Infinite Dimensional Systems

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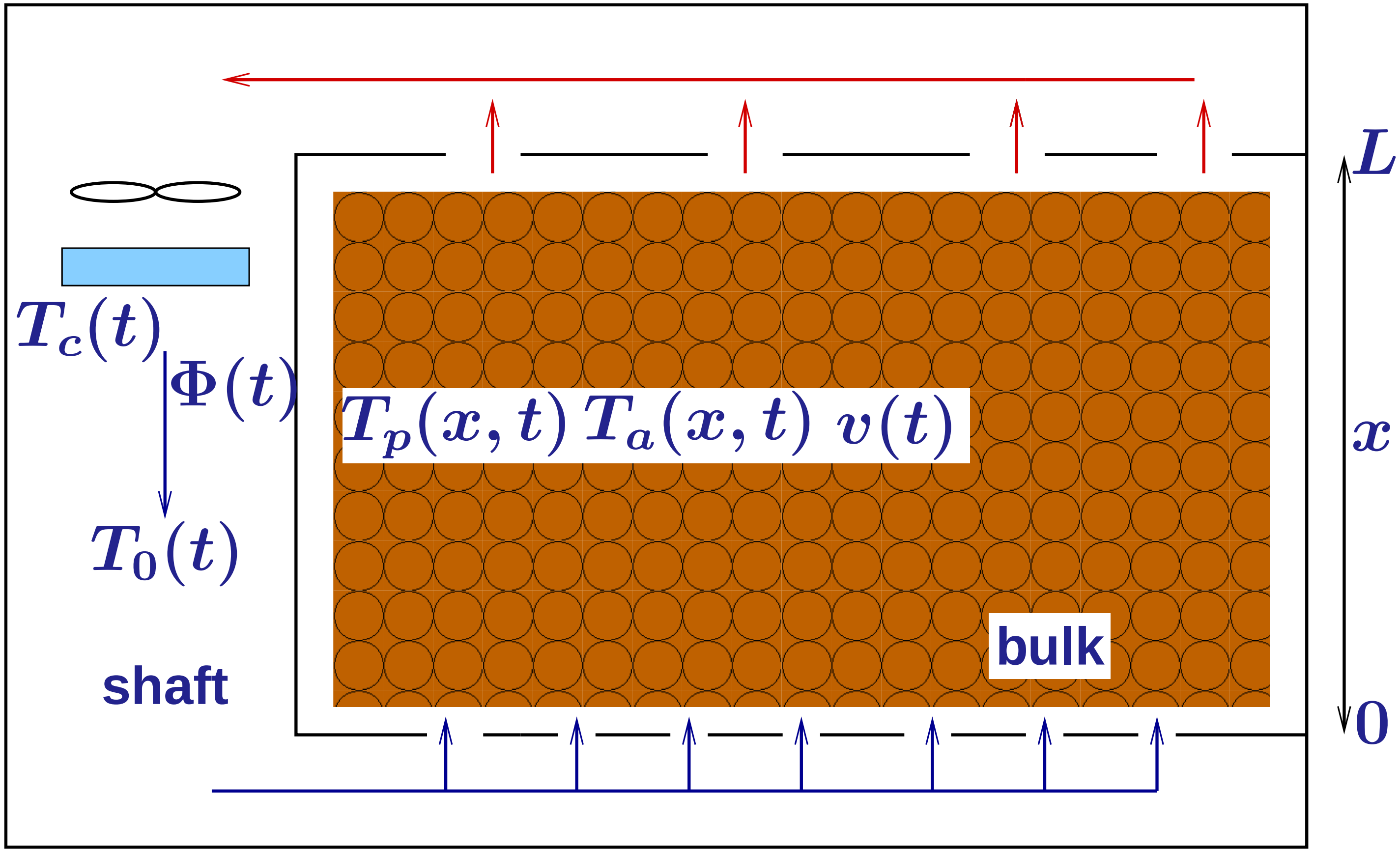
Contents

1 Motivation

1.1 Potato storage

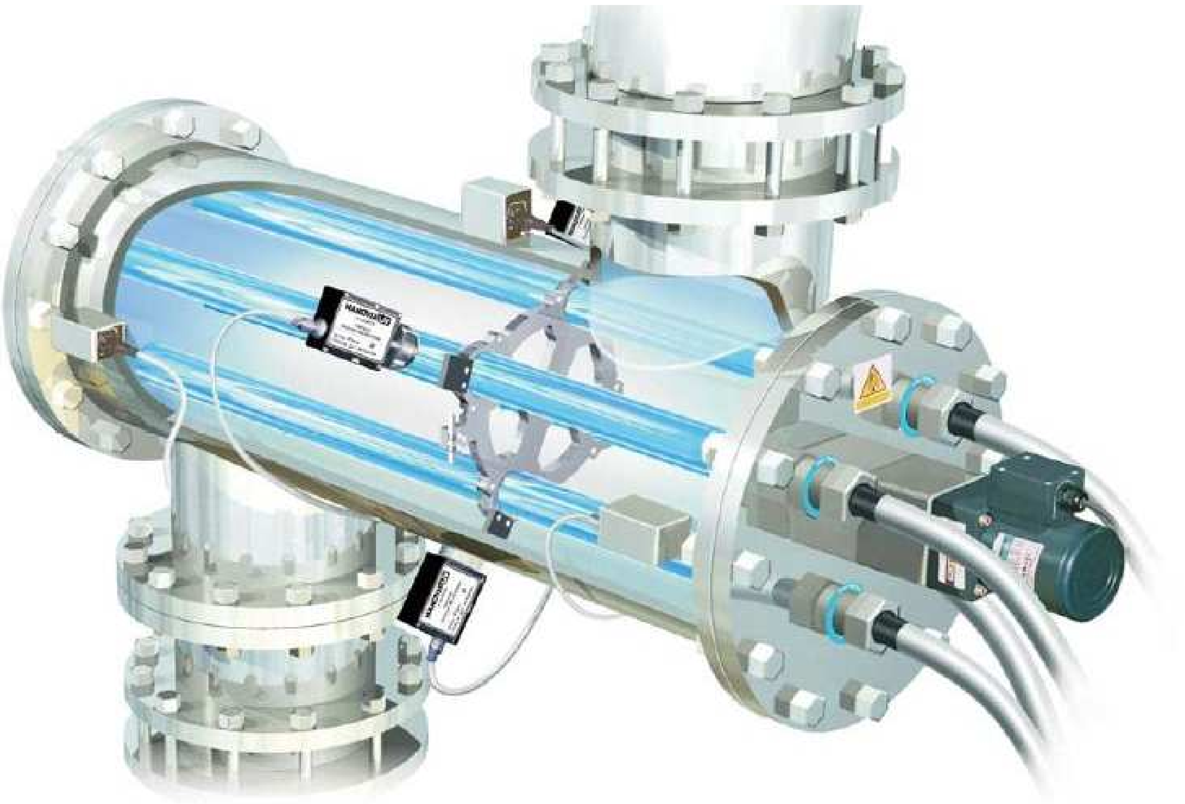
Schematic view point:



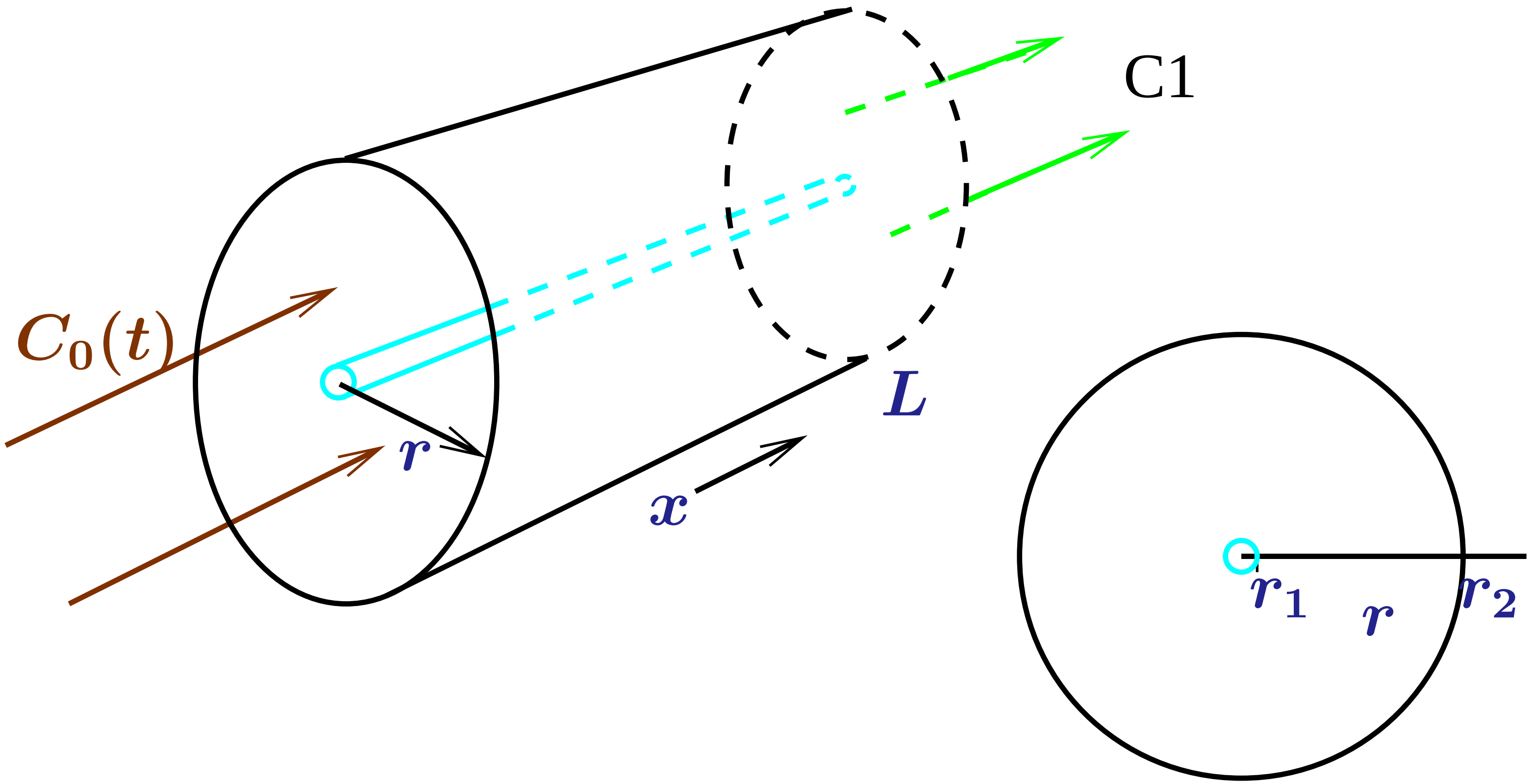
The model:

$$\begin{aligned}
\frac{dT_0}{dt}(t) &= M_1\alpha(\Phi(t))\left[T_a(L,t) - T_c(t)\right] + \\
&\quad M_2\Phi(t)\left[T_a(L,t) - T_0(t)\right] + \\
&\quad M_3\left[T_{\text{out}}(t) - T_0(t)\right] \\
\frac{\partial T_a}{\partial t}(x,t) &= -v(t)\frac{\partial T_a}{\partial x}(x,t) + M_4\left[T_p(x,t) - T_a(x,t)\right] \\
\frac{\partial T_p}{\partial t}(x,t) &= M_5 T_p(x,t) + M_6 T_a(x,t) \\
T_a(0,t) &= T_0(t).
\end{aligned}$$

1.2 UV-disinfection



Schematic view point:



The model for the velocity

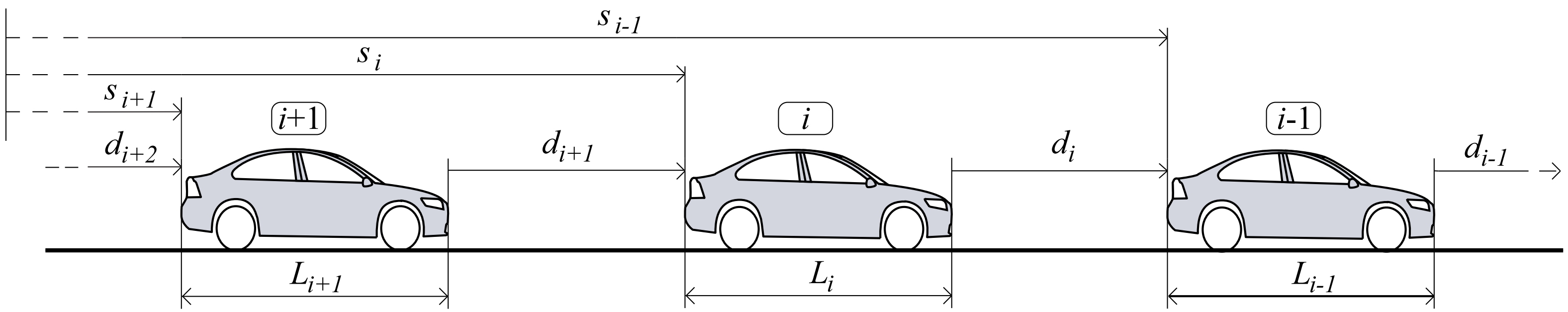
$$\begin{aligned} \frac{\partial v_x}{\partial t}(r,t) &= \beta(t) + \frac{1}{\mathrm{Re}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}(r,t)\right)\right] \\ v_x(r_1,t) &= v_x(r_2,t) = 0 \end{aligned}$$

and for the concentration

$$
\begin{aligned}
\frac{\partial C}{\partial t}(x,r,t) &= -v_x(r,t)\frac{\partial C}{\partial x}(x,r,t) + \\
&\quad \frac{1}{\mathrm{Pe}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}(x,r,t)\right) + \frac{\partial^2 C}{\partial x^2}(x,r,t)\right] \\
&\quad -\mathrm{Da}\, K(r,t)C(x,r,t) \\
\frac{\partial C}{\partial x}(0,r,t) &= 0 \\
C(x,r_1,t) &= C(x,r_2,t) = 0 \\
C(0,r,t) &= C_0(t).
\end{aligned}
$$

1.3 Platoon systems

Consider an infinite string of vehicles (a platoon)



The aim is to control the distance d_i between the cars. Every car controls its jerk, i.e, the change of its force.

Standard modeling gives the following model for the i'th car

$$\frac{d}{dt}\begin{pmatrix} d_i(t) \\ v_i(t) \\ a_i(t) \end{pmatrix} = \begin{pmatrix} v_{i-1}(t) - v_i(t) \\ a_i(t) \\ -\tau^{-1}a_i(t) + \tau^{-1}u_i(t) \end{pmatrix}, \qquad i \in \mathbb{Z}.$$

where τ is a constant.

2 Semigroups and Generators

2.1 Introduction

The aim of this part is to learn the concepts semigroups and infinitesimal generator.

The reason for this is that we want to know if the partial differential equations (p.d.e.'s) or the coupled ordinary differential equations (o.d.e.'s) have a solution.

Note there is a difference between knowing the existence of a solution and having the form/expression of the solution.

Example (Transport equation)

On the spatial domain $[0,1]$ consider the p.d.e.

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t), \qquad x \in [0,1],\ t \geq 0, \\ z(1,t) &= 0 \\ z(x,0) &= z_0(x) \qquad \text{(given)}. \end{aligned}$$

□

Example (Diffusion equation)

On the spatial domain $[0,1]$ consider the p.d.e.

$$\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t), \qquad x \in [0,1],\ t \geq 0,\\
\frac{\partial z}{\partial x}(0,t) &= 0\\
\frac{\partial z}{\partial x}(1,t) &= 0\\
z(x,0) &= z_0(x) \qquad \text{(given)}.
\end{aligned}$$

□

2.2 Semigroups

Throughout this course, we assume that

- Z **is a (complex) Hilbert space with inner product** $\langle \cdot, \cdot \rangle$**.**
- $\mathcal{L}(Z)$ **denotes the set of linear and bounded operators from** Z **to** Z**.**

Definition

A strongly continuous semigroup is an operator-valued function from $[0, \infty)$ to $\mathcal{L}(Z)$ which satisfies

- $T(0) = I$
- $T(t)T(s) = T(t+s), \quad t, s \in [0, \infty)$
- **For all $z_0 \in Z$ there holds**

$$\lim_{t \downarrow 0} T(t) z_0 = z_0.$$

□

Notation: $(T(t))_{t \geq 0}$; **Short**: C_0**-semigroup.**

To motivate this definition, think of

$$z_0 \mapsto T(t)z_0$$

as a solution of a time-invariant, linear differential equation.

