Ensemble Control

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Funded by

DFG Deutsche Forschungsgemeinschaft
SCHO 1780/1-1

I. What is Ensemble Control?

II. Reachability of linear parameter-dependent systems

III. Constructive methods

IV. What about feedback?
What is Ensemble Control?

Core task
Controlling a large, potentially infinite number of states or systems using finitely many (single) input function or feedback controller.

- A precise formulation depends on the context.
- Previous notions, e.g.
  - *Blending Problem* (Tannenbaum (1970s))
  - *Simultaneous controllability* (Ghosh (1980s))
Ensembles of states - transport of densities

Given a nonlinear system

\[ \dot{x} = f(x) + u(t)g(x) \]  \hspace{1cm} (\star)

**Task:** Steer an initial state density \( \rho_0 \in L^2(\mathbb{R}^n) \) with (\star) to a desired state density \( \rho_T \in L^2(\mathbb{R}^n) \) in finite time \( T > 0 \).

→ Open loop control problem for Liouville transport Equation

\[ \frac{\partial \rho(t, x)}{\partial t} = -\text{div}(f(x)\rho(t, x)) - u(t)\text{div}(g(x)\rho(t, x)) \]

For stochastic systems this leads to Fokker-Plank Equations

(Fleig, Grüne)

R. Brockett.
Notes on the Control of the Liouville Equation.
Ensembles of systems

Family of systems (ensemble)

\[ \dot{x} = f(x, \theta) + g(x, \theta)u(t) \]

**Task:** Steer an initial family of states \( x^0(\theta) \) to a desired family of states \( x^*(\theta) \) in finite time \( T > 0 \) by a **single** parameter-independent control \( u \).

**origin:** Quantum control for bilinear systems

\[ \dot{x} = \left( A(\theta) + u(t)B(\theta) \right)x \]

---

For $\theta \in [0, 1]$ we consider

\[
\frac{\partial x}{\partial t}(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t) \\
x(0, \theta) = x_0(\theta)
\]

Averaged controllability (Zuazua (2014))

Given $x^* \in \mathbb{R}^n$, $x_0$ and $T > 0$ find an input $u$ such that

\[
\int_0^1 \varphi(T, \theta, u) \, d\theta = x^*.
\]

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Setting the scene

One-parameter family of linear systems = **Ensemble**

\[ \frac{\partial}{\partial t} x(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t) \]

\[ x_{t+1}(\theta) = A(\theta)x_t(\theta) + B(\theta)u_t \]

\[(A, B) \in C_{n,m}(P) \iff A \in C(P, \mathbb{C}^{n \times n}), \ B \in C(P, \mathbb{C}^{n \times m})\]

- parameter space \( P \subset \mathbb{C} \) compact
- time \( t \in \mathbb{R}_+/\mathbb{N} \)
- initial values \( x(0, \theta) = 0 \)
- control inputs \( u \in L_{1,loc}^1(\mathbb{R}_+, \mathbb{C}^m) \) or \( u = (u_0, u_1, \ldots), \ u_i \in \mathbb{C}^m \)
The basic problem

**Ensemble Control Problem**

Given a family of states $f = \{ f(\theta) \in \mathbb{C}^n \mid \theta \in \mathbb{P} \}$,

is there a **parameter-independent input** $u$

steering 0 in some finite time $T \geq 0$ to $f$?

$$\forall f \ \exists u, \ T > 0 \ : \ \varphi(T, \theta, u) = f(\theta) \ ?$$
Ensembles: (In)finite dimensional systems

State space \( X = C(P, \mathbb{C}^n), L^p(P, \mathbb{C}^n) \) (separable Banach space)

Multiplication operator (bounded linear)

\[ \mathcal{M}_A : X \rightarrow X, \quad f(\theta) \mapsto A(\theta)f(\theta) \]

Input operator (bounded linear)

\[ \mathcal{M}_B : \mathbb{C}^m \rightarrow X, \quad v \mapsto B(\theta)v \]

