Model predictive control for differential-algebraic equations

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in collaboration with

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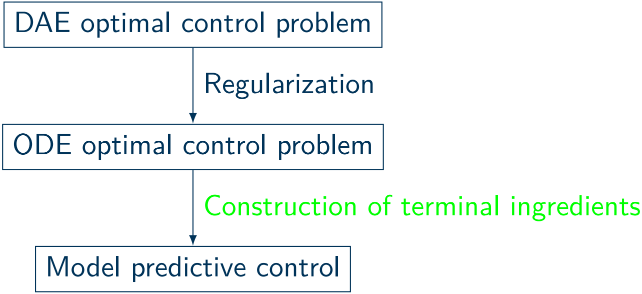
13. Elgersburg Workshop

Outline


DAE optimal control problem
Regularization
ODE optimal control problem
Construction of terminal ingredients
Model predictive control

Problem formulation

Ordinary Differential Equation (ODE)

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0,$$

with **system state** $x(t) = x(t; x^0, u) \in \mathbb{R}^d$ and **control** $u(t) \in \mathbb{R}^m$

Optimal control on an **infinite time horizon**:

Minimize the quadratic cost functional

$$J_\infty(x^0, u) := \int_{t=0}^{\infty} x(t)^\top Q x(t) + u(t)^\top R u(t) \,\mathrm{d}t$$

subject to the constraint $Fx(t) + Gu(t) \leq \mathbb{1}$.

Model predictive control

Constrained infinite horizon optimal control problems are in general computationally intractable

Remedy: Model Predictive Control (MPC)

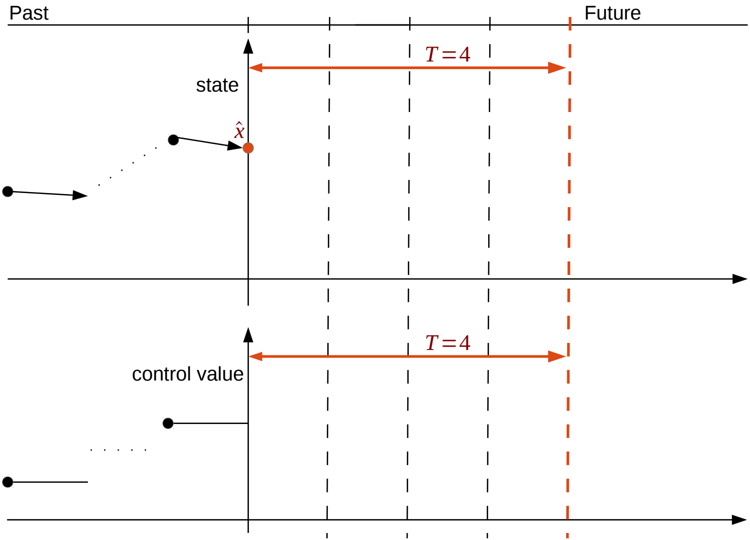
Idea: replace the original problem

$$\text{minimize} \;\; J_\infty(x^0, u) = \int_{t=0}^{\infty} x(t)^\top Q x(t) + u(t)^\top R u(t)\, \mathrm{d}t$$

by iteratively solving over a **finite** time horizon (online)

$$\text{minimize} \;\; J_T(x^0, u) = \int_{t=0}^{T} x(t)^\top Q x(t) + u(t)^\top R u(t)\, \mathrm{d}t.$$

MPC from a trajectory point of view



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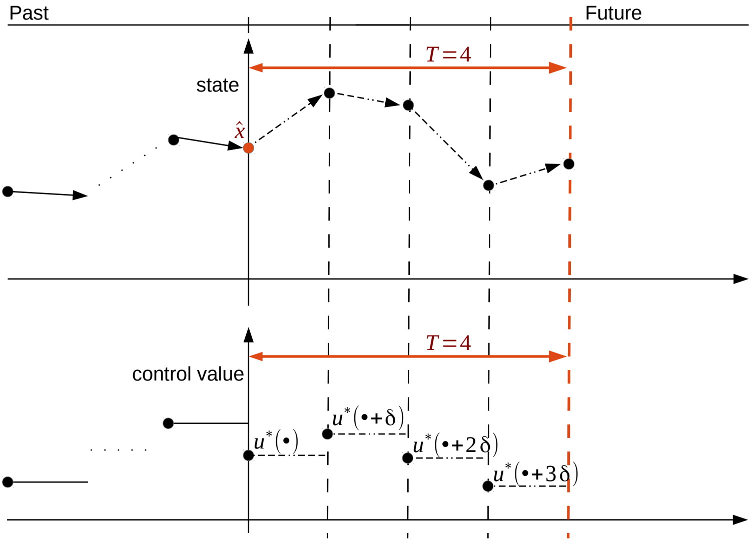