- $T(0) = I$ **Trivial.**
- **For fixed** t, $T(t)$ **is linear** **Is linearity of diff. eq.**
- $T(t)T(s) = T(t+s)$ **Is time invariance.**
- $T(t)z_0 \longrightarrow z_0$ **if** $t \downarrow 0$ **Is strong continuity.**

Example

Let A be an $n \times n$-matrix, then e^{At} is a C_0-semigroup on $\mathbb{C}^n$ or $\mathbb{R}^n$.

For instance, if

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},$$

then

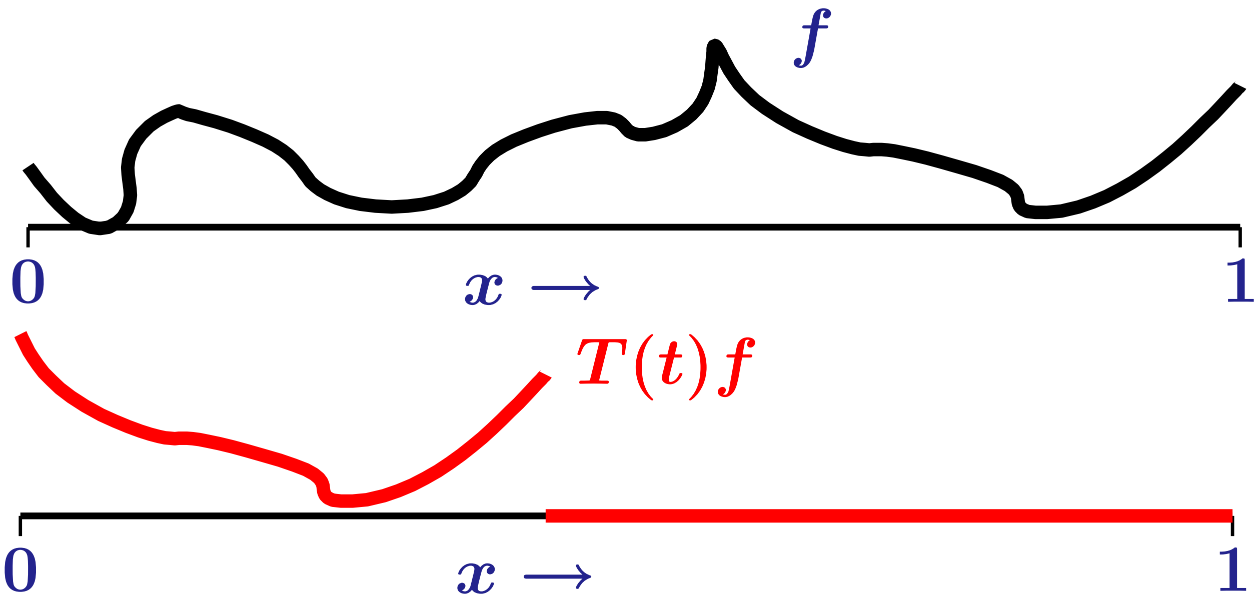
$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$

$\square$

Example

Let $Z = L^2(0,1)$ and define for $t \geq 0$ and $x \in [0,1]$

$$(T(t)f)(x) = \begin{cases} f(t+x) & x+t \leq 1 \\ 0 & x+t > 1 \end{cases}$$



□

An important property of a C_0-semigroup is that it is exponentially bounded.

Lemma

There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $t \geq 0$

$$\|T(t)\| \leq Me^{\omega t}.$$

□

The proof is as follows.

- $\sup_{t\in[0,1]} \|T(t)\| \leq M$.

  This follows from the strong continuity and the uniform boundedness theorem.

  Note $M \geq 1$.

- **Choose $t > 1$ and write $t = n + t_0$ with $t_0 \in [0, 1)$.**

$$\begin{aligned}
\|T(t)\| &= \|T(n+t_0)\| \\
&= \|T(n)T(t_0)\| \\
&\leq \|T(n)\|\|T(t_0)\| \\
&\leq M\|T(n)\| \\
&= M\|T(n-1)T(1)\| \\
&\leq M\|T(1)\|\|T(n-1)\| \\
&\leq MM\|T(n-1)\| \\
&\leq M^{n+1} = Me^{\log Mn} \\
&\leq Me^{\log Mt}.
\end{aligned}$$

2.3 Generators

Consider the finite-dimensional semigroup

$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$

Question

What is A?

Answer

Evaluate the derivative of the semigroup at $t = 0$.

Since $\frac{d}{dt}e^{At} = Ae^{At}$, we have

$$\frac{d}{dt}e^{At}\ |_{t=0} = A.$$

□

So given the (general) C_0-semigroup $(T(t))_{t\geq 0}$, we could try to find A by differentiating it at $t = 0$. However, in general $(T(t))_{t\geq 0}$ is only (strongly) continuous.

Definition

If the following limit exists,

$$\lim_{t\downarrow 0} \frac{T(t)z_0 - z_0}{t},$$

then z_0 is in the domain of A, $D(A)$.

Furthermore, for $z_0 \in D(A)$, we define

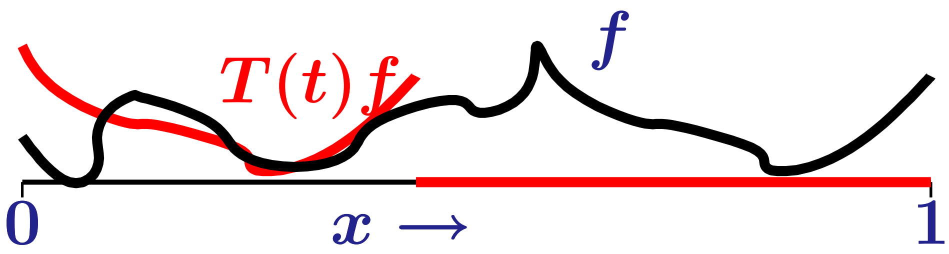
$$Az_0 = \lim_{t\downarrow 0} \frac{T(t)z_0 - z_0}{t}.$$

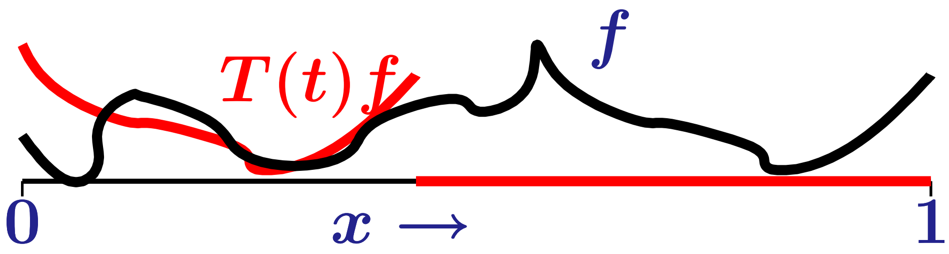
A is named the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. □

Example

Consider the shift semigroup.

$$(T(t)f)(x) = \begin{cases} f(t+x) & x+t \leq 1 \\ 0 & x+t > 1 \end{cases}$$





Choose $x < 1$, then for $t \in (0, x)$, we have that

$$\left(\frac{T(t)f - f}{t}\right)(x) = \frac{f(t+x) - f(x)}{t}.$$

Hence if the limit exists, then f must be differentiable, and the limit is the derivative.

Now take $x = 1$. For any $t > 0$, we have that

$$\left(\frac{T(t)f - f}{t}\right)(1) = \frac{0 - f(1)}{t}.$$

This limit can only exist when $f(1) = 0$.

Concluding, we have that

$$D(A) = \Big\{ f \in L^2(0,1) \mid f \text{ is absolutely continuous, with } \frac{df}{dx} \in L^2(0,1) \text{ and } f(1) = 0 \Big\}.$$

Furthermore,

$$Af = \frac{df}{dx}.$$

□

Lemma

If $z_0 \in D(A)$, then $T(t)z_0 \in D(A)$, and

$$\frac{d}{dt}[T(t)z_0] = AT(t)z_0.$$

□

Hence for $z_0 \in D(A)$, we have that $z(t) := T(t)z_0$ is a (classical) solution of the abstract differential equation

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0.$$

How does this abstract differential equation relate to a p.d.e.?

We make the following observations:

- **The function z is at every time instant an element of a Hilbert space, i.e., $z(t) \in Z$.**
- **Assume that the Hilbert space consists of functions, i.e., $f \in Z$ means a.o. $f := x \mapsto f(x)$.**
  For instance, $Z = L^2(0, 1)$.

- **This implies that $z(t)$ is for every t a function of x. We write $(z(t))\,(x) = z(x,t)$.**

- **Using this last form we see that**

$$(\dot{z}(t))\,(x) = \frac{\partial z}{\partial t}(x,t).$$

- For $A = \frac{d}{dx}$, the abstract differential equation

$$\dot{z}(t) = Az(t)$$

becomes

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial z}{\partial x}(x,t).$$

- In the domain of A are also the boundary conditions:

$$z(1,t) = 0.$$

- **For a given p.d.e. we (roughly) do the following**
  - **The state z is identified, and the p.d.e. is written as**

$$\frac{\partial z}{\partial t} = \dots$$

  - **The right-hand side defines together with the boundary conditions A.**

2.4 Which A generates a C_0-semigroup?

So we have that every semigroup has an infinitesimal generator, but you would like to know which operator A generates a C_0-semigroup.

The answer is given by the Hille-Yosida Theorem. This theorem we will not treat. We focus on two special cases

- A **bounded, and**
- $T(t)$ **a contraction**

If A is a bounded (linear) operator, i.e., there exists an $m \geq 0$ such that $\|Az\| \leq m\|z\|$ for all $z \in Z$, then A generates a C_0-semigroup, given by

$$T(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

2.5 Platoon systems

For zero input our platoon system is given by

$$\frac{d}{dt}\begin{pmatrix} d_r(t) \\ v_r(t) \\ a_r(t) \end{pmatrix} = \begin{pmatrix} v_{r-1}(t) - v_r(t) \\ a_r(t) \\ -\tau^{-1} a_r(t) \end{pmatrix}, \qquad r \in \mathbb{Z}.$$

where τ is a constant.

As state space we choose $\ell^2(\mathbb{Z}; \mathbb{C}^3)$ and as state

$$z(t) = \left(\cdots, \begin{pmatrix} d_{-1}(t) \\ v_{-1}(t) \\ a_{-1}(t) \end{pmatrix}, \begin{pmatrix} d_0(t) \\ v_0(t) \\ a_0(t) \end{pmatrix}, \begin{pmatrix} d_1(t) \\ v_1(t) \\ a_1(t) \end{pmatrix}, \cdots, \right)$$

For the sequence $(f_r)_{r\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^n)$ define the discrete Fourier transform as

$$\check{f}(\phi) = \sum_{r=-\infty}^{\infty} f_r \phi^{-r}, \qquad \phi \in \partial\mathbb{D} \quad (\underline{\text{unit circle}})$$

This Fourier transform maps $\ell^2(\mathbb{Z};\mathbb{C}^n)$ onto $L^2(\partial\mathbb{D};\mathbb{C}^n)$.

Furthermore,

$$\|(f_r)\|^2 = \sum_{r=-\infty}^{\infty} \|f_r\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|\check{f}(e^{\jmath\theta})\|^2 d\theta = \|\check{f}\|^2$$

For differential equations of the following type

$$\dot{z}_r(t) = \sum_{l=-N}^{N} A_l z_{r-l}(t), \qquad r \in \mathbb{Z}$$

the Fourier transform gives;

$$\begin{aligned} \frac{\partial \check{z}}{\partial t}(\phi, t) &= \left[\sum_{l=-N}^{N} A_l \phi^{-l} \right] \check{z}(\phi, t), \quad \phi \in \partial\mathbb{D} \\ &= \check{A}(\phi) \check{z}(\phi, t). \end{aligned}$$

Hence a parametrized finite-dimensional system.

It is easy to see that $\check{A}(\phi) = \sum_{l=-N}^{N} A_l \phi^{-l}$ defines a bounded linear operator on $L^2(\partial\mathbb{D}; \mathbb{C}^n)$.

So $\check{A}$ generates the C_0-semigroup on $L^2(\partial\mathbb{D}; \mathbb{C}^n)$

$$\left(e^{\check{A}t}\right)(\theta) = e^{\check{A}(\theta)t}, \quad \theta \in \partial\mathbb{D}.$$

Thus in the Fourier domain it is easy to find the solutions of

$$\dot{z}_r(t) = \sum_{l=-N}^{N} A_l z_{r-l}(t), \qquad r \in \mathbb{Z}$$

2.6 Contraction semigroup

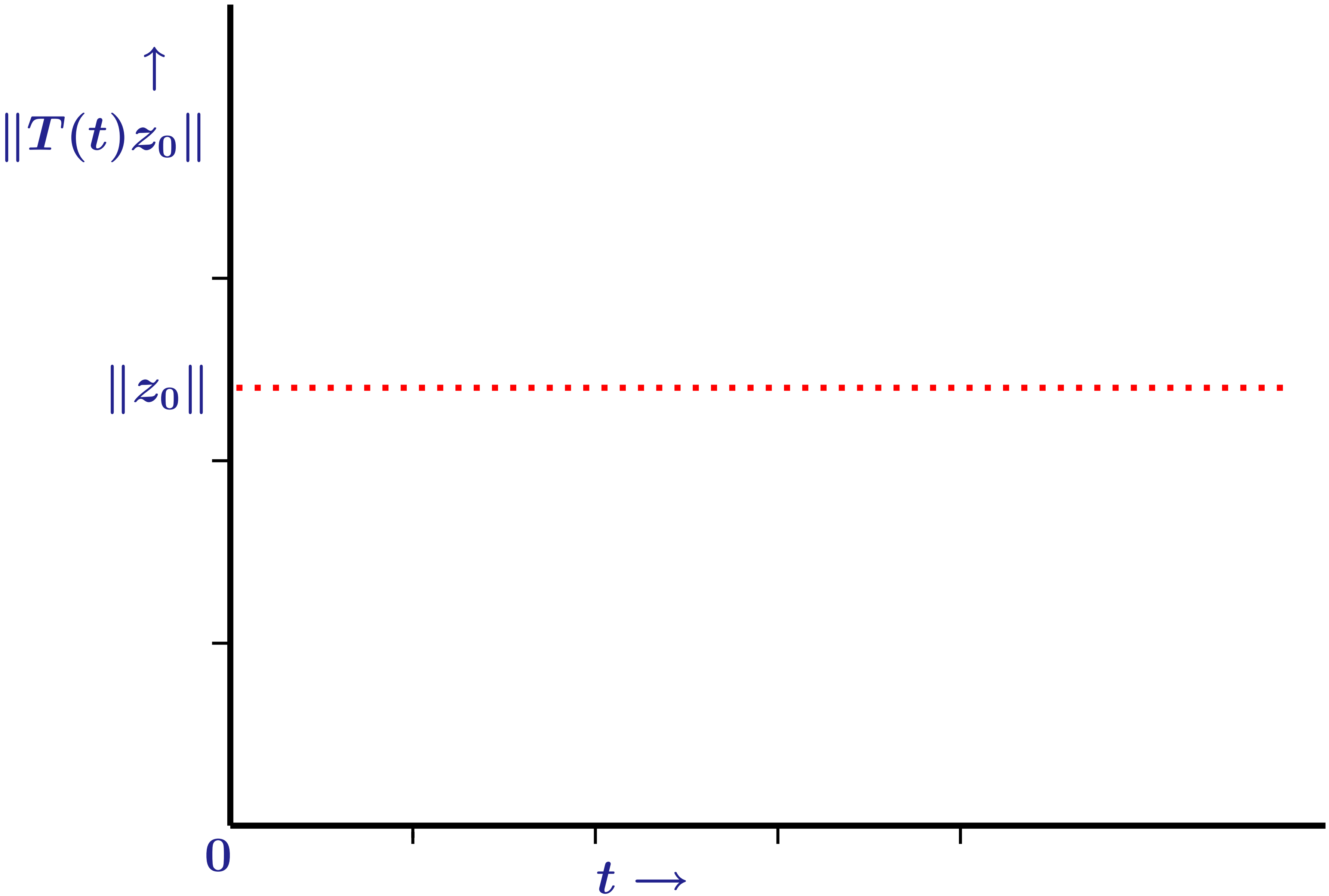
Definition

The C_0-semigroup $(T(t))_{t\geq 0}$ is contraction semigroup if

$$\|T(t)z_0\| \leq \|z_0\| \qquad \text{for all } t \geq 0.$$

□

What can we say about these semigroups?