Infinite-dimensional linear system

\[ \dot{x} = \mathcal{M}_A x + \mathcal{M}_B u \]

\[ x_{t+1} = \mathcal{M}_A x_t + \mathcal{M}_B u_t \]
Classification and a first result

→ Ensemble Control Problem = (classical) reachability problem of a special class of $\infty$-dim. linear systems
Classification and a first result

→ Ensemble Control Problem = (classical) reachability problem of a special class of $\infty$-dim. linear systems

**Triggiani (1975)**

If $\dim X = \infty$ the pair $(\mathcal{M}_A, \mathcal{M}_B)$ is never reachable.

! Crucial: inputs are parameter-independent (Baire’s Category Theorem)

R. Triggiani.
Controllability and observability in Banach spaces with bounded operators.
Ensemble reachability

\[(A, B) \in C_{n,m}(P) \text{ uniformly ensemble reachable (UER)} \iff \]

\[\forall f \in C(P, \mathbb{C}^n) \quad \forall \epsilon > 0 \quad \exists T > 0 \quad \exists u : \]

\[\sup_{\theta \in P} \| \varphi(T, \theta, u) - f(\theta) \| < \epsilon.\]

Also of interest: \(L^p\)-ensemble reachability \iff

\[
\left( \int_P \| \varphi(T, \theta, u) - f(\theta) \|^p \, d\theta \right)^{\frac{1}{p}} < \epsilon.
\]
(Well) known characterization

**Theorem**

Let \( B(\theta) = (b_1(\theta), \ldots, b_m(\theta)) \). TFAE:

(a) \((A, B)\) is ensemble reachable on \( X \).

(b) \( \text{span}\{\theta \mapsto A(\theta)^k b_j(\theta) \mid k \in \mathbb{N}_0, j = 1, \ldots, m\} = X \)

(c) The multiplication operator \( M_A \) is \( m \)-multicyclic and \( b_1, \ldots, b_m \) are cyclic vectors.

(d) \( \forall \varepsilon > 0 \ \forall f \in X \ \exists p_1, \ldots, p_m \in \mathbb{C}[z] : \)

\[
\| p_1(A) b_1 + \cdots + p_m(A) b_m - f \|_X < \varepsilon.
\]

(a) \( \iff \) (b) Triggiani (1975) (cont.-time)
are the following pairs ensemble reachable?

Example (Single input $m = 1$)

\[
A(\theta) = \begin{pmatrix}
0 & -\theta^2 & 0 \\
1 & 0 & 0 \\
0 & 0 & \theta^2 + 1
\end{pmatrix}
\quad \text{and} \quad
b(\theta) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad P = [0, 1]
\]

Example (Multi input $m = 2$)

\[
a(\theta) = \theta^2 \quad \text{and} \quad B(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad P = [-1, 1]
\]
Pointwise checkable conditions for ensemble reachability

Single-Input systems
The scalar case

\[ x^+(t, \theta) = a(\theta)x(t, \theta) + b(\theta)u(t) \]

Theorem (Dirr, S. (2018))

Let \( P \subset \mathbb{C} \) compact and contractible. Then

\[ (a, b) \in C_{1,1}(P) \quad \text{UER} \quad \iff \quad a \text{ one-to-one} \quad \iff \quad b \text{ zero-free} \]

\[ \circ \quad P = \emptyset \]

\( P \) contractible : \( \iff \) identity map on \( P \) is homotopic to a constant mapping
Mergelyan’s Theorem

Mergelyan’s Theorem (1951)
Let \( \Omega \subset \mathbb{C} \) be compact and

(i) \( \mathbb{C} \setminus \Omega \) connected

(ii) \( f \in C(\Omega, \mathbb{C}) \)

(iii) \( f \) analytic in the interior of \( \Omega \)

\[ \forall \varepsilon > 0 \exists \text{ polynomial } p : \]

\[ |f(z) - p(z)| < \varepsilon \quad \forall z \in \Omega. \]

The interior of \( \Omega \) may be empty! (Lavrentiev (1936))

