Zusammenhang und Fast-Konvexität in der linearen Systemtheorie

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Outline

Arcwise connectedness and convexity

Def.: A subset C of a real vector space V is called *convex* if for all $a, b \in C$

$$[a, b] := \{(1-t)a + tb\,;\; t \in [0,1]\} \subset C.$$

Examples: (a) Disk: $D = \{(x,y) \in \mathbb{R}^2\,;\; x^2 + y^2 = 1\}$ convex.
(b) Pointed disk: $D \setminus \{(0,0)\}$ not convex.
(c) Slotted disk: $D \setminus \{(x,0)\,;\; -1/2 \leq x \leq 1/2\}$ not convex.

Def.: A subset S of a topological space X is called *connected* if S cannot be represented as the union of two disjoint non-empty open subsets of S. S is *pathwise connected* if for all $a, b \in S$ there exists a (continuous) arc $\alpha : [0,1] \to S$ connecting $a = \alpha(0)$ with $b = \alpha(1)$.

Fact: S arcwise connected $\Rightarrow$ S connected.

Examples: (a) Pointed and slotted disks both pathwise connected.
(b) Pointed interval: $[-1,1] \setminus \{(0,0)\}$ not connected.
(c) Closed topologist's sine curve

$$S := \{(x, sin(1/x))\,;\; x \in (0,1]\} \cup \{(0,y)\,;\; y \in [-1,1]\}$$

is compact, connected, but not pathwise connected.

Local convexity and local connectedness

Def.: A subset S of a topological vector space V is called *locally convex* if every neighbourhood of a point x in S contains a convex neighbourhood U_c of x in S.

Remark: Every real normed linear space is locally convex. Open subsets and convex subsets of such a space are locally convex.

Def.: A subset S of a topological space X is called *locally (pathwise) connected* if every neighbourhood U of a point x in S contains a (pathwise) connected neighbourhood U' of x in S.

Fact: S locally convex $\Rightarrow$ S locally pathwise connected.

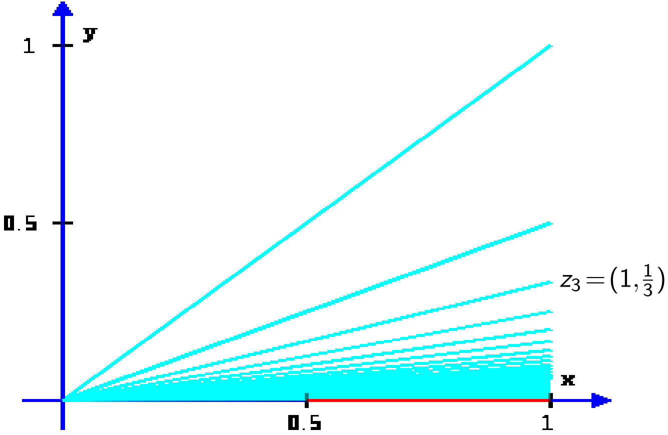
Examples: (a) Any open subset in $\mathbb{R}^N$ is loc. convex and loc. pathwise connected.

(b) The closed topologist's sine curve S is connected but not locally connected.

(c) next slide

(c) Let S_k be the segment in $\mathbb{R}^2$ between the origin and $z_k = (1, 1/k)$. The *broom space* and its closure are given by

$$S = \bigcup_{k \in \mathbb{N}} S_k \cup ([1/2, 1] \times \{0\}), \quad \overline{S} = \bigcup_{k \in \mathbb{N}} S_k \cup ([0, 1] \times \{0\})$$



The broom space S and its closure $\overline{S}$ are both connected. $\overline{S}$ is pathwise connected whereas S is not pathwise connected. Both, S and $\overline{S}$, are not locally connected.

Example: Space of Hurwitz matrices

$$\mathcal{H}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n};\ \sigma(A) \subset \mathbb{C}_-\}$$

Properties:

(i) $\mathcal{H}_n(\mathbb{K}) \subset \mathbb{K}^{n\times n}$ open subset.