↑
$\|T(t)z_0\|$
$\|z_0\|$
0
$t \longrightarrow$

We known that

$$\|T(t)z_0\|^2 = \langle T(t)z_0, T(t)z_0 \rangle.$$

For $z_0 \in D(A)$, we have that the derivative of $T(t)z_0$ equals $AT(t)z_0$.

So if we differentiate $\|T(t)z_0\|^2$, we find

$$\frac{d}{dt}\|T(t)z_0\|^2 = \langle AT(t)z_0, T(t)z_0 \rangle + \langle T(t)z_0, AT(t)z_0 \rangle.$$

At time equal to zero, we find

$$\frac{d}{dt}\left(\|T(t)z_0\|^2\right)\Big|_{t=0} = \langle Az_0, z_0\rangle + \langle z_0, Az_0\rangle.$$

We know that at $t = 0$, $\|T(t)z_0\| = \|z_0\|$, and so if $T(t)$ is a contraction semigroup, then

$$\langle Az_0, z_0\rangle + \langle z_0, Az_0\rangle = \frac{d}{dt}\|T(t)z_0\|^2\Big|_{t=0} \leq 0.$$

This holds for all $z_0 \in D(A)$.

Theorem (Lumer-Phillips)

Let A be a densely defined operator, then A generates a contraction semigroup on Z if and only if

1. $\langle Az_0, z_0\rangle + \langle z_0, Az_0\rangle \leq 0$ **for all** $z_0 \in D(A)$.
2. **The range of** $A - I$ **is the whole of** Z.

□

Example

Consider A which is given as

$$Az = \frac{dz}{dx}, \qquad x \in [0,1]$$

with the domain

$$D(A) = \Big\{ z \in L^2(0,1) \mid z \text{ is absolutely continuous, } \frac{dz}{dx} \in L^2(0,1) \text{ and } z(1) = 0 \Big\}.$$

Let us check the properties

- A **is densely defined in** $L^2(0,1)$**.**

•

$$
\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle
&= \int_0^1 \frac{dz}{dx}(x)\overline{z(x)}dx + \int_0^1 z(x)\overline{\frac{dz}{dx}}(x)dx \\
&= \int_0^1 \frac{d}{dx}\left[z(x)\overline{z(x)}\right] dx \\
&= |z(x)|^2\Big|_0^1 \\
&= 0 - |z(0)|^2 \leq 0.
\end{aligned}
$$

- To see if the range of $(A - I)$ is everything, we have for every $f \in L^2(0,1)$ to solve $(A - I)z = f$.

  This means solving

$$\frac{dz}{dx}(x) - z(x) = f(x), \qquad x \in (0,1)$$

  with boundary condition $z(1) = 0$.

  The solution of this differential equation with the given boundary value is

$$z(x) = -\int_x^1 e^{x-\xi} f(\xi) d\xi.$$

□

Example

Consider A given by

$$Az = \frac{d^2 z}{dx^2} \qquad x \in [0,1]$$

with the domain

$$D(A) = \left\{ z \in L^2(0,1) \mid z, \frac{dz}{dx} \text{ are absolutely continuous, and } \frac{dz}{dx}(0) = 0 = \frac{dz}{dx}(1) \right\}.$$

Clearly A is densely defined, and so it remains to check the other conditions.

We have that

$$\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle &= \int_0^1 \frac{d^2z}{dx^2}(x)\overline{z(x)} + z(x)\overline{\frac{d^2z}{dx^2}(x)}dx \\
&= \left[\frac{dz}{dx}(x)\overline{z(x)} + z(x)\overline{\frac{dz}{dx}(x)}\right]_0^1 - 2\int_0^1 \left|\frac{dz}{dx}(x)\right|^2 dx \\
&\leq 0
\end{aligned}$$

for $z \in D(A)$**.**

To see that

$$(A - I)z = f$$

is solvable for all $f \in L^2(0,1)$ we have to solve an o.d.e.

The solution is given by

$$z(x) = \cosh(x)z(0) + \int_0^x \sinh(x-\xi)f(\xi)d\xi,$$

with

$$z(0) = \frac{-1}{\sinh(1)} \int_0^1 \cosh(1-\xi)f(\xi)d\xi.$$

Thus A generates a contraction semigroup.

Now we know that A generates a C_0-semigroup, but can we find this semigroup?

For this we could return to the corresponding p.d.e.

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t), \qquad x \in [0,1],\ t \geq 0, \\ \frac{\partial z}{\partial x}(0,t) &= 0 \\ \frac{\partial z}{\partial x}(1,t) &= 0 \\ z(x,0) &= z_0(x) \qquad \text{(given)}. \end{aligned}$$

This we can solve by the separation of variables principle. We choose another approach.

- **A is a self-adjoint operator and the inverse of $(A - I)$ is compact.**
- **Hence A has an orthonormal basis of eigenfunctions.**

- **Solving $A\phi_n = \lambda_n \phi_n$ gives**

$$\phi_n(x) = \begin{cases} 1 & \lambda_0 = 0 \\ \sqrt{2}\cos(n\pi x) & \lambda_n = -n^2\pi^2, \qquad n \in \mathbb{N} \end{cases}$$

- **Hence the solution of**

$$\dot{z}(t) = Az(t), \qquad z(0) = \phi_n$$

is given by

$$z(t) = e^{\lambda_n t}\phi_n$$

- **This must be equal to $T(t)\phi_n$.**

- **Since $\{\phi_n, n \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis, we know that**

$$z_0 = \sum_{n=0}^{\infty} \langle z_0, \phi_n \rangle \phi_n$$

Hence

$$\begin{aligned} T(t) z_0 &= T(t) \left(\sum_{n=0}^{\infty} \langle z_0, \phi_n \rangle \phi_n \right) \\ &= \sum_{n=0}^{\infty} \langle z_0, \phi_n \rangle T(t) \phi_n \\ &= \sum_{n=0}^{\infty} \langle z_0, \phi_n \rangle e^{\lambda_n t} \phi_n. \end{aligned}$$

□

2.7 Semigroups and solutions of p.d.e.'s

We have that for any $z_0 \in D(A)$, the function $z(t) := T(t)z_0$ is the solution of

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0.$$

How for general z_0?

Note that in general $Az(t)$ has no meaning and so has $\dot{z}(t)$.

To solve this question we have first to introduce the adjoint of A.

Definition

Let A be a densely defined operator with domain $D(A)$. The domain of A^*, $D(A^*)$, is defined as consisting of those $w \in Z$ for which there exists a $v \in Z$ such that

$$\langle w, Az \rangle = \langle v, z \rangle \quad \text{for all } z \in D(A).$$

If $w \in D(A^*)$, then A^* is defined as

$$A^* w = v.$$

A^* is named the adjoint of A. □

Lemma

Let $z_0 \in Z$, and define $z(t) = T(t)z_0$. Then for every $w \in D(A^*)$, there holds that

$$\frac{d}{dt}\langle w, z(t)\rangle = \langle A^* w, z(t)\rangle.$$

□

This implies that $z(t) := T(t)z_0$ is the weak solution of $\dot{z}(t) = Az(t)$, $z(0) = z_0$.

2.8 Summary

We have seen the following

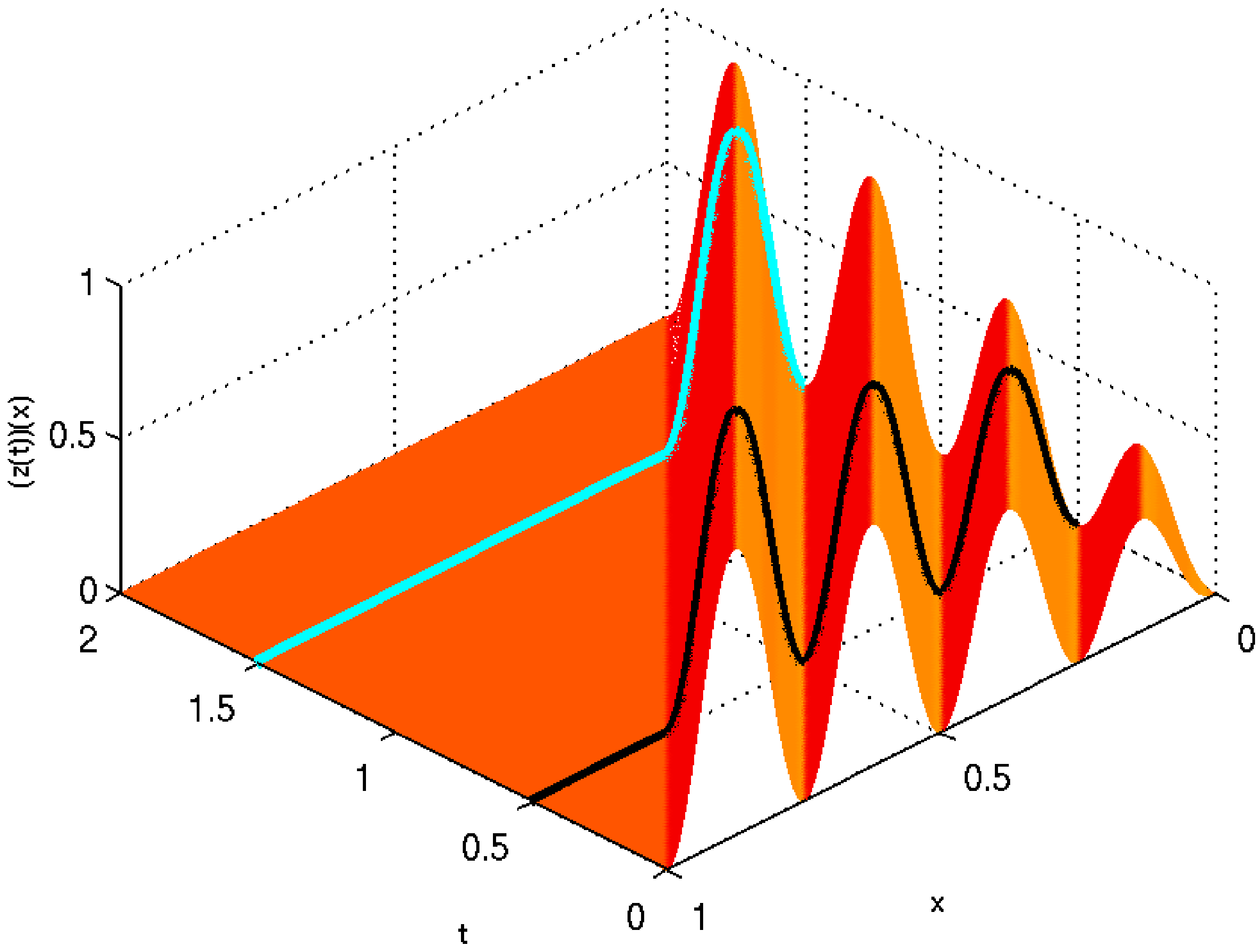
- **A p.d.e. like**

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t) \\ z(1,t) &= 0 \end{aligned}$$

will be written as

$$\dot{z}(t) = Az(t)$$

where z is the state, see figure


1
0.5
0
(z(t))(x)
2
1.5
1
0.5
0
t
1
0.5
0
x

- **The state is a function of the spatial variable.**
- **The mapping *initial state* to *state at time* t is a strongly continuous semigroup.**
- **There are conditions which tell you when the operator A generates a strongly continuous semigroup.**
- **The function $z(t) = T(t)z_0$ is always a (weak) solution of the abstract differential equation**

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0.$$

3 Inputs and Outputs

3.1 Introduction

From the previous part we know what we mean by the solution of the abstract differential equation

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0$$

However, what do we mean by the solution of

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \qquad z(0) = z_0, \\ y(t) &= Cz(t) + Du(t). \end{aligned}$$

To find the answer to this question is the aim of this part.

3.2 Inputs

In this section, we want to solve the abstract differential equation

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0.$$

We assume that

- A **generates the** C_0**-semigroup** $(T(t))_{t\geq 0}$ **on the Hilbert space** Z**.**
- $B \in \mathcal{L}(U, Z)$**.**

So we want to find a (candidate) solution for

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0.$$

Choose $t_1 > 0$ and let $t \in [0, t_1]$.

We multiply the differential equation by $T(t_1 - t)$, and bring z to the left-hand side

$$T(t_1 - t)\dot{z}(t) - T(t_1 - t)Az(t) = T(t_1 - t)Bu(t).$$

The left-hand side equals

$$\frac{d}{dt}\left[T(t_1 - t)z(t)\right] = T(t_1 - t)Bu(t).$$

Hence

$$\begin{aligned}\int_0^{t_1} T(t_1 - t)Bu(t)dt &= \left[T(t_1 - t)z(t)\right]_0^{t_1} \\ &= z(t_1) - T(t_1)z(0).\end{aligned}$$

Theorem

Consider the abstract differential equation

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0,$$

where A generates the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z, $B \in \mathcal{L}(U, Z)$, and $u \in L^1_{\text{loc}}((0,\infty); U)$. Then the (weak) solution is given by

$$z(t) = T(t)z_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau.$$

If u is continuously differentiable and $z_0 \in D(A)$, then it is the classical solution. □

Example

Consider the (controlled) p.d.e.

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t) + u(t) \\ z(1,t) &= 0. \end{aligned}$$

We can write this as

$$\dot{z}(t) = Az(t) + Bu(t)$$

with

$$Az = \frac{dz}{dx},$$

$$D(A) = \left\{ z \in L^2(0,1) \mid \frac{dz}{dx} \in L^2(0,1) \text{ and } z(1) = 0 \right\}$$

and

$$Bu = 1 \cdot u.$$

Using the semigroup generated by A, we find that the solution of the p.d.e. is given by

$$z(x,t) = z_0(x+t)\,\mathbb{1}_{[0,1]}(x+t) + \int_0^t \mathbb{1}_{[0,1]}(x+t-\tau)u(\tau)d\tau.$$

with

$$\mathbb{1}_{[0,1]}(\xi) = \begin{cases} 1 & \xi \in [0,1] \\ 0 & \xi \notin [0,1] \end{cases}$$

□

Example

We consider the heated bar which is heated uniformly in the interval $[1/2, 1]$.

$$\begin{aligned}\frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[1/2,1]}(x)u(t)\\ \frac{\partial z}{\partial x}(0,t) &= \frac{\partial z}{\partial x}(0,t) = 0.\end{aligned}$$

This we can write as

$$\dot{z}(t) = Az(t) + Bu(t)$$

with

$$(Bu)(x) = \mathbb{1}_{[1/2,1]}(x) \cdot u$$

and A as before

$$Az = \frac{d^2 z}{dx^2},$$

with domain

$$D(A) = \left\{ z \in L^2(0,1) \mid \frac{dz}{dx} \text{ and } \frac{d^2 z}{dx^2} \in L^2(0,1), \frac{dz}{dx}(0) = 0 = \frac{dz}{dx}(1) \right\}.$$

□

3.3 Outputs

Now we have solved the input problem, the understanding of the output equation

$$y(t) = Cz(t) + Du(t)$$

is easy.

