Ensemble Control

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Outline

I. **What is Ensemble Control?**

II. **Reachability of linear parameter-dependent systems**

III. **Constructive methods**

IV. **What about feedback?**

What is Ensemble Control?

Core task

Controlling a large, potentially infinite number of states or systems using finitely many (single) input function or feedback controller.

- A precise formulation depends on the context.
- Previous notions, e.g.
  - *Blending Problem* (Tannenbaum (1970s))
  - *Simultaneous controllability* (Ghosh (1980s))

Ensembles of states - transport of densities

Given a nonlinear system

$$\dot{x} = f(x) + u(t)g(x) \qquad (\star)$$

Task: Steer an initial state density $\rho_0 \in L^2(\mathbb{R}^n)$ with $(\star)$ to a desired state density $\rho_T \in L^2(\mathbb{R}^n)$ in finite time $T > 0$.

$\rightarrow$ Open loop control problem for Liouville transport Equation

$$\frac{\partial \rho(t,x)}{\partial t} = -\mathrm{div}\big(f(x)\rho(t,x)\big) - u(t)\mathrm{div}\big(g(x)\rho(t,x)\big)$$

For stochastic systems this leads to Fokker-Plank Equations (Fleig, Grüne)

R. Brockett.
Notes on the Control of the Liouville Equation.
Lecture Notes in Mathematics 2048, pp 101-129, 2012.

Ensembles of systems

Family of systems (ensemble)

$$\dot{x} = f(x,\theta) + g(x,\theta)u(t)$$

Task: Steer an initial family of states $x^0(\theta)$ to a desired family of states $x^*(\theta)$ in finite time $T > 0$ by a **single parameter-independent** control u.

origin: Quantum control for bilinear systems

$$\dot{x} = \Big(A(\theta) + u(t)B(\theta)\Big)x$$

J.-S. Li, N. Khaneja, Ensemble Control of Bloch Equations, IEEE Trans. Automatic Control 54(3):528-536, 2009.

Averaged Control

For $\theta \in [0, 1]$ we consider

$$\frac{\partial x}{\partial t}(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t)$$
$$x(0, \theta) = x_0(\theta)$$

Averaged controllability (Zuazua (2014))

Given $x^* \in \mathbb{R}^n$, x_0 and $T > 0$ find an input u such that

$$\int_0^1 \varphi(T, \theta, u)\,\mathrm{d}\theta = x^*.$$

E. Zuazua, Averaged control, Automatica 50:3077–3087, 2014.

Outline

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Setting the scene

One-parameter family of linear systems = **Ensemble**

$$\frac{\partial}{\partial t}x(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t)$$

$$x_{t+1}(\theta) = A(\theta)x_t(\theta) + B(\theta)u_t$$

$$(A,B) \in C_{n,m}(\mathbf{P}) \quad :\Longleftrightarrow \quad A \in C(\mathbf{P}, \mathbb{C}^{n\times n}),\ B \in C(\mathbf{P}, \mathbb{C}^{n\times m})$$

parameter space $\mathbf{P} \subset \mathbb{C}$ compact

time $t \in \mathbb{R}_+/\mathbb{N}$

initial values $x(0,\theta) = 0$

control inputs $u \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$ or $u = (u_0, u_1, ...),\ u_i \in \mathbb{C}^m$

The basic problem

Ensemble Control Problem

Given a family of states $f = \{f(\theta) \in \mathbb{C}^n \,|\, \theta \in \mathbf{P}\}$,

is there a **parameter-independent input** u

steering 0 in some finite time $T \geq 0$ to f?

$$\forall f \quad \exists\, u,\, T > 0 \qquad : \quad \varphi(T, \theta, u) = f(\theta) \quad ?$$

Ensembles: (In)finite dimensional systems

State space $X = C(\mathbf{P}, \mathbb{C}^n), L^p(\mathbf{P}, \mathbb{C}^n)$ (separable Banach space)