(ii) Hence locally convex, hence locally connected.

(iii) convex? NO.

$$A_0 = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \in \mathcal{H}_2(\mathbb{K}) \text{ but } \frac{1}{2}A + \frac{1}{2}B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \notin \mathcal{H}_2(\mathbb{K})$$

(iv) connected? YES. $\mathcal{H}_n(\mathbb{K})$ is even contractible (to $-I_n$).

Define the affine homotopy $F : \mathcal{H}_n(\mathbb{K}) \times [0,1] \to \mathcal{H}_n(\mathbb{K})$ by

$$F(A,t) = (1-t)A - tI_n \in [A, -I_n] \subset \mathcal{H}_n(\mathbb{K}), \qquad A \in \mathcal{H}_n(\mathbb{K}),\ t \in [0,1]$$

Then $F(A,0) = A$ and $F(A,1) = -I_n$ for all $A \in \mathcal{H}_n(\mathbb{K})$.

Space $\mathrm{Rat}_n(\mathbb{R})$ of real rational functions of degree n

$$\mathrm{Rat}_n(\mathbb{R}) = \{c(s)/d(s) \in \mathbb{R}(s)\,;\ (c,d)=1, \deg c < \deg d = n, d \text{ monic}\}$$

Let $g(s) = c(s)/d(s)$ where $c(s), d(s) \in \mathbb{R}[s]$ are of the form

$$c(s) = \sum_{k=0}^{n-1} c_k s^k; \qquad d(s) = \sum_{k=0}^{n} d_k s^k, \quad d_n = 1$$

Identifying $g(s) = \frac{c(s)}{d(s)}$ with its *coefficient vector* $(d_0, \dots, d_{n-1}; c_0, \dots c_{n-1}) \in \mathbb{R}^{2n}$ $\mathrm{Rat}_n(\mathbb{R})$ may be considered as an open subset of $\mathbb{R}^{2n}$. $\mathrm{Rat}_n(\mathbb{R})$ is obtained by removing from $\mathbb{R}^{2n}$ an algebraic variety, viz. the closed subset of vectors $(d_0, \dots, d_{n-1}; c_0, \dots c_{n-1})$ for which the resultant $\mathcal{R}(d,c)$ of the associated polynomials $d(s), c(s)$ vanishes.

Properties: (i) As an open subset of $\mathbb{R}^{2n}$, the space $\mathrm{Rat}_n(\mathbb{R})$ is locally connected and locally convex.
(ii) It is not convex and it is not even connected.

Question: How many connected components?

Cauchy indices

Def.: Local Cauchy index of a rational function $g(s) \in \mathbb{R}(s)$ at a pole $s_0 \in \mathbb{R}$ given by $C_{s_0}(g) = [\lim_{s \downarrow s_0} g(s)/|g(s)| - \lim_{s \uparrow s_0} g(s)/|g(s)|]/2.$

For $-\infty \leq a < b \leq \infty$ the sum of the local Cauchy indices at the poles of g in (a, b) is the *Cauchy index* of g on the interval (a, b) (Notation: $CI_a^b(g)$).

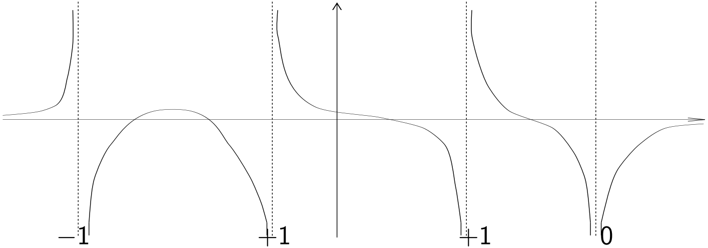


Figure : Local Cauchy indices of a function with $CI_{-\infty}^{\infty}(g) = 1$

Partial fraction decomposition of $g(s)$:

$$g(s) = \sum_{j=1}^{r} \frac{c_j(s)}{(s - \rho_j)^{m_j}} + \sum_{j=r+1}^{\ell} \frac{c_j(s)}{(s^2 - 2\rho_j s + \rho_j^2 + \omega_j^2)^{m_j}}, \quad \omega_j \neq 0\,.$$