If $C \in \mathcal{L}(Z, Y)$, and $D \in \mathcal{L}(U, Y)$, with Y a Hilbert space, then using the solution for $z(t)$ we find

$$\begin{aligned} y(t) &= C\left[T(t)z_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau\right] + Du(t) \\ &= CT(t)z_0 + \int_0^t CT(t-\tau)Bu(\tau)d\tau + Du(t). \end{aligned}$$

Example

We take our heated bar, and we measure the (average) temperature in the other half of the bar.

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[\frac{1}{2},1]}(x)u(t) \\ \frac{\partial z}{\partial x}(0,t) &= \frac{\partial z}{\partial x}(0,t) = 0. \\ y(t) &= \int_0^{\frac{1}{2}} z(x,t)dx. \end{aligned}$$

This we can written as

$$
\begin{aligned}
\dot{z}(t) &= Az(t) + Bu(t) \\
y(t) &= Cz(t)
\end{aligned}
$$

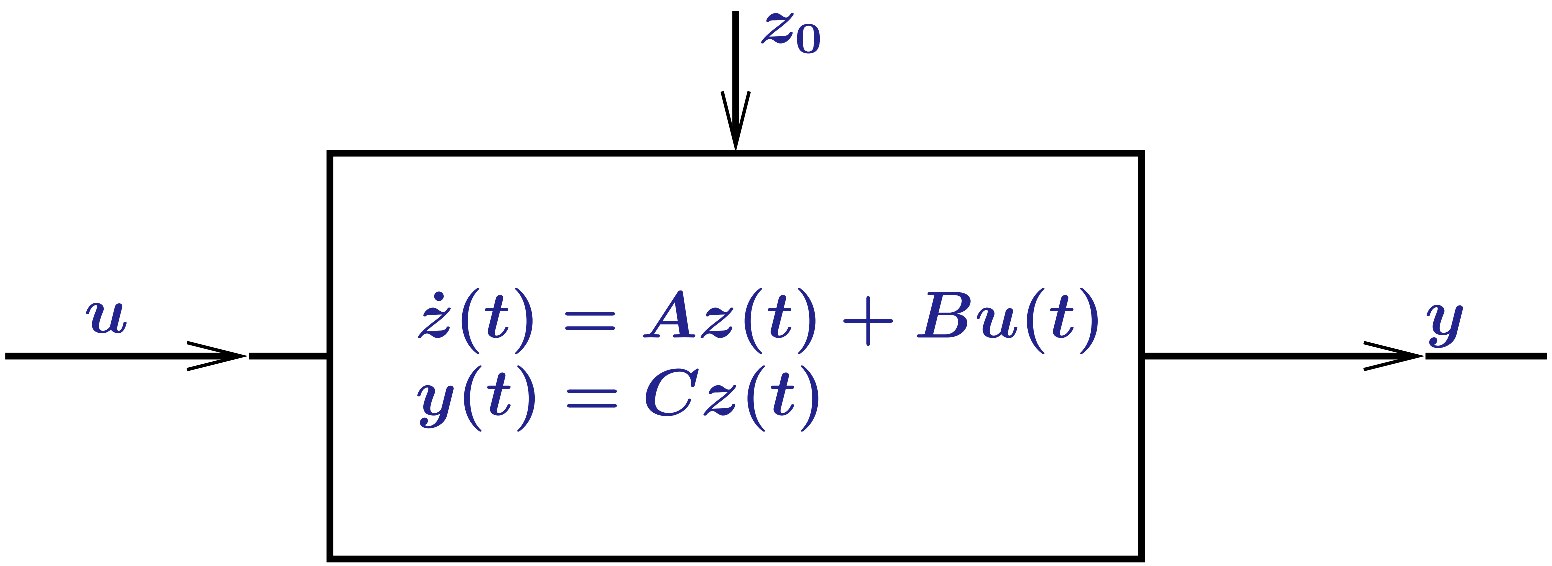
with A and B as before, and the operator $C \in \mathcal{L}(Z, \mathbb{C})$ given by

$$Cz = \int_0^{\frac{1}{2}} z(x)dx.$$

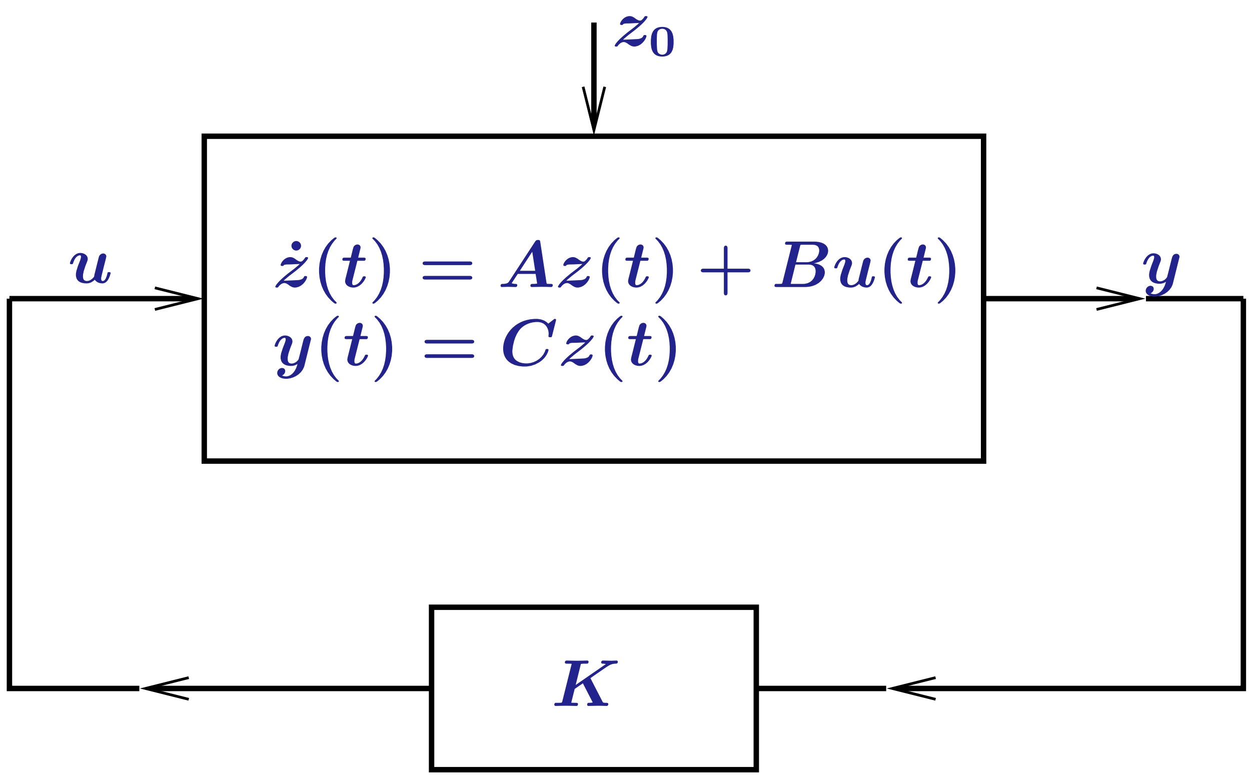
□

3.4 Feedback

We have defined the open loop system, schematically given by



However, for control we would like to close the loop.



This gives the differential equation

$$\dot{z}(t) = Az(t) + BKCz(t) = (A + BKC)\, z(t).$$

Theorem

If A generates the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z, and $B \in \mathcal{L}(U, Z)$, $F \in \mathcal{L}(Z, U)$, then $A + BF$ also generates a C_0-semigroup.

Furthermore, if we denotes this (closed-loop) semigroup by $(S(t))_{t\geq 0}$, then

$$S(t)z_0 = T(t)z_0 + \int_0^t T(t-\tau)BFS(\tau)z_0 d\tau.$$

□

3.5 Boundary control

Consider the following system

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t), \qquad x \in [0,1] \\ z(1,t) &= u(t). \end{aligned}$$

We cannot write this in the form $\dot{z}(t) = Az(t) + Bu(t)$, with $B \in \mathcal{L}(Z, \mathbb{C})$.

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t), \qquad x \in [0,1] \\ z(1,t) &= u(t). \end{aligned}$$

We perform the following trick.

Define $v(x,t) = z(x,t) - u(t)$**.**

Then v **satisfies the following partial differential equation**

$$\begin{aligned} \frac{\partial v}{\partial t}(x,t) &= \frac{\partial v}{\partial x}(x,t) - \dot{u}(t), \qquad x \in [0,1] \\ v(1,t) &= 0. \end{aligned}$$

This we can write as $\dot{v}(t) = Av(t) + B\tilde{u}(t)$**, for** $\tilde{u} = \dot{u}$**.**

Definition

The system

$$\begin{aligned}\dot{z}(t) &= \mathfrak{A}z(t), \qquad z(0) = z_0 \\ \mathfrak{B}z(t) &= u(t)\end{aligned}$$

is a boundary control system if

- **$\mathfrak{A}$ is a linear operator from $D(\mathfrak{A}) \subset Z$ to Z, $\mathfrak{B}$ is a linear operator from $D(\mathfrak{B}) \subset Z$ to U, and $D(\mathfrak{A}) \subset D(\mathfrak{B})$.**
- **The operator A defined as $Az = \mathfrak{A}z$ with domain $D(A) = D(\mathfrak{A}) \cap \ker \mathfrak{B}$ generates a C_0-semigroup on Z.**
- **The range of $\mathfrak{B}$ equals U.**

□

- **Since $\mathfrak{B}$ is surjective, we can find a B such that for all $u \in U$**
  - $Bu \in D(\mathfrak{B})$ **and**
  - $\mathfrak{B}Bu = u$.
- **Define $v(t) = z(t) - Bu(t)$. Then**
  - **If $z \in D(\mathfrak{A})$, then $v \in D(\mathfrak{A})$, and**
  - **since $\mathfrak{B}v = 0$, we have that $v \in D(A)$.**
- **v satisfies the following abstract differential equation**

$$\begin{aligned} \dot{v}(t) &= \mathfrak{A}z(t) - B\dot{u}(t) \\ &= \mathfrak{A}v(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\ &= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t). \end{aligned}$$

So if $z(t)$ is the classical solution of

$$\begin{aligned} \dot{z}(t) &= \mathfrak{A}z(t), \qquad z(0) = z_0 \\ \mathfrak{B}z(t) &= u(t), \end{aligned}$$

then $v(t) = z(t) - Bu(t)$ is a classical solution of

$$\begin{aligned} \dot{v}(t) &= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\ v(0) &= z_0 - Bu(0). \end{aligned}$$

The reserve direction also holds.

For this last differential equation:

$$\begin{aligned} \dot{v}(t) &= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\ v(0) &= z_0 - Bu(0) \end{aligned}$$

we know the solution

$$v(t) = T(t)v(0) + \int_0^t T(t-\tau)\left[\mathfrak{A}Bu(\tau) - B\dot{u}(\tau)\right] d\tau.$$

Example

We apply this to the transport equation with boundary control

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t), \qquad x \in [0,1] \\ z(1,t) &= u(t). \end{aligned}$$

Here we have that

$$\begin{aligned} \mathfrak{A}z &= \frac{dz}{dx}, \qquad D(\mathfrak{A}) = \{z \in L^2(0,1) \mid \frac{dz}{dx} \in L^2(0,1)\}. \\ \mathfrak{B}z &= z(1), \qquad D(\mathfrak{B}) = D(\mathfrak{A}). \end{aligned}$$

Let us check the assumptions.

$$\begin{aligned}\mathfrak{A}z &= \frac{dz}{dx}, \qquad D(\mathfrak{A}) = \{z \in L^2(0,1) \mid \frac{dz}{dx} \in L^2(0,1)\}.\\ \mathfrak{B}z &= z(1), \qquad D(\mathfrak{B}) = D(\mathfrak{A}).\end{aligned}$$

- **Both operators are linear.**
- $\mathfrak{A}$ **on the domain** $D(\mathfrak{A}) \cap \ker \mathfrak{B}$ **generates a** C_0**-semigroup.**
- **The range of** $\mathfrak{B}$ **is** $\mathbb{C} = U$**.**
- **Choose** $B = 1$**, then** $\mathfrak{B}Bu = u$**.**

Since $\mathfrak{A}B = 0$, we find

$$z(t) - 1 \cdot u(t) = v(t) = T(t)v_0 - \int_0^t T(t-\tau)1\dot{u}(\tau)d\tau.$$

Using the expression of the (shift) semigroup, we find that the solution of the boundary controlled p.d.e. is

$$\begin{aligned} z(x,t) - u(t) &= v_0(x+t)\,\mathbb{1}_{[0,1]}(x+t) - \\ &\quad \int_0^t \mathbb{1}_{[0,1]}(x+t-\tau)\dot{u}(\tau)d\tau. \end{aligned}$$

This we can simplify.

For $t > 1$ this expression becomes

$$
\begin{aligned}
z(x,t) - u(t) &= v_0(x+t)\,\mathbb{1}_{[0,1]}(x+t) - \\
&\quad \int_0^t \mathbb{1}_{[0,1]}(x+t-\tau)\dot{u}(\tau)d\tau \\
&= 0 - [u(\tau)]_{x+t-1}^{t} \\
&= -u(t) + u(x+t-1).
\end{aligned}
$$

For $t \in [0, 1]$ we have to distinguish two cases:

- $x \in [0, 1 - t]$:

$$\begin{aligned} z(x,t) - u(t) &= v_0(x+t)\, \mathbb{1}_{[0,1]}(x+t) - \\ &\quad \int_0^t \mathbb{1}_{[0,1]}(x+t-\tau)\dot{u}(\tau)d\tau \\ &= z_0(x+t) - u(0) - [u(\tau)]_0^t \\ &= z_0(x+t) - u(t). \end{aligned}$$

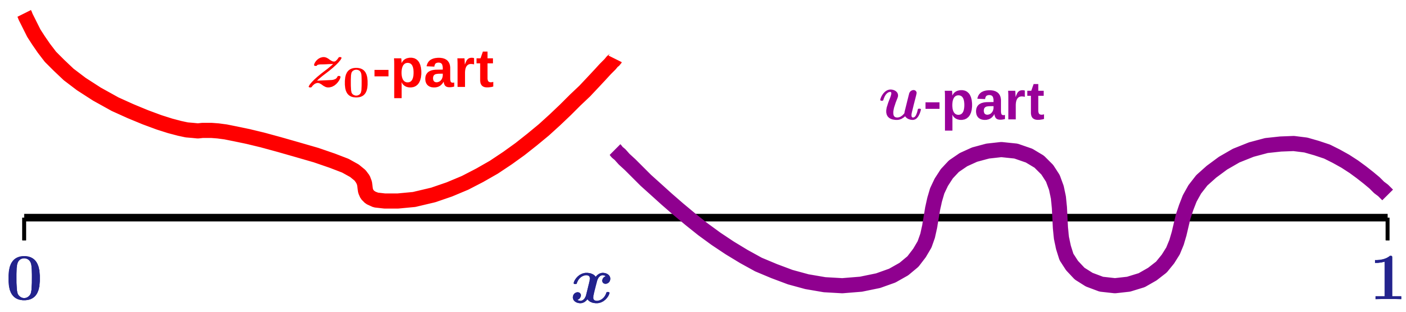
- $x \in [1 - t, 1]$:

$$\begin{aligned} z(x,t) - u(t) &= 0 - [u(\tau)]_{x+t-1}^t \\ &= -u(t) + u(x+t-1). \end{aligned}$$

This we can summarize as follows

$$z(x,t) = \begin{cases} z_0(x+t) & x+t<1 \\ u(x+t-1) & x+t>1 \end{cases}$$

The solution for $t < 1$ is depicted for all $x \in [0,1]$.