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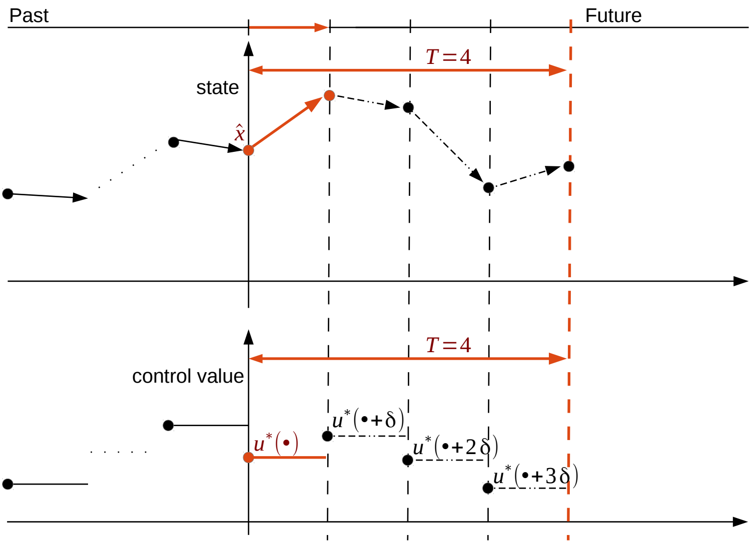

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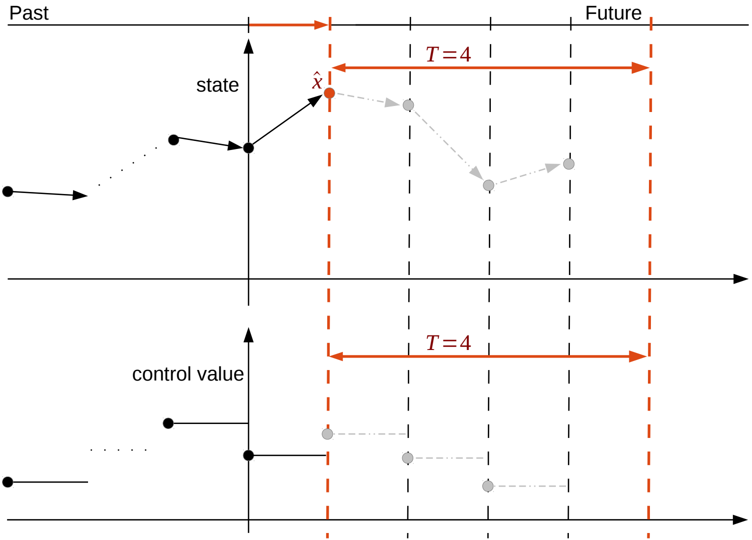

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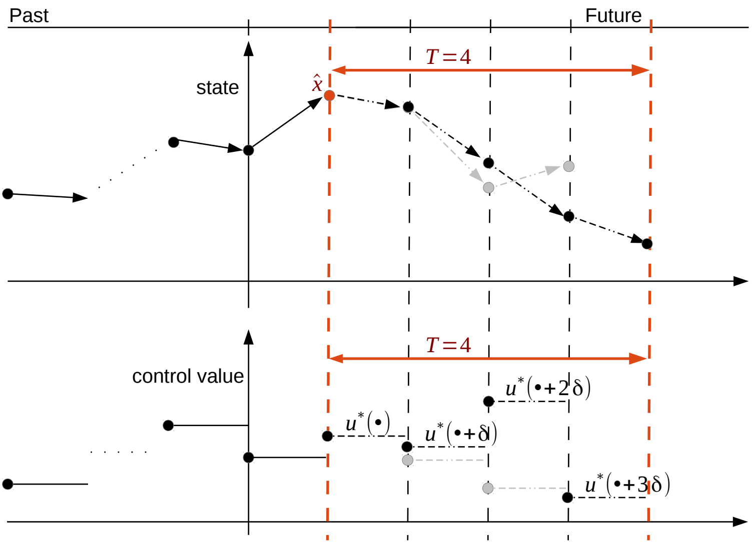