Fact: Then $CI_{-\infty}^{\infty}(g) = \sum_{j \in \underline{r},\, m_j \text{ odd}} \operatorname{sign} c_j(\rho_j).$

Connected components of $\mathrm{Rat}_n(\mathbb{R})$

Theorem [Brockett (1976)]: $\mathrm{Rat}_n(\mathbb{R})$ has $n+1$ connected components given by

$$\mathrm{Rat}(n,\nu) = \{g \in \mathrm{Rat}_n(\mathbb{R});\ CI_{-\infty}^{\infty}(g) = n - 2\nu\}, \quad \nu = 0, \ldots, n.$$

Example: $g(s) \in \mathrm{Rat}_2(\mathbb{R})$ has partial fraction decomposition of one of the forms

$(i)\, g(s) = \frac{\gamma_1}{s-\rho_1} + \frac{\gamma_2}{s-\rho_2},\ \rho_i \in \mathbb{R}, \rho_1 > \rho_2$ or $(ii)\, g(s) = \frac{c_1 s + c_0}{(s-\rho)^2 + \omega^2},\ \rho \in \mathbb{R}, \omega > 0.$

$CI_{-\infty}^{\infty} g = 0$ iff $g(s)$ of form (ii) or $\gamma_1\gamma_2 < 0$. Then $g(s)$ connectable to $g_0(s) = \frac{1}{s^2}$.

$CI_{-\infty}^{\infty} g = \pm 2$ iff $g(s)$ of form (i) and $\operatorname{sign}\gamma_i = 1$, resp. $\operatorname{sign}\gamma_i = -1$.

Then $g(s)$ connectable to

$$g_2 := \frac{1}{s+1} + \frac{1}{s+2}, \qquad \text{resp.} \qquad g_{-2} := \frac{-1}{s+1} + \frac{-1}{s+2}.$$

Example: Controllable single input systems I

Notation:

$\mathbf{L}_{n,m}(\mathbb{K}) = \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}, \quad R(A,B) := [B, AB, \ldots, A^{n-1}B]$ reachability matrix

$\mathbf{L}^c_{n,m}(\mathbb{K}) = \{(A,B) \in \mathbf{L}_{n,m}(\mathbb{K})\,;\ \operatorname{rank} R(A,B) = n\}$ set of controllable systems

Properties of $\mathbf{L}^c_{n,1}(\mathbb{K}) = \{(A,b) \in \mathbb{K}^{n\times n} \times \mathbb{K}^n\,;\ \det R(A,b) \neq 0\}$:

(i) $\mathbf{L}^c_{n,1}(\mathbb{K})$ open subset of $\mathbf{L}_{n,1}(\mathbb{K}) = \mathbb{K}^{n\times n} \times \mathbb{K}^n$.

(ii) Hence locally convex and thus locally connected.

(iii) Convex? NO.

(iv) Connected? NO if $\mathbb{K} = \mathbb{R}$ and YES if $\mathbb{K} = \mathbb{C}$. Why NO in the real case?

If $(A^0, b^0), (A^1, b^1) \in \mathbf{L}^c_{n,1}(\mathbb{R})$ with $\det R(A^0, b^0) > 0$ and $\det R(A^1, b^1) < 0$, then any connecting path $\alpha : t \mapsto (A(t), b(t))$ must pass through some $(A(t_0), b(t_0))$ with $\det R(A(t_0), b(t_0)) = 0$, i.e. $(A(t_0), b(t_0)) \notin \mathbf{L}^c_{n,1}(\mathbb{R})$.

Question: How many connected components?

