Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations

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joint work with:
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Outline

- Generic rank-one perturbation of matrices $B = A + uv^T$, effects on Jordan form.
- J-Hamiltonian matrices: $J = -J^T$ invertible, $JA = -A^T J$.
- J-symplectic matrices: $J = -J^T$ invertible, $A^T JA = J$.
- H-orthogonal matrices: $H = H^T$ invertible, $A^T HA = H$.
- H-skew-symmetric matrices: $H = H^T$ invertible, $HA = -A^T H$.

Eigenvalues and Jordan canonical form

A an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_p$.

Corresponding to λ_j Jordan blocks of sizes
$n_{j,1} \geq n_{j,2} \geq \cdots \geq n_{j,m_j}$.

Recall, Jordan block: $\mathcal{J}_n(\lambda)$ is the $n \times n$ matrix

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

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Recall, Jordan normal form:

For every matrix A there is an invertible matrix S such that $S^{-1}AS$ is a diagonal direct sum of Jordan blocks:

$$S^{-1}AS = \mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \cdots \oplus \mathcal{J}_p,$$

where

$$\mathcal{J}_j = \mathcal{J}_{n_{j,1}}(\lambda_j) \oplus \cdots \oplus \mathcal{J}_{n_{j,m_j}}(\lambda_j).$$

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Corresponding to λ_j Jordan blocks of sizes $n_{j,1} \geq n_{j,2} \geq \cdots \geq n_{j,m_j}$.

Rank one perturbation $B = A + uv^T$ (real) or $B = A + uv^*$ (complex).

Only consider *generic* u and v.

Question: what happens to Jordan structure?

Rank one perturbation:

$$\dim \operatorname{Ker}(A - \lambda_j) + 1 \geq \dim \operatorname{Ker}(B - \lambda_j) \geq \dim \operatorname{Ker}(A - \lambda_j) - 1.$$

Example $A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0)$

Lots of things *can* happen.

Example:

$$A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Now $B \sim \mathcal{J}_1(0) \oplus \mathcal{J}_4(0) \oplus \mathcal{J}_3(0)$.

Example $A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0)$

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Example:

$$B = A + uv^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $B \sim \mathcal{J}_8(0)$ for $x \neq 0$.

Example $A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0)$

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Example:

$$B = A + uv^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $B \sim \mathcal{J}_7(0) \oplus \mathcal{J}_1(0)$ for $x \neq 0$.

Example $A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0)$

Lots of things *can* happen.

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$$B = A + uv^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $B \sim \mathcal{J}_6(0) \oplus \mathcal{J}_2(0)$ for $x \neq 0$.

Example $A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0)$

Lots of things *can* happen.

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$$B = A + uv^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \end{bmatrix}$$

Now $B \sim \mathcal{J}_5(0) \oplus \text{diag}\,(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3$ are the solutions to $\lambda^3 + x = 0$ for $x \neq 0$.

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Now $B \sim \mathcal{J}_3(0) \oplus \mathrm{diag}\,(\lambda_1, \ldots, \lambda_5)$, where $\lambda_1, \ldots, \lambda_5$ are the solutions to $\lambda^5 + x = 0$ for $x \neq 0$.

Complex general A

Is there something that happens "mostly"?

And what does "mostly" mean here?

Note: the term "generic" may mean different things to different members of the audience.

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And what does "mostly" mean here?

Note: the term "generic" may mean different things to different members of the audience.

In this lecture "generic vectors u and v" will mean the following: consider the pair of complex n-vectors (u, v) as a point in $\mathbb{R}^{4n}$.

We say that a property holds for generic vectors u and v if it holds for all pairs (u, v) with the possible exception of a finite union of zero sets of polynomials in the $4n$ real coordinates of u and v (and which are not the whole of $\mathbb{R}^{4n}$).

Generic property: example

$A = \mathcal{J}_n(0)$, and $B = A + uv^*$. So

$$B = \begin{bmatrix} u_1\bar{v}_1 & 1 + u_1\bar{v}_2 & u_1\bar{v}_3 & \cdots & u_1\bar{v}_n \\ \vdots & & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & & 1 + u_{n-1}\bar{v}_n \\ u_n\bar{v}_1 & \cdots & \cdots & \cdots & u_n\bar{v}_n \end{bmatrix}.$$

Then for generic u and v B will have n different non-zero eigenvalues.

The exceptional set is here the set of vectors u, v for which either $u_n = 0$ or $v_1 = 0$ (in which case zero is an eigenvalue).