Proof is not constructive

D. Gaier.
Lectures on complex approximation.
Cyclicity of the multiplication operator

**Corollary**

Let $P \subset \mathbb{C}$ compact and contractible. Then,

$M_a$ cyclic on $C(P)$  \iff  $b$ cyclic vector

$a$ one-to-one

$b$ zero-free

$P = \emptyset$
Single input - Necessary conditions


Let $P \subset \mathbb{C}$ compact and $(A, b) \in C_{n,1}(P)$ UER. Then:

(N1) $(A(\theta), b(\theta))$ is reachable for every $\theta \in P$.

(N2) for each $\theta_1 \neq \theta_2$ one has

$$\sigma(A(\theta_1)) \cap \sigma(A(\theta_2)) = \emptyset.$$

(N3) $P$ has no interior points.

(N4) the eigenvalues are generically simple.
Theorem (Scherlein, S., Helmke (2014), Dirr, S. (2018))

Let \( P \) compact, contractible. Then, \((A, b) \in C_{n,1}(P)\) is UER if \((N1) - (N4)\) hold and

\[(S1)\] The characteristic polynomials are given by

\[z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0(\theta),\]

for some \(a_1, \ldots, a_{n-1} \in \mathbb{C} \) and some \(a_0 \in C(P, \mathbb{C}).\)
Theorem (Scherlein, S., Helmke (2014), Dirr, S. (2018))

Let $P$ compact, contractible. Then, $(A, b) \in C_{n,1}(P)$ is UER if $(N1) – (N4)$ hold and

(S1) The characteristic polynomials are given by

$$z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0(\theta),$$

for some $a_1, \ldots, a_{n-1} \in \mathbb{C}$ and some $a_0 \in C(P, \mathbb{C})$.

$$p(z) := \sum_{k=1}^{n} p_k (z^n - a_{n-1}z^{n-1} - \cdots - a_1 z)z^{k-1}$$

leads to

$$p(A(\theta))e_1 = \begin{pmatrix} p_1(a_0(\theta)) \\ \vdots \\ p_n(a_0(\theta)) \end{pmatrix}$$
Theorem (Helmke, S. (2014))

Let $P \subset \mathbb{C}$ be a Jordan arc. Then, $(A, b) \in C_{n,1}(P)$ is UER if $(N1) - (N4)$ hold and

$(S2)$ eigenvalues of $A(\theta)$ are simple $\forall \theta \in P.$

$\rightarrow \exists$ continuous change of coordinates $S(\theta) :$

$$S(\theta)^{-1} A(\theta) S(\theta) = \begin{pmatrix} a_1(\theta) & & \\ & \ddots & \\ & & a_n(\theta) \end{pmatrix} \quad \text{and} \quad S(\theta)^{-1} b(\theta) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$
Example 1 revisited

\[ A(\theta) = \begin{pmatrix} 0 & -\theta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \theta^2 + 1 \end{pmatrix}, \quad b(\theta) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad P = [0, 1] \]
Spectral selections

A set-valued map $\Gamma : \mathcal{P} \to \bigcup_{\theta \in \mathcal{P}} \sigma(A(\theta))$ is called a spectral selection if
\[
\Gamma(\theta) \subset \sigma(A(\theta)) \quad \forall \theta \in \mathcal{P}
\]

Two spectral selections $\Gamma_1$ and $\Gamma_2$ are pointwise disjoint if
\[
\Gamma_1(\theta) \cap \Gamma_1(\theta) = \emptyset \quad \forall \theta \in \mathcal{P}.
\]

$\Gamma_1$ and $\Gamma_2$ are strictly disjoint if $\Gamma_1(\mathcal{P}) \cap \Gamma_2(\mathcal{P}) = \emptyset$.