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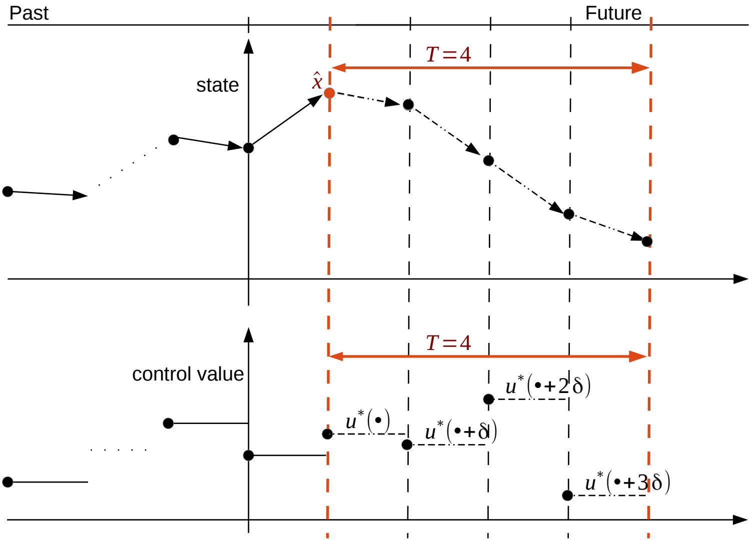

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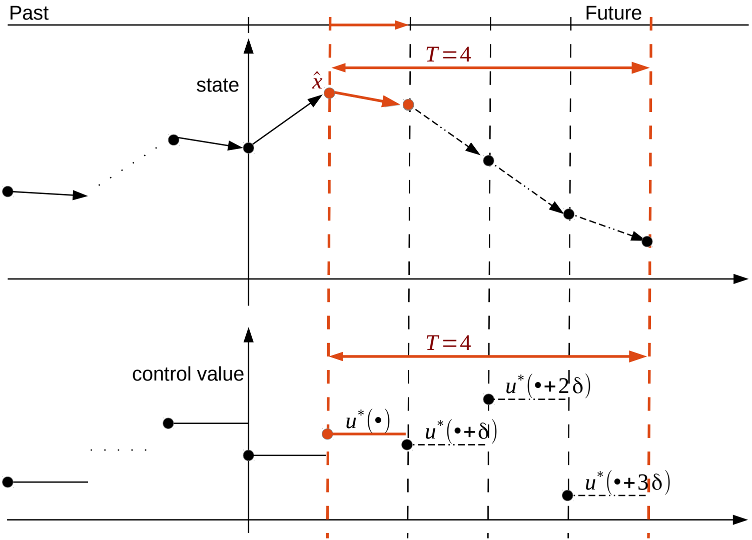

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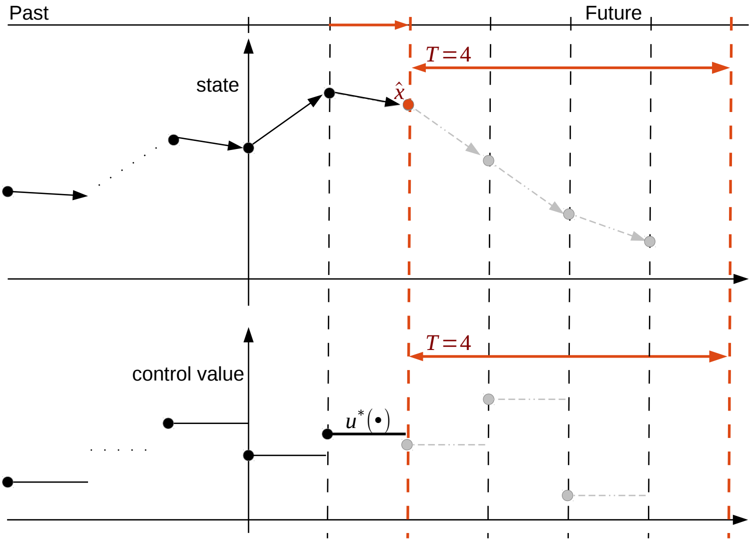