Multiplication operator (bounded linear)

$$\mathcal{M}_A \colon X \to X, \quad f(\theta) \mapsto A(\theta) f(\theta)$$

Input operator (bounded linear)

$$\mathcal{M}_B \colon \mathbb{C}^m \to X, \quad v \mapsto B(\theta) v$$

Infinite-dimensional linear system

$$\dot{x} = \mathcal{M}_A x + \mathcal{M}_B u$$
$$x_{t+1} = \mathcal{M}_A x_t + \mathcal{M}_B u_t$$

Classification and a first result

$\rightarrow$ Ensemble Control Problem $=$ (classical) reachability problem of a special class of ∞-dim. linear systems

Classification and a first result

→ Ensemble Control Problem = (classical) reachability problem of a special class of ∞-dim. linear systems

Triggiani (1975)

If $\dim X = \infty$ the pair $(\mathcal{M}_A, \mathcal{M}_B)$ is never reachable.

! Crucial: inputs are parameter-independent (Baire's Category Theorem)

R. Triggiani.
Controllability and observability in Banach spaces with bounded operators.
SIAM J. Control Optimization, 13:462–491, 1975.

Ensemble reachability

$(A, B) \in C_{n,m}(\mathbf{P})$ *uniformly ensemble reachable (UER)* $:\Longleftrightarrow$

$$\forall\, f \in C(\mathbf{P}, \mathbb{C}^n) \quad \forall\, \varepsilon > 0 \quad \exists\, T > 0 \quad \exists\, u :$$

$$\sup_{\theta \in \mathbf{P}} \|\varphi(T, \theta, u) - f(\theta)\| < \varepsilon.$$

Also of interest: L^p-ensemble reachability $\Longleftrightarrow$

$$\left(\int_{\mathbf{P}} \|\varphi(T, \theta, u) - f(\theta)\|^p \, \mathrm{d}\theta \right)^{\frac{1}{p}} < \varepsilon.$$

(Well) known characterization

Theorem

Let $B(\theta) = \big(b_1(\theta), \dots, b_m(\theta)\big)$. *TFAE:*

(a) (A, B) *is ensemble reachable on* X.

(b) $\overline{\operatorname{span}\{\theta \mapsto A(\theta)^k b_j(\theta) \mid k \in \mathbb{N}_0, j = 1, \dots, m\}} = X$

(c) *The multiplication operator* $\mathcal{M}_A$ *is m-multicyclic and* $b_1, \dots, b_m$ *are cyclic vectors.*

(d) $\forall \varepsilon > 0 \ \forall f \in X \ \exists \, p_1, \dots, p_m \in \mathbb{C}[z]$ *:*

$$\|p_1(A)\, b_1 + \dots + p_m(A)\, b_m - f\|_X < \varepsilon.$$

$(a) \iff (b)$ Triggiani (1975) (cont.-time)

However,

are the following pairs ensemble reachable?

Example (Single input $m = 1$)

$$A(\theta) = \begin{pmatrix} 0 & -\theta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \theta^2 + 1 \end{pmatrix} \quad \text{and} \quad b(\theta) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{P} = [0, 1]$$

Example (Multi input $m = 2$)

$$a(\theta) = \theta^2 \qquad \text{and} \qquad B(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad \mathbf{P} = [-1, 1]$$

Pointwise checkable conditions for ensemble reachability

Single-Input systems

The scalar case

$$x^+(t,\theta) = a(\theta)x(t,\theta) + b(\theta)u(t)$$

Theorem (Dirr, S. (2018))

Let $\mathbf{P} \subset \mathbb{C}$ *compact and contractible. Then*

$$(a,b) \in C_{1,1}(\mathbf{P}) \quad \textbf{\textit{UER}} \qquad \Longleftrightarrow \qquad \begin{array}{c} a \text{ one-to-one} \\ b \text{ zero-free} \\ \mathring{\mathbf{P}} = \emptyset \end{array}$$

P *contractible* $:\Leftrightarrow$ identity map on **P** is homotopic to a constant mapping

Mergelyan's Theorem

Mergelyan's Theorem (1951)
Let $\Omega \subset \mathbb{C}$ be compact and

(i) $\mathbb{C} \setminus \Omega$ connected

(ii) $f \in C(\Omega, \mathbb{C})$

(iii) f analytic in the interior of Ω

$\Longrightarrow$ $\forall \varepsilon > 0 \; \exists$ polynomial p :

$$|f(z) - p(z)| < \varepsilon \quad \forall z \in \Omega.$$

The interior of Ω may be empty ! (Lavrentiev (1936))