Controllable single input systems II

Def.: $(A_1, B_1), (A_2, B_2) \in \mathbf{L}_{n,m}(\mathbb{K})$ similar $((A_1, B_1) \overset{\sigma}{\sim} (A_2, B_2))$ if there exists $T \in \mathbf{Gl}_n(\mathbb{K}) = \{T \in \mathbb{K}^{n\times n};\ \det T \neq 0\}$ such that $(A_2, B_2) = (TA_1T^{-1}, TB_1)$.

Thm.: For each $(A, b) \in \mathbf{L}^c_{n,1}(\mathbb{K})$ there is a unique pair $(\tilde{A}, \tilde{b}) \overset{\sigma}{\sim} (A, b)$ of column reachability canonical form:

$$\tilde{A} = C_a := \begin{bmatrix} 0 & 0 & \cdots & \cdots & -a_0 \\ 1 & 0 & \cdots & \cdots & -a_1 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad \tilde{b} = e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

C_a companion matrix of $\chi_A(s) = p(s, a)$, $a =$ coefficient vector of $\chi_A(s)$.

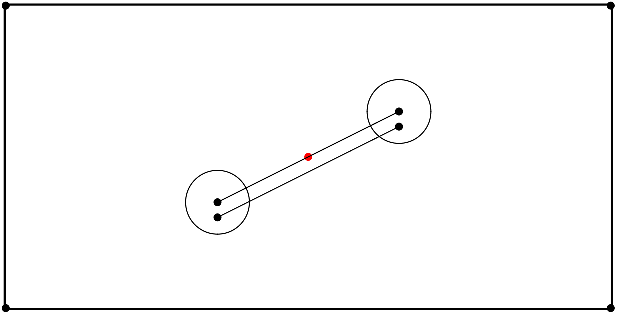
Remark: A simple calculation yields:
$f : (A, b) \mapsto (R(A, b), a)$ ($a =$ coefficient vector of χ_A) is a homeomorphism from $\mathbf{L}^c_{n,1}(\mathbb{K})$ onto $\mathbf{Gl}_n(\mathbb{K}) \times \mathbb{K}^n$ with inverse $(T, a) \mapsto (TC_aT^{-1}, Te^1)$.

Result: $\mathbf{L}^c_{n,1}(\mathbb{K}) \cong \mathbf{Gl}_n(\mathbb{K}) \times \mathbb{K}^n$.

As $\mathbf{Gl}_n(\mathbb{R})$ has two connected components and $\mathbf{Gl}_n(\mathbb{C})$ is connected,
$\mathbf{L}^c_{n,1}(\mathbb{R})$ has two connected components and $\mathbf{L}^c_{n,1}(\mathbb{C})$ is connected.

Densely convex sets

Def.: A subset E of a normed real vector space is called *densely convex* if, for every neighbourhood U_a of $a \in E$ and every neighbourhood U_b of $b \in E$ there exist $x \in U_a$, $y \in U_b$ such that $[x, y] \subset E$.



Punctured Rectangle: Not convex, but densely convex

E convex: Any two points $a, b \in E$ are connected by a straight line in E
E densely convex: First perturb, then connect

Some examples and small results

Ex.: (i) The punctured open unit disk $E = \{D \setminus \{(0,0)\}$ is densely convex.
(ii) The slotted open unit disk $E = D \setminus [-1/2, 1/2]$ is not densely convex.

Ex.: If E_0 is the set of vertices of $[-1,1]^N$ and $E = (-1,1)^N \cup E_0$, then E is densely convex, but not convex, if $N \geq 2$.

Prop.: (i) If $E \subset \mathbb{R}^N$ is densely convex, then the closure $\overline{E}$ in $\mathbb{R}^N$ is convex. Every closed densely convex subset of $\mathbb{R}^N$ is convex.

(ii) If $E \subset \mathbb{R}^N$ is open and densely convex, then E is arcwise connected.

(iii) If $E \subset \mathbb{R}^N$ is densely convex and $E \subset Y \subset \overline{E}$, then Y is densely convex.

(iv) The continuous affine image of a densely convex set is densely convex.

Caution: Intersections of densely convex sets are not necessarily densely convex!

