.3	3.6
6g	3	4.3	4.3	4.1	4.2

Outline

Linear case in its natural Hilbert space setting

$$\boxed{A(Dx)' + Bx = q, \quad G_a x(a) + G_b x(b) = \beta} \Leftrightarrow \boxed{\mathcal{T}x = y} \qquad (7)$$

Suppose: $\ker D \subseteq \ker G_a$, $\ker D \subseteq \ker G_b$

$$\mathcal{T} : H^1_D \to L^2 \times \mathbb{R}^d =: Y$$

$$Tx := A(Dx)' + Bx, \quad T_{BC}x := G_a x(a) + G_b x(b), \quad x \in H^1_D$$
$$\mathcal{T} = (T, T_{BC}) : H^1_D \to Y$$

$$L^2 := L^2((a,b), \mathbb{R}^m), \quad H^1_D := \{z \in L^2((a,b), \mathbb{R}^m) : Dx \in H^1((a,b), \mathbb{R}^k)\}$$

$$\mathcal{C}_\pi := \mathcal{C}_\pi([a,b], \mathbb{R}^m) \subset L^2, \quad \boxed{X_\pi \subset H^1_D}$$

$$U_\pi : H^1_D \to H^1_D, \quad U_\pi^2 = U_\pi, \quad \operatorname{im} U_\pi = X_\pi, \quad \ker U_\pi = X_\pi^\perp$$

$$\mathcal{R}_{\pi,M} : \mathcal{C}_\pi \times \mathbb{R}^d \subset Y \to \mathcal{C}_\pi \times \mathbb{R}^d \subset Y, \quad \mathcal{R}_{\pi,M} := (R_{\pi,M}, I_d)$$

The overdetermined least-squares collocation means now that we seek an element $x_\pi \in X_\pi$ minimizing the functional

$$\psi_{\pi,M}(x) = \| R_{\pi,M} \underbrace{(A(Dx)' + Bx - q)}_{=w} \|^2_{L^2} + |G_a x(t_0) + G_b x(t_n) - \beta|^2$$

$$= W^T \mathcal{L} W + |G_a x(t_0) + G_b x(t_n) - \beta|^2, \quad x \in X_\pi$$

$$x_\pi \in \operatorname{argmin}\{\psi_{\pi,M}(x) : \ x \in X_\pi\} = \operatorname{argmin}\{\|\mathcal{R}_{\pi,M}(\mathcal{T}x - y)\|^2 : \ x \in X_\pi\}$$

$$x_\pi = (\mathcal{R}_{\pi,M}\mathcal{T}U_\pi)^+ \mathcal{R}_{\pi,M} y$$

Theorem

Let T be regular with index $\mu > 1$, T_{BC} be accurately stated,
$y \in \operatorname{im} \mathcal{T}$, $x_* := \mathcal{T}^{-1} y$, $N \geq 1$, $M \geq N+1$, $x_\pi^+ := (\mathcal{T} U_\pi)^+ \mathcal{R}_{\pi,M}\, y$,
$0 < \sigma_1 < \cdots < \sigma_M < 1$.
Then there is a positive constant h_* such that the following hold for all meshes with $h \leq h_*$.

(a) *If A and B are constant matrices, then $x_\pi = x_\pi^+$ is uniquely determined and* $\boxed{\|x_\pi - x_*\|_{H_D^1} \leq c_* h^{\max(0,N-\mu+1)}}$.

(b) *In general, if $N \geq \mu - 1$, then x_π^+ is uniquely determined and* $\boxed{\|x_\pi^+ - x_*\|_{H_D^1} \leq c_* h^{N-\mu+1}}$.

(c) *In general, if $N \geq \mu - 1$, $M \geq N + \mu$, then x_π is uniquely determined and* $\boxed{\|x_\pi - x_*\|_{H_D^1} \leq c_* h^{N-\mu+1}}$.