Proof is not constructive

D. Gaier.
Lectures on complex approximation.
Birkhäuser, Boston, 1987.

Cyclicity of the multiplication operator

Corollary

Let $\mathbf{P} \subset \mathbb{C}$ compact and contractible. Then,

$$\begin{array}{c} \mathcal{M}_a \text{ cyclic on } C(\mathbf{P}) \\ b \text{ cyclic vector} \end{array} \iff \begin{array}{c} a \text{ one-to-one} \\ b \text{ zero-free} \\ \overset{\circ}{\mathbf{P}} = \emptyset \end{array}$$

Single input - Necessary conditions

Theorem (Helmke, S. (2014), Dirr, S. (2018))

Let $\mathbf{P} \subset \mathbb{C}$ *compact and* $(A, b) \in C_{n,1}(\mathbf{P})$ **UER**. *Then:*

(N1) $(A(\theta), b(\theta))$ *is reachable for every* $\theta \in \boldsymbol{P}$.

(N2) *for each* $\theta_1 \neq \theta_2$ *one has*

$$\sigma(A(\theta_1)) \cap \sigma(A(\theta_2)) = \emptyset.$$

(N3) $\boldsymbol{P}$ *has no interior points.*

(N4) *the eigenvalues are generically simple.*

Single-input - sufficient condition

Theorem (Scherlein, S., Helmke (2014), Dirr, S. (2018))

Let $\mathbf{P}$ *compact, contractible. Then,* $(A, b) \in C_{n,1}(\mathbf{P})$ *is* **UER** *if* $(N1) - (N4)$ *hold and*

(S1) *The characteristic polynomials are given by*

$$z^n - a_{n-1} z^{n-1} - \cdots - a_1 z - a_0(\theta),$$

for some $a_1, ..., a_{n-1} \in \mathbb{C}$ *and some* $a_0 \in C(\boldsymbol{P}, \mathbb{C})$.

Single-input - sufficient condition

Theorem (Scherlein, S., Helmke (2014), Dirr, S. (2018))

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for some $a_1, ..., a_{n-1} \in \mathbb{C}$ *and some* $a_0 \in C(\boldsymbol{P}, \mathbb{C})$.

$$p(z) := \sum_{k=1}^{n} p_k(z^n - a_{n-1}z^{n-1} - \cdots - a_1 z)z^{k-1}$$

leads to

$$p(A(\theta))e_1 = \begin{pmatrix} p_1(a_0(\theta)) \\ \vdots \\ p_n(a_0(\theta)) \end{pmatrix}$$

Single-input case - sufficient condition II

Theorem (Helmke, S. (2014))

Let $\mathbf{P} \subset \mathbb{C}$ *be a Jordan arc. Then,* $(A, b) \in C_{n,1}(\mathbf{P})$ *is* **UER** *if* $(N1) - (N4)$ *hold and*

(S2) *eigenvalues of* $A(\theta)$ *are simple* $\quad \forall\ \theta \in \boldsymbol{P}$.

$\rightarrow\ \exists$ continuous change of coordinates $S(\theta)$:

$$S(\theta)^{-1}A(\theta)S(\theta) = \begin{pmatrix} a_1(\theta) & & \\ & \ddots & \\ & & a_n(\theta) \end{pmatrix} \quad S(\theta)^{-1}b(\theta) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Example 1 revisited

Example

$$A(\theta) = \begin{pmatrix} 0 & -\theta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \theta^2 + 1 \end{pmatrix}, b(\theta) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{P} = [0, 1]$$

Spectral selections

A set-valued map $\Gamma : \mathbf{P} \to \bigcup_{\theta \in \mathbf{P}} \sigma(A(\theta))$ is called a *spectral selection* if

$$\Gamma(\theta) \subset \sigma(A(\theta)) \quad \forall\, \theta \in \mathbf{P}$$

Two spectral selections Γ_1 and Γ_2 are *pointwise disjoint* if

$$\Gamma_1(\theta) \cap \Gamma_1(\theta) = \emptyset \qquad \forall\, \theta \in \mathbf{P}.$$

Γ_1 and Γ_2 are *strictly disjoint* if $\Gamma_1(\mathbf{P}) \cap \Gamma_2(\mathbf{P}) = \emptyset$.