Hence for the control transport equation we have found a (weak) solution for all L^2-input functions. □

3.6 Summary

We have seen the following

- **A controlled p.d.e. like**

$$\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[1/2,1]}(x)u(t) \\
\frac{\partial z}{\partial x}(0,t) &= \frac{\partial z}{\partial x}(0,t) = 0 \\
y(t) &= \int_0^{\frac{1}{2}} z(x,t)dx
\end{aligned}$$

can be written as

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ y(t) &= Cz(t) + Du(t) \end{aligned}$$

with B, C, and D bounded operators.

- **We know what we mean by the solution of this (controlled) abstract differential equation, and we have a formula for it.**

- **If the control appears at the boundary, like**

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t) \\ z(1,t) &= u(t) \end{aligned}$$

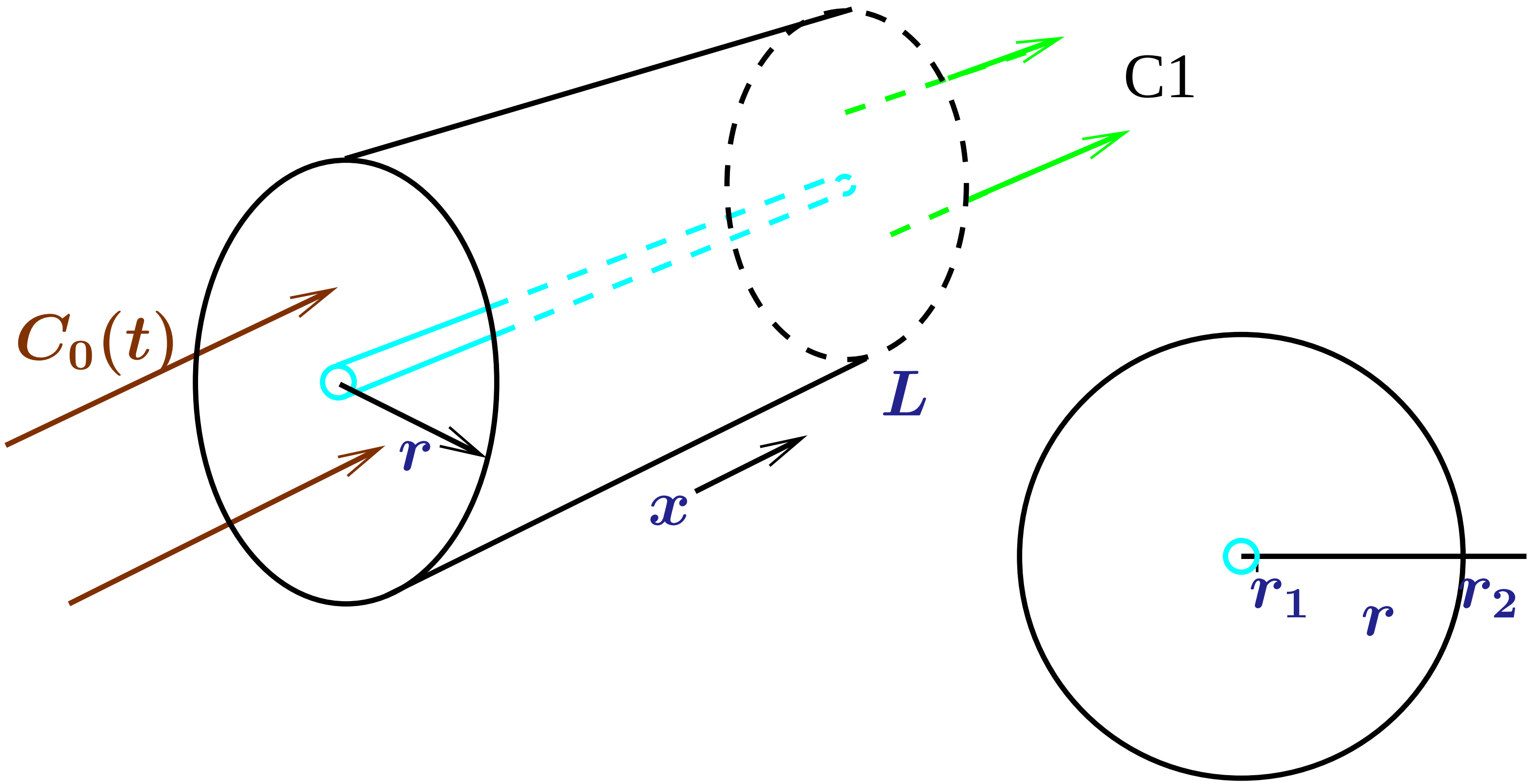
we can write this in the abstract form

$$\begin{aligned} \dot{z}(t) &= \mathfrak{A}z(t) \\ \mathfrak{B}z(t) &= u(t) \end{aligned}$$

- **We know that we have a solution (for smooth inputs) of such a boundary control system.**

With this knowledge let us revisit the UV-reactor model discussed in the first part.

Schematic view point of UV-reactor:



$$
\begin{aligned}
\frac{\partial C}{\partial t}(x,r,t) &= -v_x(r,t)\frac{\partial C}{\partial x}(x,r,t) + \\
&\quad \frac{1}{\mathrm{Pe}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}(x,r,t)\right) + \frac{\partial^2 C}{\partial x^2}(x,r,t)\right] \\
&\quad -\mathrm{Da}\,K(r,t)\,C(x,r,t) \\
\frac{\partial C}{\partial x}(0,r,t) &= 0 \\
C(x,r_1,t) &= C(x,r_2,t) = 0 \\
C(0,r,t) &= C_0(t).
\end{aligned}
$$

4 Transfer Functions

4.1 Introduction

The aim of this part is to define transfer function for systems described by partial differential equations.

We derive these transfer functions via a very simple calculation.

Furthermore, the relation with the impulse response is given.

Consider the simple ordinary differential equation

$$\dot{y}(t) + 5y(t) = 3u(t)$$

The transfer function of this system is given by

$$G(s) = \frac{3}{s+5}$$

How do you come to this?

- **Laplace transform, or**
- **Exponential solutions.**

4.2 Exponential solutions

One way for obtaining the transfer function of

$$\dot{y}(t) + 5y(t) = 3u(t)$$

is to take $u(t) = e^{st}$, $s \in \mathbb{C}$, and to try to find a solution of the same format, i.e., $y(t) = \alpha e^{st}$.

Substituting this in the differential equation, gives

$$s\alpha e^{st} + 5\alpha e^{st} = 3e^{st}.$$

Since e^{st} is non-zero, we may divide by it, and we find

$$s\alpha + 5\alpha = 3.$$

If $s \neq -5$, this is solvable;

$$\alpha = \frac{3}{s+5}.$$

Definition

Given an (abstract) differential equation in the variables $(u(t), z(t), y(t))$, where $u(t)$, $z(t)$, and $y(t)$ take their values in the (Hilbert) spaces U, Z, and Y, respectively.

Let $s \in \mathbb{C}$. If for every $u_0 \in U$, there exists a unique solution of the form $(u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$, and the mapping $u_0 \mapsto y_0$ is linear and bounded, then this mapping is called the transfer function at s, and will be denoted by $G(s)$. $\square$

Consider the abstract differential equation

$$\begin{aligned}\dot{z}(t) &= Az(t) + Bu(t)\\ y(t) &= Cz(t) + Du(t)\end{aligned}$$

with B, C, and D bounded operators.

Let $s \in \mathbb{C}$, and $u_0 \in U$. We try to find a solution of the form $(u(t), z(t), y(t)) = (u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$.

Substituting, this in the abstract differential equation gives

$$\begin{aligned}sz_0 e^{st} &= Az_0 e^{st} + Bu_0 e^{st}\\ y_0 e^{st} &= Cz_0 e^{st} + Du_0 e^{st}.\end{aligned}$$

$$\begin{aligned}(sI - A)z_0 &= Bu_0 \\ y_0 &= Cz_0 + Du_0.\end{aligned}$$

If $sI - A$ is (boundedly) invertible, then we find

$$y_0 = C(sI - A)^{-1}Bu_0 + Du_0.$$

This clearly defines a bounded linear mapping from u_0 to y_0, and so the transfer function at s is given by

$$G(s) = C(sI - A)^{-1}B + D.$$

This holds for all $s \in \rho(A)$.

Example

We take our heated bar. We heat it uniformly at one half, and we measure (half) the average temperature in the other half.

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[\frac{1}{2},1]}(x)u(t) \\ \frac{\partial z}{\partial x}(0,t) &= \frac{\partial z}{\partial x}(1,t) = 0 \\ y(t) &= \int_0^{\frac{1}{2}} z(x,t)dx. \end{aligned}$$

We obtain the transfer function, by two approaches.

Method 1.:

The p.d.e. can be written as

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ y(t) &= Cz(t) \end{aligned}$$

with $Cz = \int_0^{\frac{1}{2}} z(x)dx$, $Bu = \mathbb{1}_{[\frac{1}{2},1]}(x)u$, **and** $Az = \frac{d^2 z}{dx^2}$ **with domain**

$$D(A) = \left\{ z \in L^2(0,1) \mid \frac{dz}{dx}, \frac{d^2 z}{dx^2} \in L^2(0,1), \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0 \right\}.$$

Now we know that

$$G(s) = C(sI - A)^{-1}B.$$

Using the fact that A has an orthonormal basis of eigenfunctions, i.e., $A\phi_n = \lambda_n \phi_n$, we find that

$$(sI - A)^{-1}z = \sum_{n=0}^{\infty} \frac{1}{s - \lambda_n} \langle z, \phi_n \rangle \phi_n.$$

Hence

$$(sI - A)^{-1}Bu = \sum_{n=0}^{\infty} \frac{1}{s - \lambda_n} \langle Bu, \phi_n \rangle \phi_n$$

and

$$C(sI - A)^{-1}Bu = \sum_{n=0}^{\infty} \frac{1}{s - \lambda_n} \langle Bu, \phi_n \rangle C\phi_n.$$

Thus

$$G(s) = \sum_{n=0}^{\infty} \frac{1}{s - \lambda_n} \langle B, \phi_n \rangle C\phi_n.$$

We know that $\lambda_n = -n^2\pi^2$, n=0,1,2,. . . , and

$$\phi_n(x) = \begin{cases} 1 & n = 0 \\ \sqrt{2}\cos(n\pi x) & n \geq 1. \end{cases}$$

Using this we find that

$$\begin{aligned}
G(s) &= \sum_{n=0}^{\infty} \frac{1}{s-\lambda_n} \langle B, \phi_n \rangle C\phi_n \\
&= \frac{1}{s} \int_{1/2}^{1} 1 \cdot 1 dx \int_{0}^{1/2} 1 \cdot 1 dx + \\
&\quad \sum_{n=1}^{\infty} \frac{1}{s+n^2\pi^2} \int_{1/2}^{1} 1 \cdot \sqrt{2}\cos(n\pi x) dx \cdot \\
&\quad \int_{0}^{1/2} 1 \cdot \sqrt{2}\cos(n\pi x) dx.
\end{aligned}$$

Evaluating the integrals gives

$$G(s) = \frac{1}{4s} - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s+n^2\pi^2)}.$$

Method 2:

We try to find an exponential solution of the p.d.e. This gives the following equation

$$\begin{aligned} s z_0(x) e^{st} &= \frac{d^2 z_0}{dx^2}(x) e^{st} + \mathbb{1}_{[\frac{1}{2},1]}(x) u_0 e^{st} \\ \frac{dz_0}{dx}(0) e^{st} &= \frac{dz_0}{dx}(1) e^{st} = 0 \\ y_0 e^{st} &= \int_0^{\frac{1}{2}} z_0(x) e^{st} dx. \end{aligned}$$

Hence

$$
\begin{aligned}
sz_0(x) &= \frac{d^2 z_0}{dx^2}(x) + \mathbb{1}_{[\frac{1}{2},1]}(x)u_0 \\
\frac{dz_0}{dx}(0) &= \frac{dz_0}{dx}(1) = 0 \\
y_0 &= \int_0^{\frac{1}{2}} z_0(x)dx.
\end{aligned}
$$

The first two lines represent an o.d.e. with boundary conditions.

The solution of

$$\begin{aligned} s z_0(x) &= \frac{d^2 z_0}{dx^2}(x) + \mathbb{1}_{[\frac{1}{2},1]}(x) u_0 \\ \frac{dz_0}{dx}(0) &= \frac{dz_0}{dx}(1) = 0 \end{aligned}$$

is given as

$$\begin{aligned} z_0(x) &= \cosh(\sqrt{s}x) z_0(0) - \\ &\quad \frac{1}{\sqrt{s}} \int_0^x \sinh(\sqrt{s}(x-\xi))\, \mathbb{1}_{[1/2,1]}(\xi) u_0 d\xi \end{aligned}$$

with

$$z_0(0) = \frac{\sinh(\sqrt{s}/2) u_0}{s \sinh(\sqrt{s})} = \frac{u_0}{2s \cosh(\sqrt{s}/2)}.$$

Using this we find that

$$y_0 = \int_0^{\frac{1}{2}} z_0(x)dx$$
$$= \frac{\sinh(\sqrt{s}/2)u_0}{2s\sqrt{s}\cosh(\sqrt{s}/2)}.$$

Hence the transfer function is given by

$$G(s) = \frac{\tanh(\sqrt{s}/2)}{2s\sqrt{s}}.$$

The transfer function is unique, and so we find that

$$\frac{\tanh(\sqrt{s}/2)}{2s\sqrt{s}} = G(s) = \frac{1}{4s} - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s+n^2\pi^2)}.$$

□

Example

Consider the system with boundary control and observation

$$\begin{aligned}\frac{\partial z}{\partial t}(x,t) &= \frac{\partial z}{\partial x}(x,t)\\ z(1,t) &= u(t)\\ y(t) &= z(0,t).\end{aligned}$$

Substituting exponential functions for all signals, gives

$$\begin{aligned}sz_0(x)e^{st} &= \frac{dz_0}{dx}(x)e^{st}\\ z_0(1)e^{st} &= u_0e^{st}\\ y_0e^{st} &= z_0(0)e^{st}.\end{aligned}$$

Thus

$$\begin{aligned} sz_0(x) &= \frac{dz_0}{dx}(x) \\ z_0(1) &= u_0 \\ y_0 &= z_0(0). \end{aligned}$$

This is an ordinary differential equation with given (end) condition, u_0 and unknown (initial) condition, y_0.

The solution equals $z_0(x) = e^{s(x-1)}u_0$. Thus $y_0 = e^{-s}u_0$.

The transfer function equals

$$G(s) = e^{-s} \qquad s \in \mathbb{C}.$$

□

4.3 Impulse response

Definition

Let the (abstract) differential equation have the transfer function $G(s)$. If this transfer function exists on some right-half plane and possesses an inverse (one-sided) Laplace transform, then we call the inverse Laplace transform, the impulse response. □

Theorem

Consider the abstract differential equation

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ y(t) &= Cz(t) + Du(t) \end{aligned}$$

with B, C, and D bounded operators.

The impulse response of this differential equation equals

$$h(t) = CT(t)B + D\delta(t), \qquad t \geq 0^-.$$

□

Question

If we take the Laplace transform of the impulse response do we get the transfer function? □

Answer

No.