Model predictive control algorithm

Algorithm:

1. Measure current state $\hat{x} := x(t)$.
2. Minimize $\int_0^T x(s)^\top Q x(s) + u(s)^\top R u(s)\,\mathrm{d}s$
   subject to
   - $\dot{x}(s) = Ax(s; \hat{x}, u) + Bu(s) \quad \text{with} \quad x(0; \hat{x}, u) = \hat{x}$
   - $Fx(s; \hat{x}, u) + Gu(s) \leq \mathbb{1}$ for all $s \in [0, T]$.

   $\rightsquigarrow$ optimal control $u^\star$

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Question: What about asymptotic stability of the origin?

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Question: What about asymptotic stability of the origin?
Decrease condition: $\forall\ \tilde{x} \in \mathbb{X}_\mathrm{f}\ \exists$ admissible $\tilde{u}$ such that

$$V_\mathrm{f}(x(\delta;\tilde{x},u^\star)) \leq V_\mathrm{f}(\tilde{x}) - \int_0^\delta x(s)^\top Q x(s) + u^\star(s)^\top R u^\star(s)\,\mathrm{d}s$$

for (local) Lyapunov-function V_f.

Model predictive control algorithm

Algorithm:

1. Measure current state $\hat{x} := x(t)$.
2. Minimize $\int_0^T x(s)^\top Qx(s) + u(s)^\top Ru(s)\,\mathrm{d}s + V_\mathrm{f}(x(T;\hat{x},u))$ subject to $x(T;\hat{x},u) \in \mathbb{X}_\mathrm{f}$ and
   - $\dot{x}(s) = Ax(s;\hat{x},u) + Bu(s)$ with $x(0;\hat{x},u) = \hat{x}$
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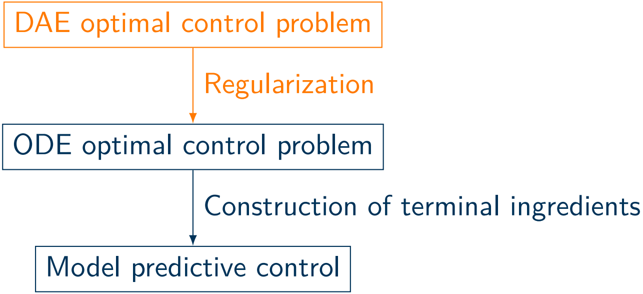
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Linear time-invariant DAEs

We consider the differential-algebraic system $[E, A, B]$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t)$$

with $E, A \in \mathbb{R}^{\ell \times n}$, $B \in \mathbb{R}^{\ell \times m}$.

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Examples

- ODE ($\ell = n$ and $E \in GL_n(\mathbb{C})$): $\dot{x}(t) = E^{-1}Ax(t) + E^{-1}Bu(t)$

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Examples

- $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u(t) \implies x(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$

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Behaviour

$$\mathfrak{B}_{[E,A,B]} = \{ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{L}^1_{\mathrm{loc}} \mid Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}} \text{ fulfils DAE a.e.} \}$$

DAE-constrained OCP

For optimization horizon $T \in (0, \infty]$, we consider the OCP

$$\text{Minimize} \quad \int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top S \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t$$

subject to

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t), \quad (Ex)(0) = Ex^0$$
$$Fx(t) + Gu(t) \leq \mathbb{1}$$

with $E, A \in \mathbb{R}^{\ell \times n}$, $B \in \mathbb{R}^{\ell \times m}$, $x^0 \in \mathbb{R}^n$, $S = S^\top \in \mathbb{R}^{(m+n)\times(m+n)}$, $F \in \mathbb{R}^{p \times n}$, and $G \in \mathbb{R}^{p \times m}$

Optimal value function $V(x^0) := \inf J(x, u)$

Equivalent representation of the DAE

System with index (at most) one [Berger & Reis '13]

$$\operatorname{im} A \subseteq \operatorname{im} E + A \ker E$$

Theorem [Benner et al. '15]
System $[E, A, B]$ is regular with index at most one $\iff \exists\, S_{\mathrm{r}}, T_{\mathrm{r}}$:

$$S_{\mathrm{r}}[sE - A, -B]\,T_{\mathrm{r}} = \left[s\begin{bmatrix} I_{\hat{n}} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right]$$

holds with $A_{22} \in GL_{n-\hat{n}}(\mathbb{R})$.