$\Gamma_1, \ldots, \Gamma_k$ is a *spectral family* if

$$\bigcup_{i=1}^{k} \Gamma_i(\theta) = \sigma(A(\theta)) \qquad \forall\, \theta \in \mathbf{P}.$$

Decomposition result

- $(A, B) \in C_{n,m}(\mathbf{P})$
- $\Gamma_1, \dots, \Gamma_k$ pairwise strictly disjoint spectral family

$\Rightarrow$(!!) $\exists$ T continuous such that (∞-dim. parallel connection)

$$T^{-1}(\theta)A(\theta)T(\theta) = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix} \quad T^{-1}(\theta)B(\theta) = \begin{pmatrix} B_1 \\ \vdots \\ B_k \end{pmatrix}$$

Theorem (Dirr, S. (2018))

Let $\mathbf{P} \subset \mathbb{C}$ *be compact and contractible.*

(a) (A, B) **UER** $\Longrightarrow$ (A_i, B_i) **UER** $\forall i = 1, \dots, k.$

(b) (A_i, B_i) **UER** & $\mathbb{C} \setminus \Gamma_i(\mathbf{P})$ *connected* $\forall\, i = 1, \dots, k$

$\Longrightarrow (A, B)$ **UER**.

Pointwise checkable conditions for ensemble reachability

Multi-Input systems

Multi-Input - scalar example

Example ($m = 2$)

$$a(\theta) = \theta^2 \qquad \text{and} \qquad B(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad \mathbf{P} = [-1, 1]$$

Multi-Input - scalar example

Example ($m = 2$)

$$a(\theta) = \theta^2 \qquad \text{and} \qquad B(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad \mathbf{P} = [-1, 1]$$

$$\forall \varepsilon > 0 \quad \forall f \in C(\mathbf{P}) \quad \exists\, p_1, p_2 \quad :$$

$$|p_1(\theta^2) + p_2(\theta^2)\,\theta - f(\theta)| < \varepsilon \quad \forall\, \theta \in \mathbf{P}$$

Multi-Input - Necessary conditions

Theorem (Helmke, S. (2014))

Let $\mathbf{P} \subset \mathbb{C}$ *compact and* $(A, b) \in C_{n,1}(\mathbf{P})$ **UER**. *Then:*

(1) $(A(\theta), B(\theta))$ *is reachable for every* $\theta \in \boldsymbol{P}$.

(2) *For each number* $s \geq m + 1$ *of distinct parameters* $\theta_1, \ldots, \theta_s \in \boldsymbol{P}$ *one has*

$$\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_s)) = \emptyset.$$

Interlude: Hermite indices

For $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ consider

$$\begin{pmatrix} b_1 & Ab_1 & \cdots A^{n-1}b_1 & \cdots & b_m & Ab_m \cdots & A^{n-1}b_m \end{pmatrix},$$

and select from left to right the first linear independent columns

$$b_1, ..., A^{h_1-1}b_1, ..., b_m, ..., A^{h_m-1}b_m.$$

and

$$h(A, B) = (h_1, ..., h_m) \qquad :\Longleftrightarrow \qquad \textit{Hermite indices},$$

where $h_i := 0$ if the column b_i has not been selected.

1. (A, B) is reachable $\iff h_1 + \cdots + h_m = n$.
2. If $\big(A(\theta), B(\theta)\big)$ depends analytically on θ the Hermite indices are generically constant.