Complex general A

A an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_p$.
Corresponding to λ_j Jordan blocks of sizes $n_{j,1} \geq n_{j,2} \geq \cdots \geq n_{j,m_j}$.
Rank one perturbation $B = A + uv^*$ (complex).
Known result (well-known???) for generic u and v the following hold:

- if $m_j > 1$ the matrix B has Jordan blocks corresponding to λ_j of sizes $n_{j,2} \geq \cdots \geq n_{j,m_j}$,
- the remaining $\sum_{j=1}^{p} n_{j,1}$ eigenvalues of B are simple.

Hörmander-Melin (1994), Savchenko (2003), Dopico-Moro (2003).

Different proof, based on ideas from systems theory: Mehl, Mehrmann, R., Rodman (2011)

Example $A = \mathcal{J}_5(0) \oplus \mathcal{J}_3(0)$

Lots of things *can* happen, but this happens *generically*.

$$B = A + uv^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $B \sim \mathcal{J}_3(0) \oplus \text{diag}\,(\lambda_1, \ldots, \lambda_5)$, where $\lambda_1, \ldots, \lambda_5$ are the solutions to $\lambda^5 + x = 0$ for $x \neq 0$.

Add structure in indefinite inner product: what happens?

First the J-Hamiltonian case. A is a *complex* matrix, $J = -J^T$ invertible, and $JA = -A^T J$.

Example: $J = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix}$ and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^T \end{bmatrix}$,
with $A_{12} = A_{12}^T$ and $A_{21} = A_{21}^T$.

Such matrices appear in many applications,
e.g., in optimal control theory.

Complex Hamiltonian case: what happens?

A is a *complex* matrix, $J = -J^T$ invertible, and $JA = -A^T J$.

There is a canonical form for the pair of matrices (A, J) under transformations $(A, J) \mapsto (S^{-1}AS, S^T JS)$.
In particular: Jordan blocks with zero eigenvalue and odd size come in coupled pairs. Corresponding block in J.

$$A = \mathcal{J}_{2m+1}(0) \oplus \mathcal{J}_{2m+1}(0)^T, \qquad J = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix}.$$

J-Hamiltonian rank one perturbation is of the form $B = A + uu^T J$.

When the largest Jordan block with zero eigenvalue is of odd size, what happens? It comes with "a sister".

Impossible: only one of those is lost, while the other remains.

Generic rank one perturbations of Hamiltonians

Surprise, surprise. One block *becomes one larger*.

So the canonical form will then contain a block of size $\mathcal{J}_{2m+2}(0)$. There are m additional non-zero simple eigenvalues of B. Thus, if

$$A = \mathcal{J}_{2m+1}(0) \oplus \mathcal{J}_{2m+1}(0)^T, \qquad J = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix},$$

then for generic vectors u

$$B = A + uu^T J \sim \mathcal{J}_{2m+2}(0) \oplus \mathcal{J}_1$$

where $\mathcal{J}_1$ has m simple non-zero eigenvalues.

This is quite different from the result of Hörmander-Melin/Savchenko/Dopico-Moro.

All other cases/eigenvalues for the Hamiltonian case do behave as in the unstructured case. Results rely heavily on canonical form.

J-symplectic matrices

$J = -J^T$, $A^T JA = J$, complex matrices. Structured rank one perturbations:

$$B = (I + uu^T J)A,$$

then $B^T JB = J$ and rank $B - A$ is one.

Again canonical form exists. In this case eigenvalues on the unit circle are special ones, and in particular, odd size Jordan blocks with eigenvalues ± 1 come in coupled pairs.

$$A = \begin{bmatrix} \mathcal{J}_{2m+1}(1) & 0 \\ 0 & (\mathcal{J}_{2m+1}(1)^T)^{-1} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix}.$$

Like in J-Hamiltonian case: generic rank one perturbation makes the block one larger.

$$B \sim \mathcal{J}_{2m+2}(1) \oplus \tilde{J}$$

where $\tilde{J}$ has $2m$ distinct eigenvalues, not equal to 1.

H-orthogonal matrices

$H = H^T$ and $A^T H A = H$, complex matrices.
Structured rank one perturbations:

$$B = (I - \frac{2}{u^T H u} u u^T H) A.$$

Again canonical form exists. In this case eigenvalues on the unit circle are special ones, and in particular, *even* size Jordan blocks with eigenvalues ± 1 come in coupled pairs.

$$A = \begin{bmatrix} \mathcal{J}_{2m}(1) & 0 \\ 0 & (\mathcal{J}_{2m}(1)^T)^{-1} \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & I_{2m} \\ I_{2m} & 0 \end{bmatrix}.$$

Like in previous cases: generic rank one perturbation makes the block one larger.