$\Gamma_1, \ldots, \Gamma_k$ is a spectral family if
\[
\bigcup_{i=1}^{k} \Gamma_i(\theta) = \sigma(A(\theta)) \quad \forall \theta \in \mathcal{P}.
\]
Decomposition result

- \((A, B) \in C_{n,m}(P)\)
- \(\Gamma_1, \ldots, \Gamma_k\) pairwise strictly disjoint spectral family

\[ \Rightarrow (!!) \exists T \text{ continuous such that } (\infty\text{-dim. parallel connection}) \]

\[
T^{-1}(\theta)A(\theta)T(\theta) = \begin{pmatrix}
A_1 \\
\vdots \\
A_k
\end{pmatrix} \quad T^{-1}(\theta)B(\theta) = \begin{pmatrix}
B_1 \\
\vdots \\
B_k
\end{pmatrix}
\]

Theorem (Dirr, S. (2018))

Let \(P \subset \mathbb{C}\) be compact and contractible.

(a) \((A, B) \ UER \quad \Rightarrow \quad (A_i, B_i) \ UER \quad \forall i = 1, \ldots, k.\)

(b) \((A_i, B_i) \ UER \quad \& \quad \mathbb{C} \setminus \Gamma_i(P) \text{ connected} \quad \forall i = 1, \ldots, k \]
\[
\Rightarrow (A, B) \ UER.
\]
Pointwise checkable conditions for ensemble reachability

Multi-Input systems
Example $(m = 2)$

\[ a(\theta) = \theta^2 \quad \text{and} \quad B(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad P = [-1, 1] \]
Example \((m = 2)\)

\[
a(\theta) = \theta^2 \quad \text{and} \quad B(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad \mathbb{P} = [-1, 1]
\]

\[
\forall \varepsilon > 0 \quad \forall f \in \mathcal{C}(\mathbb{P}) \quad \exists \ p_1, p_2 : \quad |p_1(\theta^2) + p_2(\theta^2) \theta - f(\theta)| < \varepsilon \quad \forall \ \theta \in \mathbb{P}
\]
Theorem (Helmke, S. (2014))

Let $P \subset \mathbb{C}$ compact and $(A, b) \in C_{n,1}(P)$ UER. Then:

1. $(A(\theta), B(\theta))$ is reachable for every $\theta \in P$.

2. For each number $s \geq m + 1$ of distinct parameters $\theta_1, \ldots, \theta_s \in P$ one has

$$\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_s)) = \emptyset.$$
Interlude: Hermite indices

For \((A, B) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m}\) consider

\[
(b_1, Ab_1 \cdots A^{n-1}b_1 \cdots b_m, Ab_m \cdots A^{n-1}b_m),
\]

and select from left to right the first linear independent columns

\[
b_1, \ldots, A^{h_1-1}b_1, \ldots, b_m, \ldots, A^{h_m-1}b_m.
\]

and

\[
h(A, B) = (h_1, \ldots, h_m) \iff \text{Hermite indices},
\]

where \(h_i := 0\) if the column \(b_i\) has not been selected.

1. \((A, B)\) is reachable \(\iff h_1 + \cdots + h_m = n.\)

2. If \((A(\theta), B(\theta))\) depends analytically on \(\theta\) the Hermite indices are generically constant.
Interlude: Hermite canonical form

If \((A, B)\) is reachable, the invertible transformation

\[ T = (b_1, \ldots, A^{h_1-1}b_1, \ldots, b_m, \ldots, A^{h_m-1}b_m) \]

yields the Hermite canonical form

\[ T^{-1}AT = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{mm} \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} b_1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & b_m \end{pmatrix} \]

\(T\) simultaneous block-triangularizes \((A, B)\)
Theorem (Dirr, S. (2018))

Let \( P \) compact. Then, \((A, B) \in C_{n,m}(P)\) is UER if

1. \((A(\theta), B(\theta))\) reachable \( \forall \theta \in P \).

2. Hermite indices of \((A(\theta), B(\theta))\) are constant.

3. The corresponding subpairs \((A_{ii}, b_i)\) are UER.
Remarks on the proof

- The Decomposition/Parallel Connection Theorem does not apply here.