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Observation: $\binom{x}{u} \in \mathfrak{B}_{[E,A,B]} \iff z := T_{\mathrm{r}}^{-1}x$ solves ODE

$$\dot{z}_1(t) = (A_{11} - A_{12}A_{22}^{-1}A_{21})z_1(t) + (B_1 - A_{12}A_{22}^{-1}B_2)u(t)$$

and $z_2(t)$ is given by $-A_{22}^{-1}A_{21}z_1(t) + B_2u(t)$.

Regularization by [Berger & Van Dooren '15]

How to get the system regular with index at most one?

Theorem [Berger & Van Dooren '15]
$\exists \hat{T}$ orthogonal, $U(s) = sU_1 + U_0$ unimodular:

$$[sE - A, -B]\hat{T} = U(s)\begin{bmatrix} 0 & 0 \\ sE_{\mathrm{r}} - A_{\mathrm{r}} & -B_{\mathrm{r}} \end{bmatrix}$$

such that $[E_{\mathrm{r}}, A_{\mathrm{r}}, B_{\mathrm{r}}]$ is regular index 1

Pros & Cons:

+ No controllability assumptions necessary
+ Numerically stable algorithm to find transformation matrices and regularized system
– $\hat{T}$ "mingles" state and input

Smoothness of solutions

Lemma

- $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathfrak{B}_{[E,A,B]} \implies \hat{T}^{-1}\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathfrak{B}_{[E_\mathrm{r},A_\mathrm{r},B_\mathrm{r}]}$
- If $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathfrak{B}_{[E_\mathrm{r},A_\mathrm{r},B_\mathrm{r}]}$ fulfils additional smoothness condition

$$\left(U_0\left[\begin{smallmatrix} 0 & 0 \\ E_\mathrm{r} & 0 \end{smallmatrix}\right] - U_1\left[\begin{smallmatrix} 0 & 0 \\ A_\mathrm{r} & B_\mathrm{r} \end{smallmatrix}\right]\right)\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{W}^{1,1}_{\mathrm{loc}}$$

$$\implies \hat{T}\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathfrak{B}_{[E,A,B]}$$

Example

$$\underbrace{\left[s\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]}_{x_2 \in \mathcal{W}^{1,1}_{\mathrm{loc}}} = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{no smoothness condition}}$$

Initial values

Problem: transformation $\hat{T}$ mingles state and input:

$$\binom{x}{u} \in \mathfrak{B}_{[E,A,B]} \Longrightarrow \binom{z}{v} := \hat{T}^{-1}\binom{x}{u} \in \mathfrak{B}_{[E_{\mathrm{r}},A_{\mathrm{r}},B_{\mathrm{r}}]}$$

$\rightsquigarrow$ knowledge of $u(0)$ might be necessary to initialize $z(0)$

Theorem

For regular DAEs

$$[I_{\hat{n}}, 0]\,\hat{T}^{-1}\begin{pmatrix} \{x^0\} \\ \mathbb{R}^m \end{pmatrix} = \left\{[I_{\hat{n}}, 0]\,\hat{T}^{-1}\begin{pmatrix} x^0 \\ 0 \end{pmatrix}\right\}$$

$\rightsquigarrow$ only x^0 is necessary to initialize $z(0)$

Conjecture The knowledge of x^0 is enough for singular DAEs as well
$\rightsquigarrow$ analyze system by means of the Kronecker form

Outline

DAE optimal control problem

↓ Regularization

ODE optimal control problem

↓ Construction of terminal ingredients

Model predictive control

Solution of the unconstrained OCP

DAE-OCP can be solved by an ODE-OCP:

Theorem

$$V(x^0) = \inf \int_0^T \begin{pmatrix} z_1(t) \\ v(t) \end{pmatrix}^\top \begin{bmatrix} \hat{Q} & \hat{H} \\ \hat{H}^\top & \hat{R} \end{bmatrix} \begin{pmatrix} z_1(t) \\ v(t) \end{pmatrix} \mathrm{d}t$$

$$\text{s. t. } \dot{z}_1(t) = (A_{11} - A_{12}A_{22}^{-1}A_{21})z_1(t) + (B_1 - A_{12}A_{22}^{-1}B_2)v(t)$$

$$z_1(0) \in [I_{\hat{n}}, 0]\, T_{\mathrm{r}}^{-1}\, \hat{T}^{-1} \begin{pmatrix} \{x^0\} \\ \mathbb{R}^m \end{pmatrix}$$

Remarks

- $\begin{bmatrix} \hat{Q} & \hat{H} \\ \hat{H}^\top & \hat{R} \end{bmatrix}$ calculated algebraically from the DAE-OCP using the transformation matrices T_{r}, $\hat{T}$
- Subspace constraints for $z_1(0)$ can be replaced by initial condition $z_1(0) = [I_{\hat{n}}, 0]\, T_{\mathrm{r}}^{-1}\, \hat{T}^{-1} \begin{pmatrix} x^0 \\ 0 \end{pmatrix}$ if $[E, A, B]$ is regular

Feasibility and regularity of the OCP

When does the OCP have an optimal solution?

Assumptions (from ODE optimal control [Lancaster & Rodman '95])

1. $\begin{bmatrix} \hat{Q} & \hat{H} \\ \hat{H}^\top & \hat{R} \end{bmatrix} \succeq 0$
2. $(\hat{A}, \hat{B}) = (A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ is stabilizable
3. $\hat{R} > 0$
4. $(\hat{A}, \hat{Q})$ is observable
5. $\mathrm{rk}\begin{bmatrix} \hat{Q} & \hat{H} \\ \hat{H}^\top & \hat{R} \end{bmatrix} = \mathrm{rk}\, \hat{Q} + \hat{R}$

$\rightsquigarrow$ The algebraic Riccati equation

$$\hat{A}^\top \hat{P} + \hat{P}\hat{A} + \hat{Q} - (\hat{P}\hat{B} + \hat{H})\hat{R}^{-1}(\hat{P}\hat{B} + \hat{H})^\top = 0$$

has a unique solution $\hat{P} > 0$

Theorem DAE-OCP fulfils assumptions

$$\Longrightarrow \exists \binom{x^*}{u^*} \in \mathfrak{B}_{[E,A,B]} : (Ex^*)(0) = Ex^0, \quad J(x^*, u^*) = V(x^0)$$

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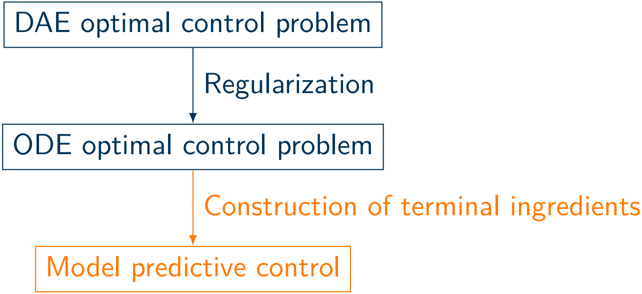
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   s.t.
   - $\frac{\mathrm{d}}{\mathrm{d}t}(E\bar{x})(s) = A\bar{x}(s) + B\bar{u}(s), \quad (E\bar{x})(0) = \hat{x}$
   - $F\bar{x}(s) + G\bar{u}(s) \leq \mathbb{1}$

   $\rightsquigarrow$ optimal solution $\begin{pmatrix} \bar{x}^* \\ \bar{u}^\star \end{pmatrix}$
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Needs to fulfil decrease condition

$$V_\mathrm{f}\left(\begin{pmatrix} x^*(\delta) \\ u^*(\delta) \end{pmatrix}\right) \leq V_\mathrm{f}(x^0) - \int_0^\delta \begin{pmatrix} x^*(t) \\ u^*(t) \end{pmatrix}^\top \hat{S} \begin{pmatrix} x^*(t) \\ u^*(t) \end{pmatrix} \mathrm{d}t$$

to guarantee asymptotic stability

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   - $\hat{x}(T) \in \mathbb{X}_\mathrm{f}$

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to guarantee asymptotic stability

Singular DAEs: a cautionary example

Example

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + u(t)$$

Any state feedback $\mu(x) = kx = \begin{pmatrix}k_1 & k_2\end{pmatrix} x$ leads to unconstrained solutions that don't fulfil state and input constraints:

$$\dot{x}_2(t) = k_2 x_2(t) + (1 + k_1)\, x_1(t)$$

No constraints on x_1!