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Like in previous cases: generic rank one perturbation makes the block one larger.

$$B \sim \mathcal{J}_{2m+1}(1) \oplus \tilde{J}$$

where $\tilde{J}$ has $2m$ distinct eigenvalues, not equal to 1.
Extra surprise: if -1 is not already an eigenvalue of A, then -1 is always an eigenvalue of B!

H-positive real matrices

Recall: $H = H^T$ real and A is H-positive means

$$HA + A^T H \geq 0.$$

No canonical form, but a lot of information on the structure is available. Rank one H-positive real matrices do exist.

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$$HA + A^T H \geq 0.$$

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Special case H-skew-symmetric matrices

$$HA + A^T H = 0.$$

Canonical form exists. Rank one H-skew symmetric matrices do not exist. Yet, we may consider perturbation by a rank one H-positive real matrix.

H-positive real matrices. First result

Any rank k H-positive real matrix is of the following form: $UEU^T H$ for some $n \times k$ U and $k \times k$ E such that $E^T + E \geq 0$.

Theorem *Let A be H-positive real. Then, for any $n \times k$ matrix U and for any invertible $k \times k$ matrix E with $E + E^T > 0$, the H-positive real rank-k perturbation ,*

$$B(t) = A + tUE^T U^T H \qquad t > 0$$

has the property that all the pure imaginary eigenvalues of $B(t)$ are eigenvalues of A.

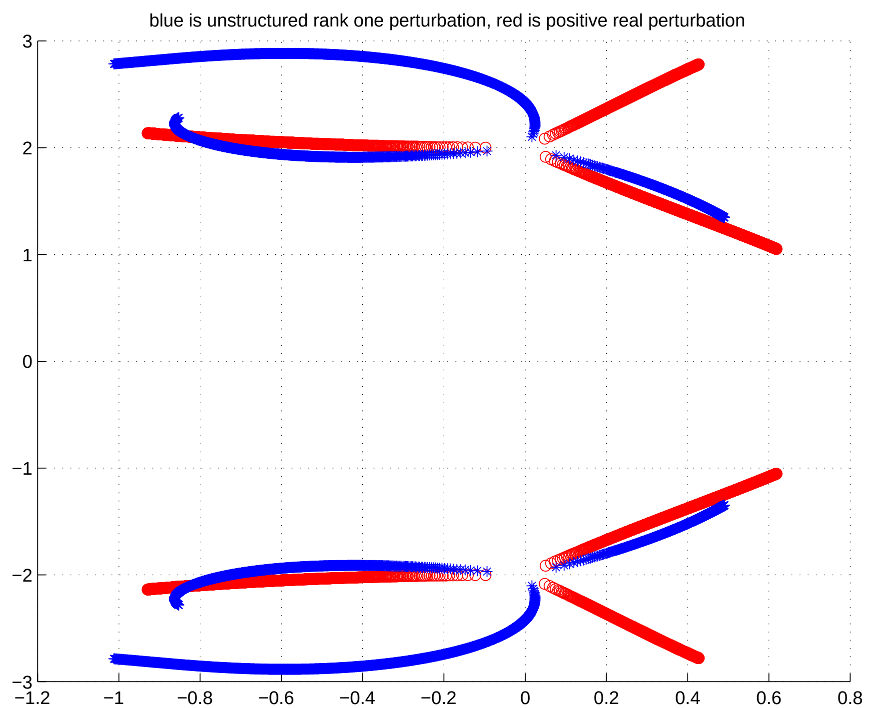
So: "new" eigenvalues are not on the imaginary axis.