Interlude: Hermite canonical form

If (A, B) is reachable, the invertible transformation

$$T = (b_1, ..., A^{h_1-1}b_1, ..., b_m, ..., A^{h_m-1}b_m)$$

yields the Hermite canonical form

$$T^{-1}AT = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_m \end{pmatrix}$$

! T simultaneous block-triangularizes (A, B)

Multi-input case - sufficient condition III

Theorem (Dirr, S. (2018))

Let $\mathbf{P}$ *compact. Then,* $(A, B) \in C_{n,m}(\mathbf{P})$ *is* **UER** *if*

1. $(A(\theta), B(\theta))$ *reachable* $\forall \theta \in \boldsymbol{P}$.
2. *Hermite indices of* $(A(\theta), B(\theta))$ *are constant.*
3. *The corresponding subpairs* (A_{ii}, b_i) *are* **UER**.

Remarks on the proof

- The Decomposition/Parallel Connection Theorem does not apply here.
- Proof is based on the fact that a pair (A, B) of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

  is **UER** if the diagonal pairs are **UER.**

  Warning: The converse in not true ($m = 1$)

Canonical forms !?

Pointwise reachability $\not\Rightarrow$ $T(\theta)$ continuous.

E.g. for

$$A(\theta) = \begin{pmatrix} 0 & g(\theta) \\ 1 & 0 \end{pmatrix} \quad B(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $g(\theta) := \theta \sin(\frac{1}{\theta})$ and $g(0) = 0$ one has

$$\begin{pmatrix} b_1 & A(\theta)b_1 & b_2 & A(\theta)b_2 \end{pmatrix} = \begin{pmatrix} 0 & g(\theta) & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$\mapsto$ Hermite (Kronecker) indices might have infinitely many jumps

$\Rightarrow$ canonical form ?!?

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First comments

Two possible approaches are based on:

(1) Complex approximation theory
(Runge, Walsh, Weierstrass)

(2) Linear integral equations
(Interpolation, projection method)

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Two possible approaches are based on:

(1) Complex approximation theory
(Runge, Walsh, Weierstrass)

(2) Linear integral equations
(Interpolation, projection method)

Let $(A, B) \in C_{n,m}(\mathbf{P})$, $T > 0$ and $\mathcal{R}\colon L^2([0, T], \mathbb{R}^m) \to C(\mathbf{P}, \mathbb{R}^n)$,

$$\mathcal{R}u\,(\theta) = \int_0^T e^{A(\theta)(T-s)} B(\theta) u(s)\,\mathrm{d}s$$

Moment collocation

Let $(A, B) \in C_{n,m}(\mathbf{P})$ **UER**, then choose distinct moments

$$\theta_1, ..., \theta_N$$

and solve the equation

$$\begin{pmatrix} \int_0^T e^{A(\theta_1)(T-s)} B(\theta_1) u(s)\, \mathrm{d}s \\ \vdots \\ \int_0^T e^{A(\theta_N)(T-s)} B(\theta_N) u(s)\, \mathrm{d}s \end{pmatrix} = \begin{pmatrix} f(\theta_1) \\ \vdots \\ f(\theta_N) \end{pmatrix}.$$

Let u_N denote its minimum L^2-norm solution and

$$\Delta_N := \sup_{\theta \in P} \left(\inf_{\theta_i} |\theta - \theta_i| \right).$$

Then, by Nashed and Wahba (1974) one has

$$\lim_{\Delta_N \to 0} \|u_N - \mathcal{R}^\dagger f\|_{L^2} = 0.$$

current task: Development of consensus methods

Discrete-time systems

$$x_{t+1}(\theta) = A(\theta)x_t(\theta) + b(\theta)u_t$$

For inputs $u_0, \ldots, u_{T-1}$ the solution is given by

$$\varphi(T, \theta, u) = \sum_{k=0}^{T-1} A(\theta)^k b(\theta) u_{T-1-k} = \left(\sum_{k=0}^{T-1} u_{T-1-k} A(\theta)^k \right) b(\theta)$$

For $\varepsilon > 0$ and $f \in C(\mathbf{P}, \mathbb{C}^n)$ one has

$$\|\varphi(T, \theta, u) - f(\theta)\| = \|p(A(\theta))\, b(\theta) - f(\theta)\| < \varepsilon$$

Sufficient cond's allow for a reduction to problems of the form

$$|p(x) - f(a^{-1}(x))| < \varepsilon$$

Complex approximation

$$|p(x) - f(a^{-1}(x))| < \varepsilon$$

- is solved based on (complex) approximation theory
- Proofs yield constructive methods except where *Mergelyan*'s Theorem is used (*Lavrentiev* ??).
- $\mathbf{P} = [a, b] \rightarrow$ Bernstein polynomials (Weierstrass)
- **P** Jordan arc $\rightarrow$ Cauchy integral formula (Runge/Walsh)

Outline

What about feedback?