Example

Let $Z = \ell^2(\mathbb{Z})$, and consider the following differential equation

$$\begin{aligned}
\frac{dz}{dt}(n,t) &= z(n+1,t), \qquad n \in \mathbb{Z} \setminus \{0\} \\
\frac{dz}{dt}(0,t) &= z(1,t) + u(t) \\
y(t) &= z(1,t).
\end{aligned}$$

Substituting the exponential functions, leads to

$$\begin{aligned} s z_0(n) &= z_0(n+1), \qquad n \in \mathbb{Z} \setminus \{0\} \\ s z_0(0) &= z_0(1) + u_0 \\ y_0 &= z_0(1). \end{aligned}$$

From the first equation, we obtain

$$z_0(n) = \begin{cases} s^{n-1} z_0(1) & n \geq 1 \\ s^n z_0(0) & n \leq 0. \end{cases}$$

Since we must have that $z_0 \in \ell^2(\mathbb{Z})$, we find that

$$z_0(n) = \begin{cases} s^{n-1} z_0(1) = 0 & n \geq 1 \text{ and } |s| \geq 1 \\ s^n z_0(0) = 0 & n \leq 0 \text{ and } |s| \leq 1. \end{cases}$$

Using the equation

$$s z_0(0) = z_0(1) + u_0,$$

we find for $|s| > 1$

$$z_0(n) = \begin{cases} 0 & n \geq 1 \\ s^{n-1} u_0 & n \leq 0 \end{cases}$$

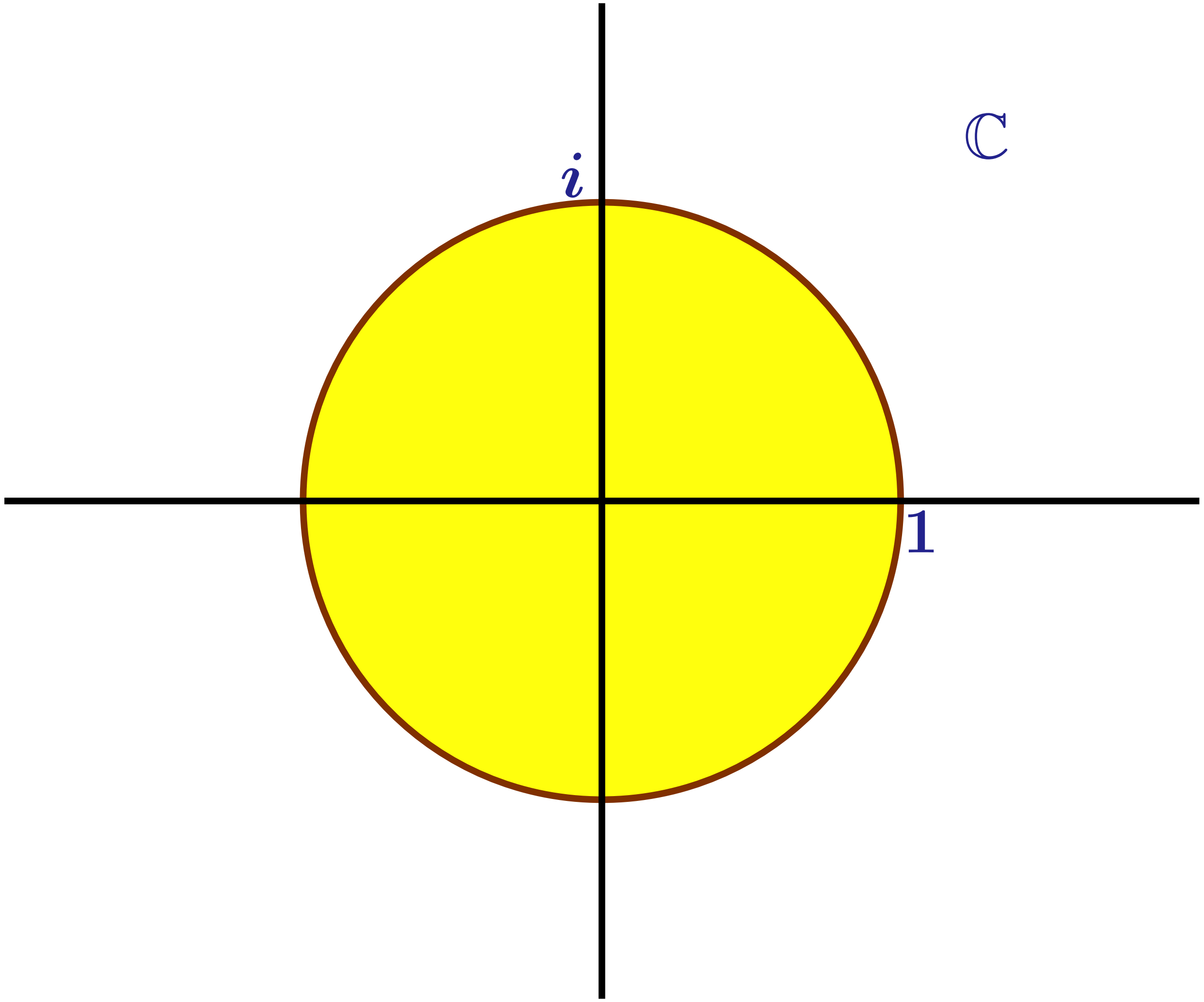
For $|s| < 1$

$$z_0(n) = \begin{cases} -s^{n-1}u_0 & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

So we have that

$$y_0 = z_0(1) = \begin{cases} 0 & |s| > 1 \\ -u_0 & |s| < 1. \end{cases}$$

The transfer function is given by



Yellow means $G(s) = -1$. White means $G(s) = 0$.

The inverse Laplace transform of $G(s)$ gives $h(t) = 0$.

If we take of this the Laplace transform, then it is obviously zero everywhere, and thus not equals to $G(s)$ for all s. □

Remark

The system can be written in the standard format $\dot{z}(t) = Az(t) + Bu(t), y(t) = Cz(t) + Du(t)$ with all operators bounded. □

Theorem

Consider the system

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ y(t) &= Cz(t) + Du(t), \end{aligned}$$

where B, C, and D are bounded operators, and A generates the C_0-semigroup $(T(t))_{t\geq 0}$. If this semigroup satisfies

$$\|T(t)\| \leq Me^{\omega t}, \qquad \text{for all } t > 0,$$

then the transfer function and the Laplace transform of the impulse response exist in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \omega\}$. Furthermore, they are equal on this set. □

Example (revisited)

In our previous example we have that the semigroup is bounded by e^t, and so the theorem tells us that on the right-half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$ the transfer function and the Laplace transform of the impulse response are the same.

Since $h(t) = 0$ and since $G(s) = 0$ on this right-half plane, we are in agreement with the theorem.

We also see that (in general) a larger half-plane is not possible. □

Theorem

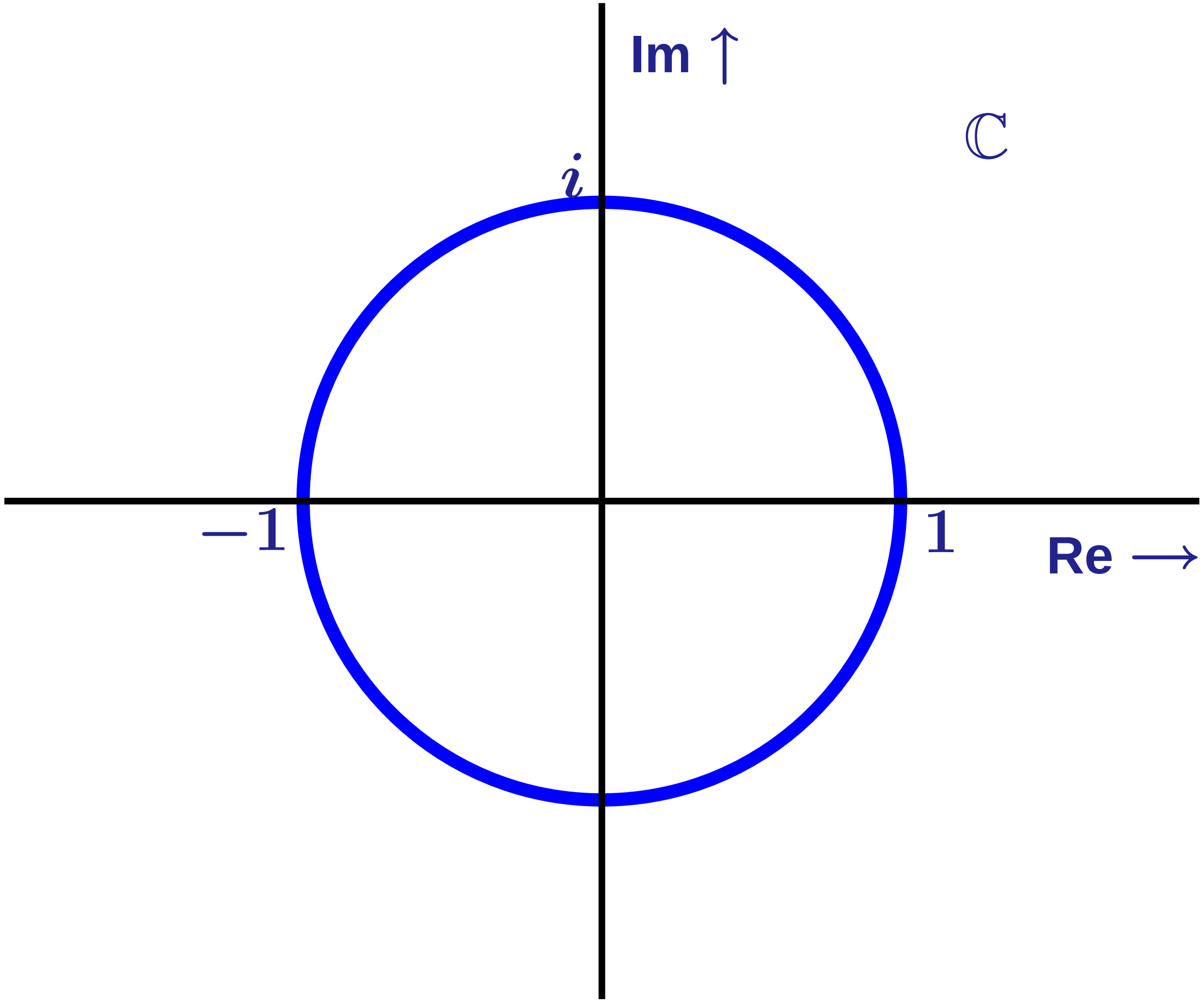
If the transfer function $G(s)$ is a meromorphic function on $\mathbb{C}$, then the meromorphic continuation of the Laplace transform of the impulse response is equal to $G(s)$ on $\mathbb{C}$. □

4.4 Summary

In this part we have seen the following

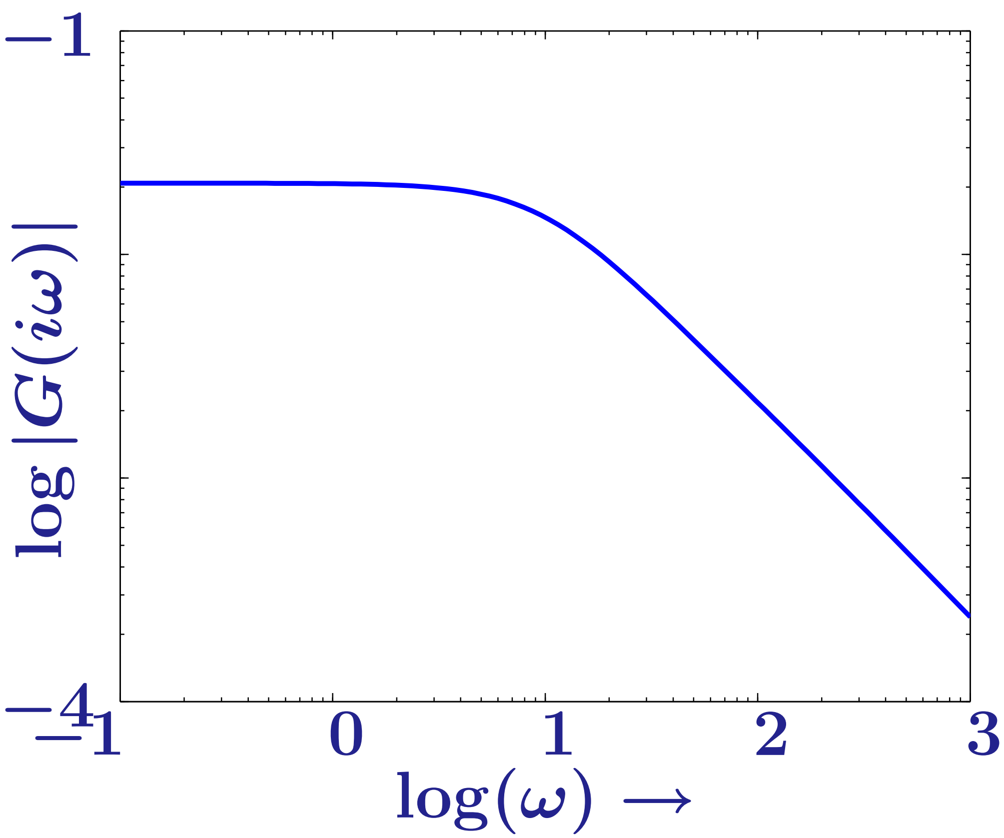
- **Transfer functions are easy to find by using exponential solutions.**
- **You don't have to write a p.d.e. as an abstract differential equation in order to obtain the transfer function.**
- **It is not hard to find the transfer function of a system with boundary control and observation .**
- **The Laplace transform of the impulse response only equals the transfer function on some right-half plane.**
- **For scalar transfer function the design rules of Bode and Nyquist still apply.**

Here we show the Nyquist plot of e^{-st}.



The (magnitude) Bode plot of

$$G(s) = -2\sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s+n^2\pi^2)}$$



5 Stability and Stabilizability

5.1 Introduction

In this part we want to define stability and show how these notions relate to similar notions for ordinary differential equations.

Furthermore, we show how we can stabilize systems.

5.2 Stability notions

There are several stability notions for the abstract differential equation

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0.$$

We assume that A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z.

Definition

The semigroup $(T(t))_{t\geq 0}$ is exponentially stable if there exists $M, \omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}, \qquad t \geq 0.$$

□

Definition

The semigroup $(T(t))_{t\geq 0}$ is strongly stable if for all $z_0 \in Z$

$$\lim_{t\to\infty} T(t)z_0 = 0.$$

□

5.3 Stability of platoon models

Consider the the platoon model

$$\dot{z}(t) = \check{A}z(t), \qquad z(0) = z_0 \in Z$$

where $Z = L^2(\partial\mathbb{D}; \mathbb{C})$, and $\left(\check{A}z(t)\right)(\phi) = \check{A}(\phi)z(\phi,t)$, $\phi \in \partial\mathbb{D}$.

Theorem

Assume that $\check{A}$ is a continuous function of ϕ. The above system is exponentially stable if and only if there exists a $\delta > 0$ such that for all $\phi \in \partial\mathbb{D}$ the eigenvalues of $\check{A}(\phi)$ have real part less or equal to $-\delta$.

Example

Consider our platoon model

$$\frac{d}{dt}\begin{pmatrix} d_r(t) \\ v_r(t) \\ a_r(t) \end{pmatrix} = \begin{pmatrix} v_{r-1}(t) - v_r(t) \\ a_r(t) \\ -\tau^{-1}a_r(t) \end{pmatrix}, \qquad r \in \mathbb{Z}.$$

For this model $\check{A}$ is given by

$$\check{A}(\phi) = \begin{pmatrix} 0 & \phi^{-1} - 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau^{-1} \end{pmatrix}, \quad \phi \in \partial\mathbb{D}.$$

It is clear that this is not exponentially stable (also not strongly stable). □

5.4 Exponential stability and Lyapunov equations

The following theorem relates exponential stability with the solution of a Lyapunov equation.

Theorem

The following are equivalent.

- $(T(t))_{t\geq 0}$ **is exponentially stable.**
- **For all** $z_0 \in Z$ **we have that** $T(t)z_0 \in L^2((0,\infty); Z)$.
- **There exists a positive** $L \in \mathcal{L}(Z)$ **satisfying the following Lyapunov equation**

$$\langle Az, Lz \rangle + \langle Lz, Az \rangle = -\langle z, z \rangle \qquad z \in D(A).$$

□

Some remarks concerning the Lyapunov equation.