H-positive real matrices. First result continued

So, for every $t > 0$ and any vector u all points of

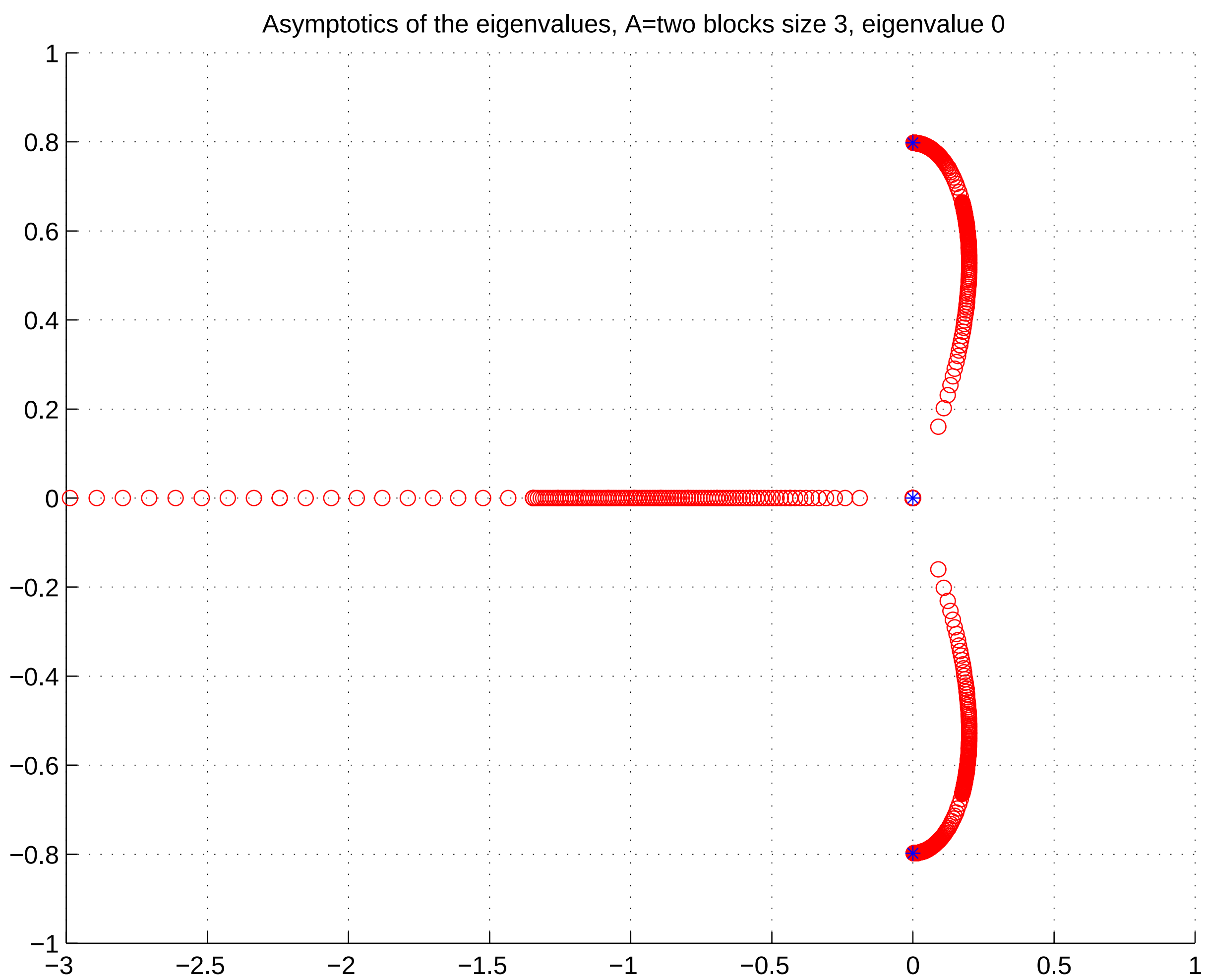
$$\sigma(A + tuu^T H) \setminus \sigma(A)$$

are not on the imaginary axis. Again quite contrary to general case.



H-positive real matrices. First result continued

However, for $t \to \infty$ they may approximate the imaginary axis.



H-skew symmetric case: Example (complex)

$$A = \mathcal{J}_2(0) \oplus \mathcal{J}_2(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then $HA + A^T H = 0$. Consider generic *complex* rank one perturbation

$$B = A + uu^* H.$$

Then B has a Jordan block of size two with eigenvalue zero (and two non-zero eigenvalues).

Same as in unstructured case.

H-skew symmetric case: Example (real)

$$A = \mathcal{J}_2(0) \oplus \mathcal{J}_2(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Consider generic *real* structured rank one perturbation

Same B, but now $u \in \mathbb{R}^4$ instead of complex,

$$B = A + uu^T H$$

For generic u the matrix $B = A + uu^T H$ has a Jordan block of size three with eigenvalue zero (and one non-zero eigenvalue).