Overview: Some constrained convexity concepts

Def.: Let $\alpha \in (0,1)$. A subset E of a real vector space is called *α-convex* if $(1-\alpha)x + \alpha y \in E$ for all $x, y \in E$.
$\alpha = 1/2$: "midpoint convexity": *J. von Neumann* (1935), "midpoint convex functions" Bernstein and Doetsch (1915)

Def.: Let $\emptyset \neq \Delta \subset (0,1)$. A subset E of a real vector space is called *Δ- convex* if $(1-\alpha)x + \alpha y \in E$ for all $\alpha \in \Delta$ and $x, y \in E$. Every E which is Δ- convex for some Δ is called *quasiconvex*. *Green and Gustin* (1950)

Def.: A subset E of a normed real vector space is called *closely convex* if $[x, y] \subset \operatorname{cl} E$ for all $x, y \in E$. *Blaga and Kolumban* (1994)

Fact: convex $\Rightarrow$ α-convex $\Rightarrow$ quasiconvex $\Rightarrow$ closely convex.

Def.: A subset E of a normed real vector space is called *almost convex* if for any finite set $\{a_1, \ldots, a_N\} \subset E$ and every $\varepsilon > 0$ there exists $\{z_1, \ldots, z_N\} \subset E$ such that $\|a_i - z_i\| < \varepsilon$ for $i \in \underline{N}$ and $\operatorname{conv}\{z_1, \ldots, z_N\} \subset E$. *Himmelberg* (1972)

almost convex implies densely convex

Overview: Applications of constrained convexity

1. Study of additive functions: $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) + f(y) = f(x + y)$

Remark: Every additive function is midpoint convex. Every convex function on $\mathbb{R}$ is continuous, but midpoint convex functions f may be discontinuous.
Cauchy (1821): Solution for continuous functions: $f(x) = f(1)x$.
Hamel (1905): General solution via a basis of $\mathbb{R}$ over the field $\mathbb{Q}$ (Hamel basis)
Frechet (1913), Measurable additive functions $f : \mathbb{R} \to \mathbb{R}$ are continuous.
Newly proved by *Sierpinski* (1920), *Banach* (1920),*Kac* (1936/37),...

2. Fixed point theory *Himmelberg* (1972), *Idzik* (1988), *Park* (1995)

3. Monotone operators *Minty* (1970), *Rockafellar* (1970), *Bauschke* (2007), *Phelps and Simons* (1998), *Simons* (2008)

4. Optimization *Bot,Grad and Wanka* (2006-2008), *Frenk and Kassay* (2004), (2007)

5. Mathematical economics, multiobjective optimization and game theory
R. John (2005), *Martinez-Legaz* (2005)

Dense convexity of $\mathbf{L}^c_{n,m}(\mathbb{R})$ with $m \geq 2$ input channels: strategy

Notation: $(A, B) \in \mathbf{L}_{n,m}(\mathbb{R})$ where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $B = [b^1, \ldots, b^m] \in \mathbb{R}^{n\times m}$.
$R(A,B) = [B, AB, \ldots, A^{n-1}B]$, $\mathbf{L}^c_{n,m}(\mathbb{R}) = \{(A,B) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}; \operatorname{rank} R(A,B) = n\}$.

Straight Lines in $\mathbf{L}_{n,m}(\mathbb{R})$: $(A, B) + t(X, Y) = (A + tX, B + tY)$, $t \in \mathbb{R}$.
$(X, Y) \in \mathbf{L}_{n,m}(\mathbb{R})$ where $X \in \mathbb{R}^{n\times n}$, $Y = [y^1, \ldots, y^m] \in \mathbb{R}^{n\times m}$.

Aim: Prove that $\mathbf{L}^c_{n,m}(\mathbb{R})$ is densely convex if $m \geq 2$.