Lemma

For a positive operator $L \in \mathcal{L}(Z)$ the following are equivalent:

- L **satisfies**

$$\langle Az, Lz\rangle + \langle Lz, Az\rangle = -\langle z, z\rangle \qquad \forall z \in D(A).$$

- L **satisfies**

$$\langle Az_1, Lz_2\rangle + \langle Lz_1, Az_2\rangle = -\langle z_1, z_2\rangle \qquad \forall z_1, z_2 \in D(A).$$

- L **maps the domain of** A **into the domain of** A^***, and** L **satisfies**

$$A^*L + LA = -I \qquad \text{on } D(A).$$

Proof of the equivalence between the last two items.

Assume that $z_0 \in L^2((0,\infty);Z)$ for all $z_0 \in Z$. For $z_1, z_2 \in Z$, define

$$\langle z_1, Lz_2\rangle = \int_0^\infty \langle T(t)z_1, T(t)z_2\rangle dt.$$

This operator is well-defined, self-adjoint, and positive. Furthermore, for $z_1, z_2 \in D(A)$

$$\langle Az_1, Lz_2\rangle + \langle z_1, LAz_2\rangle$$
$$= \int_0^\infty \langle T(t)Az_1, T(t)z_2\rangle dt + \int_0^\infty \langle T(t)z_1, T(t)Az_2\rangle dt.$$

Hence

$$
\begin{aligned}
&\langle Az_1, Lz_2\rangle + \langle z_1, LAz_2\rangle \\
&\quad = \int_0^\infty \langle T(t)Az_1, T(t)z_2\rangle dt + \int_0^\infty \langle T(t)z_1, T(t)Az_2\rangle dt \\
&\quad = \int_0^\infty \frac{d}{dt}\left[\langle T(t)z_1, T(t)z_2\rangle\right] dt \\
&\quad = 0 - \langle z_1, z_2\rangle.
\end{aligned}
$$

If the Lyapunov equation has a positive solution, then

$$\int_0^{t_1} \|T(t)z_0\|^2 dt$$

$$\begin{aligned} &= \int_0^{t_1} \langle T(t)z_0, T(t)z_0\rangle dt = \int_0^{t_1} \langle T(t)z_0, I T(t)z_0\rangle dt \\ &= -\int_0^{t_1} \langle AT(t)z_0, LT(t)z_0\rangle dt \\ &\quad -\int_0^{t_1} \langle T(t)z_0, LAT(t)z_0\rangle dt \\ &= -\int_0^{t_1} \frac{d}{dt}\left[\langle T(t)z_0, LT(t)z_0\rangle\right] dt \\ &= -\langle T(t_1)z_0, LT(t_1)z_0\rangle + \langle z_0, Lz_0\rangle. \end{aligned}$$

Hence for all $t_1 > 0$

$$\int_0^{t_1} \|T(t)z_0\|^2 dt \leq \|L\| \|z_0\|^2$$

and so $T(t)z_0 \in L^2((0,\infty);Z)$.

Thus we have proved the equivalence between the last two items.

Proof of the equivalence between the first two items.

If $T(t)z_0 \in L^2((0,\infty), Z)$ for all z_0, then

- $\|T(t)\| \leq M$ for all $t > 0$
- The following estimate holds

$$
\begin{aligned}
t\|T(t)z_0\|^2 &= \int_0^t \|T(t)z_0\|^2 dt \\
&= \int_0^t \|T(t-\tau)T(\tau)z_0\|^2 d\tau \\
&\leq M^2 \int_0^t \|T(\tau)z_0\|^2 d\tau \leq K\|z_0\|^2.
\end{aligned}
$$

Thus

$$\|T(t)z_0\|^2 \leq \frac{K}{t}\|z_0\|^2.$$

Now using the semigroup property, we find that $(T(t))_{t\geq 0}$ is exponentially stable. □

There are examples of unstable semigroups for which the infinitesimal generator A has no spectrum in the set $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$.

Hence, in general, we cannot conclude stability by only looking at the spectrum of A. However,

Theorem

The semigroup $(T(t))_{t\geq 0}$ is exponentially stable if and only if

$$\sup_{\{s\in\mathbb{C}\mid\mathrm{Re}(s)>0\}} \|(sI - A)^{-1}\| < \infty.$$

□

5.5 Strong stability

We may also conclude exponential stability when there exists a positive L such that

$$A^*L + LA \leq -\varepsilon I, \qquad \varepsilon > 0.$$

Question

Is the semigroup strongly stable when there exists an $L \in \mathcal{L}(Z)$, $L > 0$ such that

$$A^*L + LA < 0 \qquad \text{on } D(A)?$$

Answer

No.

Example

Take the Hilbert space

$$Z = \left\{ f : [0, \infty) \mapsto \mathbb{C} \mid \int_0^\infty |f(x)|^2 \left[e^{-x} + 1\right] dx < \infty \right\}.$$

It inner product is given by:

$$\langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)} \left[e^{-x} + 1\right] dx.$$

Note that Z "is" $L^2(0, \infty)$.

As semigroup, we choose the shift:

$$
\begin{aligned}
(T(t)f)(x) &= \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t) \end{cases} \\
&= f(x-t)\,\mathbb{1}_{[0,\infty]}(x-t).
\end{aligned}
$$

There holds

$$\begin{aligned}
\|T(t)f\|^2 &= \int_0^\infty |f(x-t)\,\mathbb{1}_{[0,\infty]}(x-t)|^2\left[e^{-x}+1\right]dx \\
&= \int_0^\infty |f(\xi)|^2\left[e^{-(\xi+t)}+1\right]d\xi \\
&\geq \int_0^\infty |f(\xi)|^2 d\xi \\
&\geq \frac{1}{2}\int_0^\infty |f(\xi)|^2\left[e^{-\xi}+1\right]d\xi \\
&= \frac{1}{2}\|f\|^2.
\end{aligned}$$

Hence $(T(t))_{t\geq 0}$ is not strongly stable.

The infinitesimal generator is given by

$$Af = -\frac{df}{dx}$$

with domain

$$D(A) = \Big\{ f \in Z \mid f \text{ is absolutely continuous } \frac{df}{dx} \in Z, \text{ and } f(0) = 0 \Big\}.$$

Next we evaluate

$$\langle Az, z\rangle + \langle z, Az\rangle$$

for $z \in D(A)$ **and** $z \neq 0$**.**

For $A = -\frac{d}{dx}$ with boundary condition $z(0) = 0$, we find

$$\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle \\
&= \int_0^\infty (-1)\frac{dz}{dx}(x)\overline{z(x)}\left[e^{-x}+1\right]dx + \\
&\quad \int_0^\infty z(x)(-1)\overline{\frac{dz}{dx}(x)}\left[e^{-x}+1\right]dx \\
&= -\int_0^\infty \frac{d}{dx}\left(|z(x)|^2\right)\left[e^{-x}+1\right]dx.
\end{aligned}$$

Hence

$$
\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle \\
&= -\left[|z(x)|^2\left[e^{-x}+1\right]\right]_0^\infty + \\
&\quad \int_0^\infty |z(x)|^2\left[-e^{-x}\right]dx \\
&= 0 - \int_0^\infty |z(x)|^2 e^{-x} dx \\
&< 0.
\end{aligned}
$$

Concluding, we have constructed an infinitesimal generator A for which the semigroup is not strongly stable, but the Lyapunov inequality

$$\langle Az, z\rangle + \langle z, Az\rangle < 0, \qquad z \in D(A), z \neq 0$$

holds. Note that $L = I$. □

Remark

If the trajectories, $t \mapsto T(t)z_0$, are pre-compact, then this Lyapunov inequality implies strong stability. □

Theorem

Let $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on the Hilbert space Z. If

- **The semigroup is uniformly bounded, i.e., $\|T(t)\| \leq M$**
- **A has no eigenvalues on the imaginary axis**
- **The spectrum on the imaginary axis is countable,**

then $(T(t))_{t\geq 0}$ is strongly stable.

5.6 Stabilizability

In this section we study the question whether there exist a feedback, $u = Fz$ which stabilizes the system

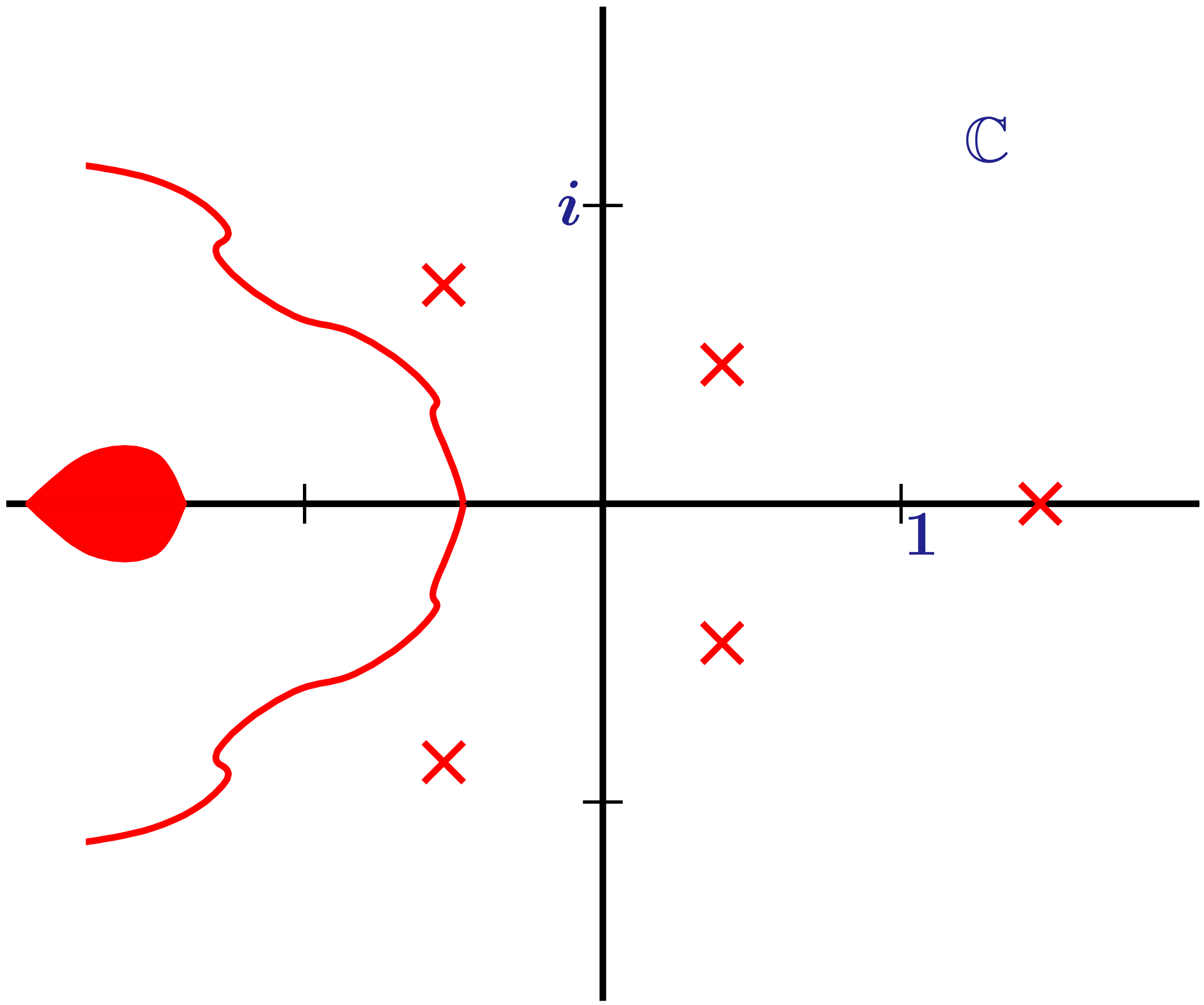
$$\dot{z}(t) = Az(t) + Bu(t).$$

That is, the operator $A + BF$ generates an exponentially stable semigroup.

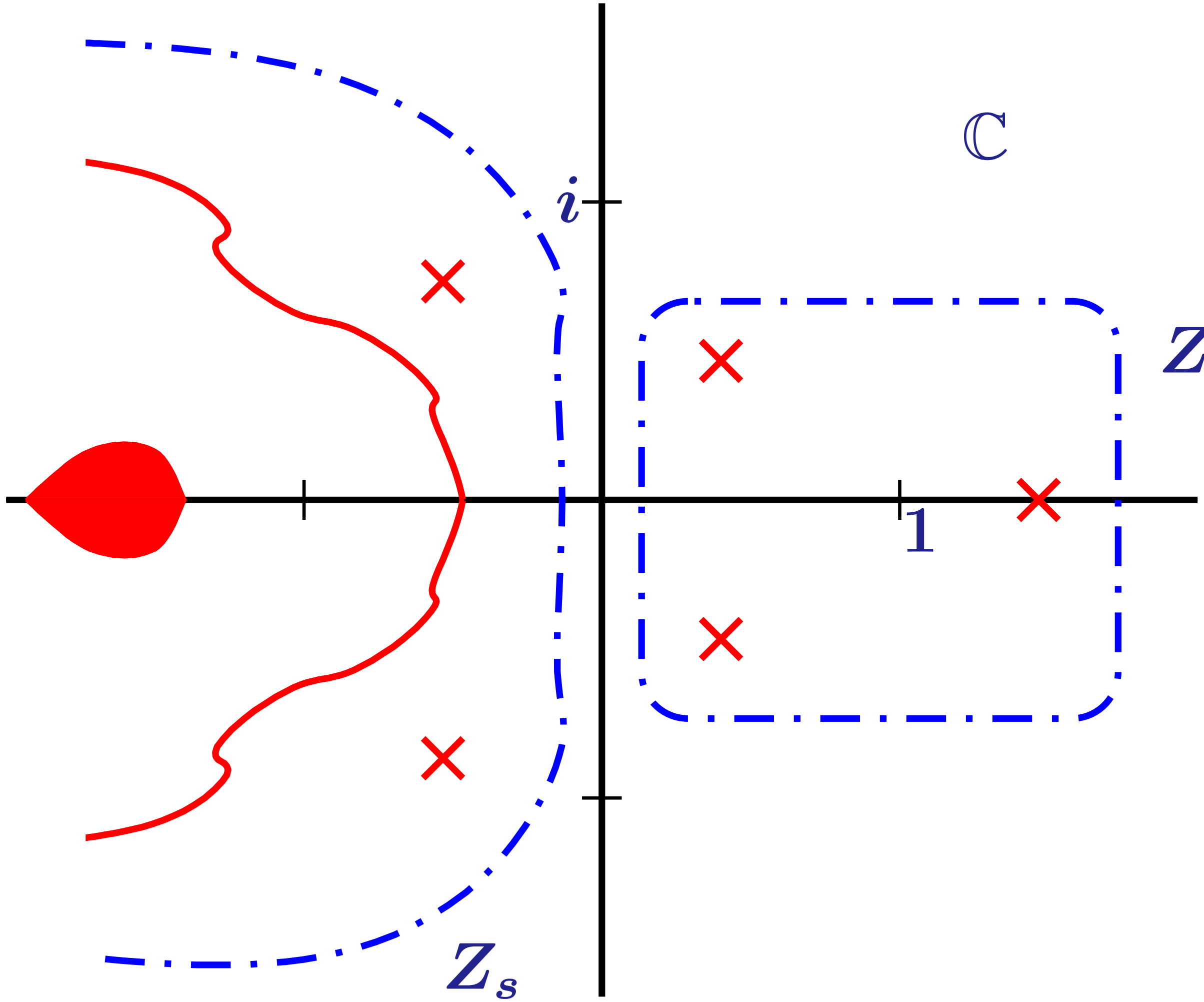
Theorem

Consider $\dot{z}(t) = Az(t) + Bu(t)$, with $B \in \mathcal{L}(\mathbb{C}^m, Z)$. The following are equivalent.

- **There exists an $F \in \mathcal{L}(Z, \mathbb{C}^m)$ such that $A + BF$ generates an exponentially stable semigroup on Z.**
- **Z can be decomposed as $Z = Z_u \oplus Z_s$ with**
  - $\dim(Z_u) < \infty$.
  - **For all $t > 0$ we have that $T(t)Z_u \subset Z_u$ and $T(t)Z_s \subset Z_s$.**
  - **$T(t)|_{Z_s}$ is exponentially stable.**
  - **The system restricted to Z_u is controllable.**



Red denotes the spectrum.