- Proof is based on the fact that a pair \((A, B)\) of the form

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}
\]

is \textbf{UER} if the diagonal pairs are \textbf{UER}.

\textbf{Warning:} The converse is not true \((m = 1)\)
Pointwise reachability $\nRightarrow T(\theta)$ continuous.

E.g. for

$$A(\theta) = \begin{pmatrix} 0 & g(\theta) \\ 1 & 0 \end{pmatrix} \quad B(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $g(\theta) := \theta \sin(\frac{1}{\theta})$ and $g(0) = 0$ one has

$$(b_1 \quad A(\theta)b_1 \quad b_2 \quad A(\theta)b_2) = \begin{pmatrix} 0 & g(\theta) & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$\mapsto$ Hermite (Kronecker) indices might have infinitely many jumps

$\Rightarrow$ canonical form $?!?$
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Two possible approaches are based on:

(1) Complex approximation theory
   (Runge, Walsh, Weierstrass)

(2) Linear integral equations
   (Interpolation, projection method)
Two possible approaches are based on:

(1) Complex approximation theory
    (Runge, Walsh, Weierstrass)

(2) Linear integral equations
    (Interpolation, projection method)

Let \((A, B) \in C_{n,m}(\mathbb{P}), \ T > 0\) and \(\mathcal{R} : L^2([0, T], \mathbb{R}^m) \to C(\mathbb{P}, \mathbb{R}^n),\)

\[
\mathcal{R}u(\theta) = \int_0^T e^{A(\theta)(T-s)}B(\theta)u(s) \, ds
\]
Moment collocation

Let \((A, B) \in C_{n,m}(P)\) \(\text{UER}\), then choose distinct moments
\[
\theta_1, \ldots, \theta_N
\]
and solve the equation
\[
\begin{pmatrix}
\int_0^T e^{A(\theta_1)(T-s)} B(\theta_1) u(s) \, ds \\
\vdots \\
\int_0^T e^{A(\theta_N)(T-s)} B(\theta_N) u(s) \, ds
\end{pmatrix}
= \begin{pmatrix}
f(\theta_1) \\
\vdots \\
f(\theta_N)
\end{pmatrix}.
\]

Let \(u_N\) denote its minimum \(L^2\)-norm solution and
\[
\Delta_N := \sup_{\theta \in P} \left( \inf_{\theta_i} |\theta - \theta_i| \right).
\]

Then, by Nashed and Wahba (1974) one has
\[
\lim_{\Delta_N \to 0} \|u_N - \mathcal{R}^{\dagger} f\|_{L^2} = 0.
\]

**current task:** Development of consensus methods
Discrete-time systems

\[ x_{t+1}(\theta) = A(\theta)x_t(\theta) + b(\theta)u_t \]

For inputs \( u_0, \ldots, u_{T-1} \) the solution is given by

\[ \varphi(T, \theta, u) = \sum_{k=0}^{T-1} A(\theta)^k b(\theta) u_{T-1-k} = \left( \sum_{k=0}^{T-1} u_{T-1-k} A(\theta)^k \right) b(\theta) \]

For \( \varepsilon > 0 \) and \( f \in C(P, \mathbb{C}^n) \) one has

\[ \| \varphi(T, \theta, u) - f(\theta) \| = \| p(A(\theta)) b(\theta) - f(\theta) \| < \varepsilon \]

Sufficient cond’s allow for a reduction to problems of the form

\[ | p(x) - f(a^{-1}(x)) | < \varepsilon \]
Complex approximation

$$|p(x) - f(a^{-1}(x))| < \varepsilon$$

- is solved based on (complex) approximation theory
- Proofs yield constructive methods except where Mergelyan’s Theorem is used (Lavrentiev ??).
- $P = [a, b] \rightarrow$ Bernstein polynomials (Weierstrass)
- $P$ Jordan arc $\rightarrow$ Cauchy integral formula (Runge/Walsh)
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What about feedback?