Proof: $N \geq 1,\ M \geq N+1$

A, B constant (HMTWW(2017), HMT(2018)):

$$\|(\mathcal{T}U_\pi)^+\|_{Y\to H^1_D} \leq c_+ h_\pi^{-\min(N,\mu-1)}, \quad \mathcal{R}_{\pi,M}\mathcal{T}U_\pi = \mathcal{T}U_\pi.$$

General case (HMT(2018),HM(2019)):

$$\|(\mathcal{T}U_\pi)^+\|_{Y\to H^1_D} \leq c_+ h_\pi^{-(\mu-1)},$$

$$\|\mathcal{R}_{\pi,M}\mathcal{T}U_\pi - \mathcal{T}U_\pi\|_{H^1_D\to Y} \leq C_{AB} h_\pi^{M-N-\frac{1}{2}}$$

if $M \geq N+\mu$: $\|(\mathcal{R}_{\pi,M}\mathcal{T}U_\pi)^+\|_{Y\to H^1_D} \leq C_+ h_\pi^{-(\mu-1)}$

Nonlinear case in the Hilbert-space setting

$$\boxed{f((Dx)'(t), x(t), t) = 0, t \in [a, b], \quad g(x(a), x(b)) = 0} \Leftrightarrow \boxed{\mathcal{F}x = 0} \quad (8)$$

Suppose: $g(u, v) = g(D^+Du, D^+Dv)$, x_* is a sufficiently smooth solution residing in a regularity region, $\operatorname{dom} \mathcal{F} = \mathcal{B}_{H^1_D}(x_*, \rho)$, BC "accurate".

$\mathcal{F} : \operatorname{dom} \mathcal{F} \subset H^1_D \to Y$ is Gâteaux-differentiable, but not Fréchet!

$$\|\mathcal{F}'(x) - \mathcal{F}'(\bar{x})\| \leq L h_\pi^{-\frac{1}{2}} \|x - \bar{x}\| \quad \text{if } x, \bar{x} \in \operatorname{dom} \mathcal{F} \cap X_\pi.$$

$$\|\mathcal{F}'(x) - \mathcal{F}'(\bar{x})\| \leq L h_\pi^{-\frac{1}{2}} (\|x - \bar{x}\| + \hat{L} h_\pi^N) \quad \text{if } x \in \operatorname{dom} \mathcal{F} \cap X_\pi, \ \bar{x} \in \operatorname{dom} \mathcal{F} \text{ sufficiently smooth.}$$

Newton-like iteration with bounded outer inverses: $x_{k+1} = x_k + z_{k+1}$

$$z_{k+1} = \operatorname{argmin}\{\|\mathcal{F}'(x_k)z + \mathcal{F}x_k\|^2 : z \in X_\pi\} = -(\mathcal{F}'(x_k)U_\pi)^+ \mathcal{F}x_k$$

Theorem

If $N \geq 2\mu - 1$ *and* $x_0 \in X_\pi$ *is sufficientlly close to* x_* *(* h_π *sufficiently small), then the iteration is feasible and there is a number* k_π *such that*

$$\boxed{\|x_{k+1} - x_*\| \leq 3c_* h_\pi^{N-\mu+1}, \quad k \geq k_\pi}.$$

Multilevel approach: $h_{\pi_{i+1}} = qh_{\pi_i}$, $0 < q < 1$, $X_{\pi_i} \subset X_{\pi_{i+1}}$

$$\|x^{[\pi_i]}_{k_{\pi_i}+1} - x_*\| \to 0 \quad (i \to 0).$$

Outline

Conclusion and open questions

- ✓ The experiments and hitherto theoretical contributions give rise to the conjecture that next to the existing derivative-array based methods there is further potential toward a reliable direct numerical treatment of DAEs.
- ✓ The basic procedure of overdetermined least-squares collocation is totally simple.
  In view of a reliable implementation there remain open questions, e.g., the practical choice of N, M, stepsize control, window-techniques.
- ✓ The theoretical foundation needs further serious efforts, e.g., to understand the nice test results with low-level degree N.

✓

Thank you for your attention !

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