⇝ State feedback not sufficient to construct terminal region

Construction of terminal region and costs

Idea

Construct terminal region and costs for ODE-OCP using the algebraic Riccati equation, transfer them to the DAE-OCP

Theorem With *terminal region*

$$\mathbb{X}_\mathrm{f} := \left\{ [I_{\hat{n}}, 0]\, \hat{T}\, T_\mathrm{r} \begin{bmatrix} I_{\hat{n}} \\ -\hat{R}^{-1}(\hat{B}^\top \hat{P} + \hat{H}) \end{bmatrix} \hat{x} \;\middle|\; \hat{x} \in \mathbb{R}^{\hat{n}} \wedge \hat{x}^\top \hat{P} \hat{x} \leq \rho \right\},$$

where

$$\rho := \lambda_{\min}(\hat{P}) \left\| \begin{bmatrix} F & G \end{bmatrix} \hat{T}\, T_\mathrm{r} \begin{bmatrix} I_{\hat{n}} \\ -\hat{R}^{-1}(\hat{B}^\top \hat{P} + \hat{H}) \end{bmatrix} \right\|_\infty^{-2}$$

and *terminal costs*

$$V_\mathrm{f} : \mathbb{X}_\mathrm{f} \to \mathbb{R}_{\geq 0}, \quad V_\mathrm{f}(x^0) := V(x^0),$$

the closed MPC loop is asymptotically stable.

Example

$$\text{Minimize} \int_0^T \|x(t)\|^2 + \|u(t)\|^2 \mathrm{d}t$$

$$\text{s.t.} \ \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$(Ex)(0) = Ex^0$$

$$-1 \leq x_i(t) \leq 1, i \in \{1, \dots, 5\}, \quad -1 \leq u(t) \leq 1$$

Regularization $\rightsquigarrow$ $x_1 = 0$, $x_2 = 0$, x_4 input, x_3, x_5 real states

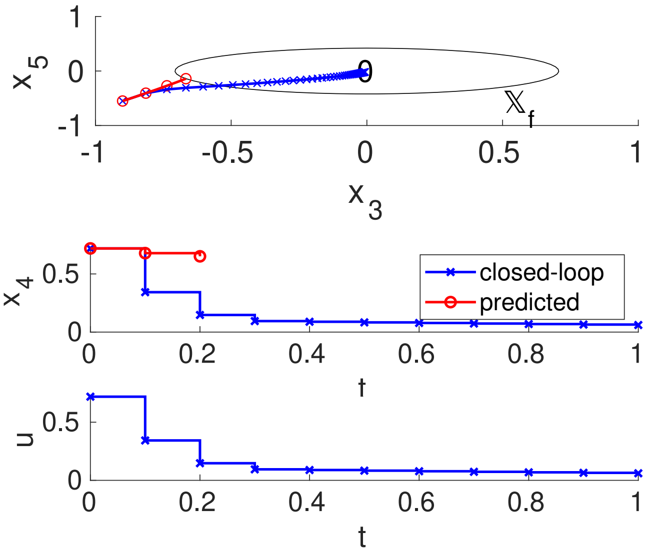
Equivalent ODE-OCP

$$\text{Minimize} \int_0^T \|z_1(t)\|^2 + \|v(t)\|^2 \mathrm{d}t$$

$$\text{s.t.} \ \dot{z}_1(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} z_1(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} v(t), \ z_1(0) = \begin{pmatrix} x_3^0 \\ x_5^0 \end{pmatrix}$$

$$-1 \leq z_i(t) \leq 1, \quad -1 \leq v_i(t) \leq 1, \quad i \in \{1, 2\}$$

Terminal region and closed-loop performance



Conclusions & Outlook

Optimal control for DAEs

1. Regularization to index one system by [Berger & Van Dooren '15]
2. Transformation to ODE-OCP

Model Predictive Control

- Inclusion of state and input constraints
- Terminal region and costs for asymptotic stability

Future work

- Prove conjecture: initial condition for x^0 is enough
- Extend results to nonlinear systems