H-skew symmetric case: canonical form

Special feature in the canonical form for the real case: even size blocks with zero eigenvalue come in pairs.

$$A = \mathcal{J}_{2m}(0) \oplus \mathcal{J}_{2m}(0), \qquad H = \begin{bmatrix} 0 & \Sigma_{2m} \\ \Sigma_{2m}^T & 0 \end{bmatrix}$$

where

$$\Sigma_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{k-2} & \cdots & 0 & 0 \\ (-1)^{k-1} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

H-skew symmetric case

Theorem *Let* $A \in \mathbb{R}^{n\times n}$ *and* H *be an invertible real symmetric matrix, such that* $HA + A^T H = 0$. *Consider a generic rank-one perturbation of the following form* $B = A + uu^T H, \quad u \in \mathbb{R}^n$.

1. *If the largest Jordan block corresponding to eigenvalue zero is of even size* $2m$ *it is part of a pair. Then* B *has a Jordan block of size* $2m + 1$ *with eigenvalue zero.*

2. *In all other cases the Jordan structure of* B *is as in the case of an unstructured rank one perturbation.*

Finally

Many more results: e.g.,
- small rank perturbations for other structured classes (Batzke and Mehl),
- behaviour of eigenvalues as function of a parameter (with Wojtylak)
- M-matrices (with Bierkens),
- quaternion case (with Mehl)
- operator case (already in Hörmander-Melin), very nice work of de Snoo-Winkler-Wojtylak, Trunk et al. (several papers)
- pencil case, nice paper by Trunk and Gernandt

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Thank you for your attention

References unstructured rank one perturbations

L. Hörmander, A. Melin, A remark on perturbations of compact operators, *Math. Scand* 75 (1994), 255–262.

F. Dopico and J. Moro, Low rank perturbation of Jordan structure, *SIAM J. Matrix Anal. Appl.* 25 (2003), 495–506.

S.V. Savchenko, On a generic change in the spectral properties under perturbation by an operator of rank one, [Russian] *Mat. Zametki* 74 (2003), 590–602; [English] *Math. Notes* 74 (2003), 557–568.

S.V. Savchenko, On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank, [Russian] *Funkcional. Anal. i Prilozhen.* 38 (2004), 85–88; [English] *Funct. Anal. Appl.* 38 (2004), 69–71.

References unstructured rank one perturbations

A.C.M. Ran and M. Wojtylak, Eigenvalues of rank one perturbations of unstructured matrices. *Linear Algebra and its Applications* 437 (2012) 589–600.

Chr. Mehl, V. Mehrmann, A.C.M. Ran and L. Rodman, Jordan forms of real and complex matrices under generic rank one perturbations. *Operator and Matrices* 7 (2013), 381–398.

Chr. Mehl and A.C.M. Ran, Low rank perturbations of quaternion matrices. *Electronic Linear Algebra* 32 (2017) 514-530.

J. Bierkens and A.C.M. Ran, A singular M-matrix perturbed by a nonnegative rank one matrix has positive principal minors; is it D-stable? *Linear Algebra and its Applications* 457 (2014), 191–208.

References structured case

Chr. Mehl, V. Mehrmann, A.C.M. Ran and L. Rodman, Eigenvalue perturbation theory of structured matrices under generic structured rank one perturbations. *Linear Algebra and its Applications*, 435 (2011), 687–716.

C. Mehl, V. Mehrmann, A.C.M. Ran, L. Rodman, Perturbation theory of selfadjoint matrices and sign characteristics under generic structured rank one perturbations. *Linear Algebra and its Applications* 436 (2012), 4027-4042.

J.H. Fourie, G.J. Groenewald, D.B. Janse van Rensburg, and A.C.M. Ran, Rank one perturbations of H-positive real matrices. *Linear Algebra and its Applications* 439 (2013), 653–674.

References structured case

C. Mehl, V. Mehrmann, A.C.M. Ran, L. Rodman, Eigenvalue perturbation theory of symplectic, orthogonal, and unitary matrices under generic structured rank one perturbations. *BIT Numerical Mathematics* 54 (2014), 219–255.

C. Mehl, V. Mehrmann, A.C.M. Ran, L. Rodman, Eigenvalue perturbation theory of structured real matrices and their sign characteristics under generic structured rank-one perturbations. *Linear and Multilinear Algebra* 64 (2016), no. 3, 527 – 556.

L. Batzke, C. Mehl, A.C.M. Ran, L. Rodman, Generic rank-k perturbations of structured matrices. Proceedings IWOTA 2014, OT 255 (2016), 27–48.