Strategy: 1. Study lines $(A, B) + \mathbb{R}(X, Y)$ instead of segments $[(A_1,B_1), (A_2,B_2)]$.
2. Examine for which starting points (A, B) and which directions (X, Y) the straight lines $(A, B) + \mathbb{R}(X, Y)$ remain in $\mathbf{L}^c_{n,m}(\mathbb{R})$.
3. Start with system $(A, B) \in \mathbf{L}^c_{n,m}(\mathbb{R})$ which is controllable by the first two channels separately (generic condition).
4. Consider directions $(X, Y) \in \mathbf{L}^c_{n,m}(\mathbb{R})$ with the same property.
5. Find conditions for the data A, B, X, b^1, y^1 such that $A+tX$ is cyclic for all $t \in \mathbb{R}$ and $(A+tX, b^1+ty^1)$ looses controllability only at a finite number of $t \in \mathbb{R}$.
6. After constructing the data A, b^1, X, y^1, choose $b^2, y^2 \in \mathbb{R}^n$ such that $(A+tX, b^2+ty^2)$ is controllable where $(A+tX, b^1+ty^1)$ is not.
7. Use the resultant to show that these constructions work generically.

Straight lines in $\mathbf{L}_{n,1}(\mathbb{R})$

For any $(A, b), (X, y) \in \mathbf{L}_{n,1}(\mathbb{R})$ consider the line $\{(A + tX, b + ty)\,;\ t \in \mathbb{R}\}$.

$$(A + tX, b + ty) \text{ is controllable } \Leftrightarrow p(t; A, b, X, y) = \det R(A + tX, b + ty) \neq 0.$$

$$p(t; A, b, X, y) = \det[b + ty, (A + tX)(b + ty), \dots, (A + tX)^{n-1}(b + ty)].$$

The j-th column of $R(A + tX, b + ty)$ is a vector polynomial of degree $\leq j$ in t.
Hence $p(t; A, b, X, y)$ is of degree $\leq 1 + \cdots + n = \binom{n+1}{2}$ in t.

Lemma: $p(t; A, b, X, y) = \det R(A + tX, b + ty)$ is a polynomial in t of degree $\deg_t p(t; A, b, X, y) \leq N := \binom{n+1}{2}$ with coefficients which are polynomials in the entries of (A, b, X, y):

$$p(t; A, b, X, y) = c_N(A, b, X, y)t^N + \cdots + c_1(A, b, X, y)t + c_0(A, b, X, y)$$

The coefficient of t^N in $p(t; A, b, X, y)$ is $\det R(X, y)$ and so

$$\deg_t p(t; A, b, X, y) = N \quad \Leftrightarrow \quad \det R(X, y) \neq 0 \quad \Leftrightarrow \quad (X, y) \in L^c_{n,1}(\mathbb{R}).$$

$T(A, b, X, y) := \{t \in \mathbb{R}\,;\ \det R(A + tX, b + ty) = 0\}$ is finite with at most N elements. For every $t \in \mathbb{R}$ with $|t| \geq t(A, b, X, y) := max\{|t|\,;\ t \in T(A, b, X, y)\}$ we have $(A + tX, b + ty) \in L^c_{n,1}(\mathbb{R})$.

The resultant approach I

For $(A, B, X, Y) \in \mathbf{L}_{n,m}(\mathbb{R}) \times \mathbf{L}_{n,m}(\mathbb{R})$, $m \geq 2$, consider two polynomials in t

$$p_1(t;A,B,X,Y) = \det R(A+tX, b^1+ty^1),\ \ p_2(t;A,B,X,Y) = \det R(A+tX, b^2+ty^2).$$

By lemma both polynomials are of degree $\leq N := \binom{n+1}{2}$ in t:

$$p_1(t; A, B, X, Y) = a_N(A, b^1, X, y^1)t^N + \cdots + a_1(A, b^1, X, y^1)t + a_0(A, b^1, X, y^1)$$
$$p_2(t; A, B, X, Y) = b_N(A, b^2, X, y^2)t^N + \cdots + b_1(A, b^2, X, y^2)t + b_0(A, b^1, X, y^2)$$

If $(X, y^1), (X, y^2) \in L^c_{n,1}(\mathbb{R})$, then $\deg_t p_1 = \deg_t p_2 = N$.
If, for a given $t \in \mathbb{R}$, $p_1(t; A, B, X, Y) \neq 0$ or $p_2(t; A, B, X, Y) \neq 0$, then $(A + tX, b^1 + ty^1)$ or $(A + tX, b^2 + ty^2)$ are controllable.