Basic idea:

To overcome the strong and non robust on the Hermite indices and the spectra use controllers of the form

$$u(t,x) := K\mathcal{C}x + u(t),$$

where

- $K \in \mathbb{C}^{m\times p}$ output feedback gain
- $\mathcal{C}$ output operator, e.g.

(1) sampling operator: $\theta_1, ..., \theta_N \in \mathbf{P}$

$$\mathcal{C}f = \begin{pmatrix} f(\theta_1) \\ \vdots \\ f(\theta_N) \end{pmatrix}$$

(1) averaging operator: $\mathcal{C}f = \int_{\mathbf{P}} C(\theta)f(\theta)\,\mathrm{d}\theta.$

Illustrative example - $\mathcal{C} = \mathrm{Id}$

Example (controlled harmonic oscillator)

Let $\mathbf{P} = [-1, 1]$ and $g \in C^1(\mathbf{P}, \mathbb{R})$

$$\ddot{x} + \theta^2 x = g(\theta)u(t, x) = g(\theta)\big(kx(t, \theta) + u(t)\big), \quad k \in \mathbb{R}.$$

Question: Is the pair

$$A_k(\theta) := \begin{pmatrix} 0 & 1 \\ kg(\theta) - \theta^2 & 0 \end{pmatrix}, \quad b_g(\theta) := \begin{pmatrix} 0 \\ g(\theta) \end{pmatrix}.$$

UER for some $k \in \mathbb{R}$?

Illustrative example

Proposition (S. (2018))

Let $g \in C^1(\mathbf{P}, \mathbb{R})$ be zero-free and strictly monotone. Then

(a) $\exists\ k^* = k^*(g) \in \mathbb{R}\ :\ (A_k, b_g)$ ***UER*** $\forall k > k^*$.

(b) *if* $f \in \mathrm{Lip}(\mathbf{P}, \mathbb{R}) \quad \exists\ \tilde{k} \in \mathbb{R}$ *s.t.*

$\forall k > \tilde{k}\ \exists\ C \in \mathbb{R}$ *polynomials* $(p_n)_n\ (n \geq 3)\ :$

$$\|p_n(A_k)b_g - f\|_\infty \leq C\sqrt{\frac{\log n}{n}}.$$

Mixed open-loop and feedback controllers

Task: ($\mathcal{C} = \mathrm{Id}$)

Find conditions on (A, B) s.t. $\exists\ K \in \mathbb{C}^{m\times n}$ so that

(1) $\theta_1 \neq \theta_2 \implies \sigma(A(\theta_1) + B(\theta_1)K) \cap \sigma(A(\theta_2) + B(\theta_2)K) = \emptyset$

(2) $(A(\theta) + B(\theta)K, B(\theta))$ has constant Hermite indices.

and one of following condition holds

(3) The characteristic polynomials of $A(\theta) + B(\theta)K$ are

$$z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0(\theta)$$

(3') $A(\theta) + B(\theta)K$ has simple eigenvalues for each $\theta \in \mathbf{P}$

Mixed open-loop and feedback controllers

Using Ackermann's formula it follows immediately

Proposition

Suppose $(A, b) \in C_{n,1}(\mathbf{P})$ *is pointwise reachable.*

$\Rightarrow\ \exists\ k(\theta)$ *continuous :* $(A + bk, b)$ ***UER*** *.*

In a nutshell

- Key idea of ensemble control
- Pointwise necessary and sufficient conditions for uniform ensemble reachability for parameter-dependent linear systems
- Methods from (complex) approximation theory, complex analysis, functional analysis and (plane) topology
- In some cases the inputs can be constructed
- There are many other problems which have not been addressed up to now, e.g. ...

Thank you for your attention !

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