Basic idea:

To overcome the strong and non robust on the Hermite indices and the spectra use controllers of the form

\[ u(t, x) := KC x + u(t), \]

where

- \( K \in \mathbb{C}^{m \times p} \) output feedback gain
- \( C \) output operator, e.g.

1. sampling operator: \( \theta_1, \ldots, \theta_N \in \mathbf{P} \)

\[ Cf = \begin{pmatrix} f(\theta_1) \\ \vdots \\ f(\theta_N) \end{pmatrix} \]

1. averaging operator: \( Cf = \int_{\mathbf{P}} C(\theta)f(\theta) \, d\theta. \)
Illustrative example - $\mathcal{C} = \text{Id}$

Example (controlled harmonic oscillator)

Let $P = [-1, 1]$ and $g \in C^1(P, \mathbb{R})$

$$\ddot{x} + \theta^2 x = g(\theta)u(t, x) = g(\theta)(kx(t, \theta) + u(t)), \quad k \in \mathbb{R}.$$ 

**Question:** Is the pair

$$A_k(\theta) := \begin{pmatrix} 0 & 1 \\ kg(\theta) - \theta^2 & 0 \end{pmatrix}, \quad b_g(\theta) := \begin{pmatrix} 0 \\ g(\theta) \end{pmatrix},$$

**UER** for some $k \in \mathbb{R}$?
Proposition (S. (2018))

Let $g \in C^1(P, \mathbb{R})$ be zero-free and strictly monotone. Then

(a) $\exists k^* = k^*(g) \in \mathbb{R} : (A_k, b_g)^{UER} \forall k > k^*$.

(b) if $f \in \text{Lip}(P, \mathbb{R})$ $\exists \tilde{k} \in \mathbb{R}$ s.t.

$\forall k > \tilde{k} \exists C \in \mathbb{R}$ polynomials $\left(p_n\right)_n$ $(n \geq 3)$ :

$$\|p_n(A_k)b_g - f\|_{\infty} \leq C\sqrt{\frac{\log n}{n}}.$$
Mixed open-loop and feedback controllers

Task: \((C = \text{Id})\)

Find conditions on \((A, B)\) s.t. \(\exists K \in \mathbb{C}^{m \times n}\) so that

\[(1)\] \(\theta_1 \neq \theta_2 \implies \sigma(A(\theta_1) + B(\theta_1)K) \cap \sigma(A(\theta_2) + B(\theta_2)K) = \emptyset\)

\[(2)\] \((A(\theta) + B(\theta)K, B(\theta))\) has constant Hermite indices.

and one of following condition holds

\[(3)\] The characteristic polynomials of \(A(\theta) + B(\theta)K\) are

\[z^n - a_{n-1}z^{n-1} - \cdots - a_1z - a_0(\theta)\]

\[(3')\] \(A(\theta) + B(\theta)K\) has simple eigenvalues for each \(\theta \in \mathbb{P}\)
Mixed open-loop and feedback controllers

Using Ackermann’s formula it follows immediately

**Proposition**

Suppose \((A, b) \in C_{n,1}(\mathbf{P})\) is pointwise reachable.

\[ \Rightarrow \exists \ k(\theta) \text{ continuous}: (A + bk, b) \text{ UER}. \]
In a nutshell

- Key idea of ensemble control
- Pointwise necessary and sufficient conditions for uniform ensemble reachability for parameter-dependent linear systems
- Methods from (complex) approximation theory, complex analysis, functional analysis and (plane) topology
- In some cases the inputs can be constructed
- There are many other problems which have not been addressed up to now, e.g. ...
Thank you for your attention!

U. Helmke and M. Schönlein.
Uniform ensemble controllability for one-parameter families of time-invariant linear systems.

M. Schönlein and U. Helmke.
Controllability of ensembles of linear dynamical systems.

G. Dirr and M. Schönlein
Uniform and $L^q$-ensemble reachability of parameter-dependent linear systems.

M. Schönlein.
Ensemble reachability of parametric harmonic oscillators via mixed open-loop and feedback control.
PAMM 2018.