If $p_1(t; A, B, X, Y)$ and $p_2(t; A, B, X, Y)$ have no joint zeros $t \in \mathbb{R}$, then

$$\{(A + tX, B + tY)\,;\ t \in \mathbb{R}\} \subset \mathbf{L}^c_{n,m}(\mathbb{R}).$$

In particular, we need that $A + tX$ is cyclic for all $t \in \mathbb{R}$.

Question: Does this hold generically for all $(A, B), (X, Y) \in L^c_{n,1}(\mathbb{R})$?

The resultant approach II

$p_1(t; A, B, X, Y)$ and $p_2(t; A, B, X, Y)$, as polynomials in t, have no joint factor in $\mathbb{R}[t]$ if and only if their resultant does not vanish.

$$\mathcal{R}(A, B, X, Y) = \mathcal{R}_t(p_1(t; A, B, X, Y), p_2(t; A, B, X, Y))$$

$$= \det \begin{bmatrix} a_N(A, b^1, X, y^1) & \cdots & \cdots & a_0(A, b^1, X, y^1) & & \\ & \ddots & & & \ddots & \\ & & a_N(A, b^1, X, y^1) & \cdots & \cdots & a_0(A, b^1, X, y^1) \\ b_N(A, b^2, X, y^2) & \cdots & \cdots & b_0(A, b^2, X, y^2) & & \\ & \ddots & & & \ddots & \\ & & b_N(A, b^2, X, y^2) & \cdots & \cdots & b_0(A, b^2, X, y^2) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} N \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} N \text{ rows} \end{matrix}$$

Problem 1: Can we prove that the resultant $\mathcal{R}(A, B, X, Y)$ does not vanish identically on $\mathbf{L}_{n,m}(\mathbb{R}) \times \mathbf{L}_{n,m}(\mathbb{R})$?

Problem 2: If $p_1(t; A, B, X, Y)$ and $p_2(t; A, B, X, Y)$ have no joint factors in $\mathbb{R}[t]$, then they do not have joint roots in $\mathbb{R}$, but the converse is not true!

Results

Thm.: There exists a proper real algebraic variety $\mathcal{V}$ in $\mathbf{L}_{n,m}(\mathbb{R}) \times \mathbf{L}_{n,m}(\mathbb{R})$ such that all $(A, B, X, Y) \notin \mathcal{V}$ have the properties:
(i) $\deg_t p_1(t; A, B, X, Y) = \deg_t p_2(t; A, B, X, Y) = N = \binom{n+1}{2}$.

(ii) $A + tX$ is cyclic for all $t \in \mathbb{R}$.

(iii) $p_1(t; A, B, X, Y)$ and $p_2(t; A, B, X, Y)$ have no joint factors over $\mathbb{R}$.

In particular, the line $\{(A + tX, B + tY)\,;\ t \in \mathbb{R}\}$ is contained in $\mathbf{L}^c_{n,m}(\mathbb{R})$ for all $(A, B, X, Y) \in \mathbf{L}_{n,m}(\mathbb{R}) \times \mathbf{L}_{n,m}(\mathbb{R}) \setminus \mathcal{V}$.

Cor.: $\mathbf{L}^c_{n,m}(\mathbb{R})$ is generically convex in the sense that there exists a proper real algebraic variety $\mathcal{W}$ in $\mathbf{L}_{n,m}(\mathbb{R}) \times \mathbf{L}_{n,m}(\mathbb{R})$ such that for all $(A_1, B_1, A_2, B_2) \in \mathbf{L}^c_{n,m}(\mathbb{R}) \times \mathbf{L}^c_{n,m}(\mathbb{R}) \setminus \mathcal{W}$ the segment $[(A_1, B_1), (A_2, B_2)]$ is contained in $\mathbf{L}^c_{n,m}(\mathbb{R})$.

Cor.: $\mathbf{L}^c_{n,m}(\mathbb{R})$ is densely convex and hence, as an open subset of $\mathbf{L}_{n,m}(\mathbb{R})$, is connected.

Cor.: The set $\mathfrak{C}_n(\mathbb{R})$ of cyclic matrices in $\mathbb{R}^{n\times n}$ is densely convex.

Canonical forms

Def.: Let $X \neq \emptyset$ be a set provided with an equivalence relation $\sim$. A map $\Gamma : X \to X$ is called a *canonical form* for $\sim$, if it satisfies

(i) $\quad \Gamma(x) \sim x \quad \text{for all} \quad x \in X;$

(ii) $\quad \forall x, x' \in X : \quad x \sim x' \iff \Gamma(x) = \Gamma(x').$

Example: For $m = 1$ we constructed the column reachability canonical form $\Gamma : (A, b) \mapsto (\tilde{A}, \tilde{b}) = (C_a, e^1)$ on $\mathbf{L}^c_{n,1}(\mathbb{R})$. This canonical form is continuous.

Problem of Kalman and Hazewinkel: Does there exist a continuous canonical form on $X = \mathbf{L}^c_{n,m}(\mathbb{R})$ for the similarity relation $\overset{\sigma}{\sim}$, if $m \geq 2$?

Lemma: If $\Gamma : \mathbf{L}^c_{n,m}(\mathbb{R}) \to \mathbf{L}^c_{n,m}(\mathbb{R})$ is a continuous canonical form for $\overset{\sigma}{\sim}$, then as in the single-input case:

$$\mathbf{Gl}_n(\mathbb{R}) \times \Gamma(\mathbf{L}^c_{n,m}(\mathbb{R})) \cong \mathbf{L}^c_{n,m}(\mathbb{R}).$$

Thm. [Hazewinkel and Kalman]: If $m \geq 2$, there does not exist a continuous canonical form for the similarity relation $\overset{\sigma}{\sim}$ on $\mathbf{L}^c_{n,m}(\mathbb{R})$.

Proof: $\mathbf{L}^c_{n,m}(\mathbb{R})$ is open and densely convex, hence pathwise connected. Therefore $\mathbf{L}^c_{n,m}(\mathbb{R})$ cannot be homeomorphic to $\mathbf{Gl}_n(\mathbb{R}) \times \Gamma(\mathbf{L}^c_{n,m}(\mathbb{R}))$ (not connected).

Summary

We gave some selected examples of connectedness results in linear systems theory.

- The connected components of the space $\mathrm{Rat}_n(\mathbb{R})$ of real rational functions of fixed degree n have been determined (Brockett's Theorem).
- The connected components of the space $\mathbf{L}^c_{n,1}(\mathbb{R})$ of controllable single-input systems (A, b) have been determined.

New concepts and results:

- New convexity properties have been proposed(densely convex and generically convex) and related to constrained convexity notions from the literature.
- The set $\mathbf{L}^c_{n,m}(\mathbb{R})$ of canonicaly input pairs is generically convex.
  As an easy consequence of this result the Theorem of Kalman and Hazewinkel about the non-existence of a continuous canonical form for the similarity action on $\mathbf{L}^c_{n,m}(\mathbb{R})$, if $m \geq 2$, is obtained.
- The set $\mathfrak{C}_n(\mathbb{R})$ of cyclic matrices in $\mathbb{R}^{n\times n}$ is densely convex.

References

D. Hinrichsen and A.J. Pritchard.
Mathematical Systems Theory I
Springer-Verlag, Berlin, 2005.

R. T. Rockafellar.
Convex Analysis.
Princeton University Press, Princeton, New Jersey, 1970.

Bochnak, J. and Coste, M. and Roy, M.-F.,
Géométrie algébrique réelle,
Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge Band 12, Springer-Verlag (1987).

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