ℂ
i
Z_u
1
Z_s

Example

We consider the heated bar. We heat it uniformly at one half, and we measure (half) the average temperature in the other half.

$$\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[\frac{1}{2},1]}(x)u(t)\\
\frac{\partial z}{\partial x}(0,t) &= \frac{\partial z}{\partial x}(1,t) = 0\\
y(t) &= \int_0^{\frac{1}{2}} z(x,t)dx.
\end{aligned}$$

We know that the corresponding A has the eigenvalues

$\lambda_n = -n^2\pi^2, n = 0, 1, 2, \ldots$, and the eigenfunctions

$$\phi_n(x) = \begin{cases} 1 & n = 0 \\ \sqrt{2}\cos(n\pi x) & n = 1, 2, \ldots \end{cases}$$

This A has an unstable eigenvalue, $\lambda_0 = 0$. Now we define

$$Z_u = \mathrm{span}\{\phi_0\}$$

$$Z_s = \mathrm{span}_{n=1,2,\ldots}\{\phi_n\} = Z_u^{\perp}.$$

Let us check the conditions.

- $\dim(Z_u) = 1 < \infty$.
- $T(t)Z_u \subset Z_u$ and $T(t)Z_s \subset Z_s$.
- $$T(t)|_{Z_s} = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle \cdot, \phi_n\rangle \phi_n.$$

  This is exponentially stable.

- A restricted to Z_u is $A_u = [0]$, and the projection of B onto Z_u is $B_u = \langle B, \phi_0\rangle = [1/2]$.

  The pair (A_u, B_u) is controllable.

Hence the system is stabilizable.

You can stabilize it by stabilizing the finite-dimensional part, and not effecting the stable part.

A feedback that works is

$$Fz = -k\langle z, \phi_0\rangle\phi_0,$$

with $k > 0$**.** □

5.7 Stabilization of the platoon model

After Fourier transform our platoon model becomes

$$\frac{\partial \check{z}}{\partial t}(\phi, t) = \check{A}(\phi)\check{z}(\phi, t) + \check{B}(\phi)\check{u}(\phi, t),$$

with $\phi \in \partial\mathbb{D}$, and

$$\check{A}(\phi) = \begin{pmatrix} 0 & \phi^{-1} - 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau^{-1} \end{pmatrix}, \quad \check{B}(\phi) = \begin{pmatrix} 0 \\ 0 \\ \tau^{-1} \end{pmatrix}.$$

Note that $U = L^2(\partial\mathbb{D})$. So infinite-dimensional.

Suppose that we apply a state feedback of the type $\check{u}(\phi, t) = \check{F}(\phi)\check{z}(\phi, t)$ with

$$\check{F}(\phi) = \begin{pmatrix} f_1(\phi) & f_2(\phi) & f_3(\phi) \end{pmatrix}$$

Then $\check{A}(\phi) + \check{B}(\phi)\check{F}(\phi)$ equals

$$\begin{pmatrix} 0 & \phi^{-1} - 1 & 0 \\ 0 & 0 & 1 \\ \tau^{-1} f_1(\phi) & \tau^{-1} f_2(\phi) & \tau^{-1} f_3(\phi) - \tau^{-1} \end{pmatrix}.$$

Exponentially stable?

5.8 Transfer function and stability

One can check stability by looking at the transfer function. We can do that for the state-space system.

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ y(t) &= Cz(t) + Du(t). \end{aligned}$$

Theorem

Consider the (standard) state space system with B, C, and D bounded operators. Let the input and output space be finite-dimensional.

Assume further that there exists a $F \in \mathcal{L}(Z, U)$ and a $K \in \mathcal{L}(Y, Z)$ such that $A + BF$ and $A + KC$ generate exponentially stable semigroups.

The semigroup generated by A is exponentially stable if and only if the transfer function $G(s) = C(sI - A)^{-1}B + D$ is analytic and bounded in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$. □

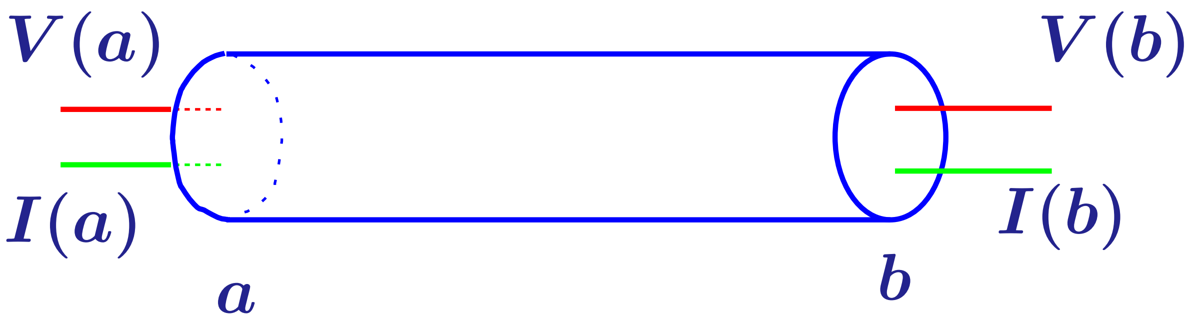
5.9 Summary

- **We have introduce different notions of stability for a C_0-semigroup.**
- **For exponential stability there are nice equivalent conditions.**
- **For strong stability there are nice sufficient conditions, but they are not sufficient.**
- **If the input is finite-dimensional, then there is a complete characterizing of stabilizability.**

6 Linear Port Hamiltonian Systems

Warning: x will denote the state from the state space X, and ζ denotes the spatial variable.

6.1 A typical example



Consider the transmission line on the spatial interval $[a, b]$

$$\frac{\partial Q}{\partial t}(\zeta, t) = -\frac{\partial}{\partial \zeta} \frac{\phi(\zeta, t)}{L(\zeta)}$$
$$\frac{\partial \phi}{\partial t}(\zeta, t) = -\frac{\partial}{\partial \zeta} \frac{Q(\zeta, t)}{C(\zeta)}.$$

We write $x_1 = Q$ (charge) and $x_2 = \phi$ (flux), and we find that

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(\zeta,t) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix} \frac{1}{C(\zeta)}x_1(\zeta,t) \\ \frac{1}{L(\zeta)}x_2(\zeta,t) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix}\begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix}\right]$$
$$= P_1\frac{\partial}{\partial \zeta}(\mathcal{H}x)(\zeta,t)$$

So we have, $J = P_1\frac{\partial}{\partial \zeta}$ (skew-symmetric) and the Hamiltonian (energy) equals $H = \frac{1}{2}\int_a^b x^T\mathcal{H}x d\zeta$ (positive).

To use this structure we differentiate the Hamiltonian (energy) along trajectories.

$$\begin{aligned}
\frac{dH}{dt}(t) &= \frac{1}{2}\int_a^b \frac{\partial x}{\partial t}(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) dz + \\
&\quad \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) \frac{\partial x}{\partial t}(\zeta,t) dz \\
&= \frac{1}{2}\int_a^b \left(P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x)(\zeta,t) \right)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta + \\
&\quad \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta,t) (P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x)(\zeta,t) d\zeta \\
&= \frac{1}{2}\int_a^b \frac{\partial}{\partial \zeta} \left[(\mathcal{H}x)^T (\zeta,t) P_1 (\mathcal{H}x)(\zeta,t) \right] d\zeta \\
&= \frac{1}{2} \left[(\mathcal{H}x)^T (\zeta,t) P_1 (\mathcal{H}x)(\zeta,t) \right]_a^b ,
\end{aligned}$$

where we have used the symmetry of P_1 and $\mathcal{H}$.

So we have that the time-change of Hamiltonian satisfies

$$\frac{dH}{dt}(t) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b. \tag{1}$$

That is the change of internal energy goes via the boundary (ports).

Note the we only used that P_1 and $\mathcal{H}$ are symmetric. We did not need the specific form of P_1 or $\mathcal{H}$.

The balance equation (1) also holds for the system

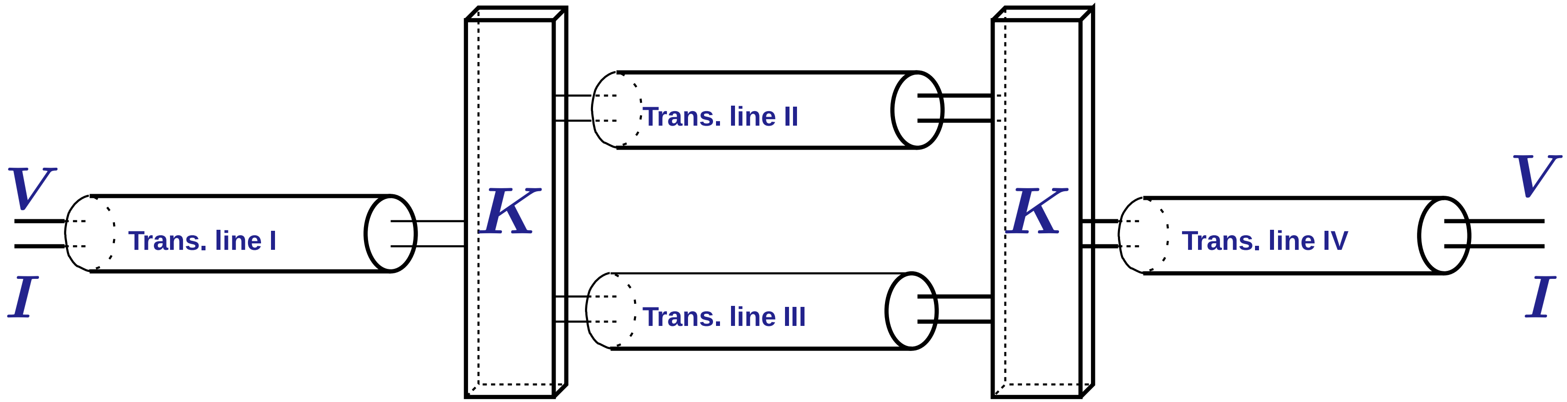
$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta,t) + P_0\left[\mathcal{H}x\right](\zeta,t) \tag{2}$$

with P_0 anti-symmetric, i.e., $P_0^T = -P_0$.

Many (hyperbolic) systems can be written in this format.

Example

Consider transmission lines in a network



In the coupling parts K, we have that Kirchhoff laws holds. Hence charge flowing out of the transmission line I, enters II and III, etc.

The P_1 of the big system is the diagonal matrix, build from the uncoupled P_1's (which are all the same). The $\mathcal{H}$ of the coupled system is the diagonal matrix of the uncoupled $\mathcal{H}$'s.

The coupling is written down as boundary conditions of the pde.

7 Homogeneous solutions of P.H.S.

Consider the p.d.e. of our Port-Hamiltonian system

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta, t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta, t)\right].$$

with energy (Hamiltonian)

$$E(t) = \frac{1}{2}\int_a^b x(\zeta, t)^T \mathcal{H}(\zeta)\overline{x(\zeta, t)}d\zeta.$$

We assume that

- P_1 **is real and symmetric, i.e.,** $P_1^T = P_1$,
- P_0 **is real and anti-symmetric, i.e.,** $P_0^T = -P_0$,
- $\mathcal{H}(\zeta)$ **is a positive (real) symmetric matrix, uniformly satisfying** $0 < mI \le \mathcal{H}(\zeta) \le MI$, **and continuously differentiable.**

The energy

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta)\overline{x(\zeta,t)}d\zeta.$$

can be seen as a weighted L^2-norm.

Associated to this p.d.e. we define the state space

$$X = L^2((a,b);\mathbb{C}^n)$$

with inner product

$$\langle f, g \rangle = \frac{1}{2}\int_a^b f(\zeta)^T \mathcal{H}(\zeta)\overline{g(\zeta)}d\zeta.$$

and the differential operator

$$\mathfrak{A}x := P_1 \frac{d}{d\zeta}\left[\mathcal{H}x\right] + P_0\left[\mathcal{H}x\right]$$

with domain

$$D(\mathfrak{A}) = \{x \in X \mid x \text{ is absolutely continuous, and } \frac{dx}{d\zeta} \in X\}.$$

Thus the squared norm of x equals the energy of this state.

The following balance equation holds

Lemma

On the space X, the operator $\mathfrak{A}x = P_1 \frac{d}{d\zeta}\left[\mathcal{H}x\right] + P_0\left[\mathcal{H}x\right]$ satisfies

$$\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}x\rangle$$
$$= \frac{1}{2}\left[(\mathcal{H}x)^T(b)P_1\overline{(\mathcal{H}x)(b)} - (\mathcal{H}x)^T(a)P_1\overline{(\mathcal{H}x)(a)}\right]$$

for $x \in D(\mathfrak{A})$. □

Proof (for $P_0 = 0$) Using the definition of the inner product, we find

$$\begin{aligned}
&\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}x\rangle \\
&\quad = \frac{1}{2}\int_a^b \left[P_1 \frac{\partial}{\partial\zeta}\left(\mathcal{H}x\right)(\zeta)\right]^T \mathcal{H}(\zeta)\overline{x(\zeta)}d\zeta + \\
&\qquad \frac{1}{2}\int_a^b x(\zeta)^T\mathcal{H}(\zeta)\overline{\left[P_1 \frac{\partial}{\partial\zeta}\left(\mathcal{H}x\right)(\zeta)\right]}d\zeta.
\end{aligned}$$

Using the fact that $P_1, \mathcal{H}(\zeta)$ are real symmetric, we write the last

expression as

$$\frac{1}{2}\int_a^b \left[\frac{d}{d\zeta}(\mathcal{H}x)(\zeta)\right]^T P_1 \mathcal{H}(\zeta)\overline{x(\zeta)} +$$

$$[\mathcal{H}(\zeta)x(\zeta)]^T \left[P_1 \frac{d}{d\zeta}(\mathcal{H}\overline{x})(\zeta)\right] d\zeta$$

$$= \frac{1}{2}\int_a^b \frac{d}{d\zeta}\left[(\mathcal{H}x)^T(\zeta) P_1 \overline{(\mathcal{H}x)(\zeta)}\right] d\zeta$$

$$= \frac{1}{2}\left[(\mathcal{H}x)^T(b) P_1 \overline{(\mathcal{H}x)(b)} - (\mathcal{H}x)^T(a) P_1 \overline{(\mathcal{H}x)(a)}\right].$$

□

Theorem

Consider the operator A which is defined as

$$Ax = \mathfrak{A}x = P_1 \frac{\partial}{\partial\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x)$$

on the domain $D(A) = D(\mathfrak{A}) \cap \ker \mathfrak{B}$, where

$$\mathfrak{B}x = M \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix}.$$

If M is $n \times 2n$-matrix of full rank, then A generates a contraction semigroup on X if and only if

$$\langle Ax, x\rangle + \langle x, Ax\rangle \leq 0 \quad \text{for all } x \in D(A).$$

□

Hence by posing n boundary conditions, we only have to check the inequality.

Since we have the power balance this can be done quickly.

If $\mathcal{H}$ is non-constant, it can be very hard (impossible) to obtain the expression for the C_0-semigroup.

Example

Consider the transmission line.

$$\mathfrak{A}x = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{d}{d\zeta} \left[\begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix} \begin{pmatrix} x_1(\zeta) \\ x_2(\zeta) \end{pmatrix} \right]$$

and

$$\begin{aligned} \langle \mathfrak{A}x, x \rangle + \langle x, \mathfrak{A} \rangle &= \frac{1}{2} \left[(\mathcal{H}x)^T (b) P_1 \overline{(\mathcal{H}x)(b)} - (\mathcal{H}x)^T (a) P_1 \overline{(\mathcal{H}x)(a)} \right] \\ &= V(a)I(a) - V(b)I(b), \end{aligned}$$

since $x_1/C = Q/C = V$ **and** $x_2/L = \phi/L = I$.

If we choose $V(a) = 0$ and $V(b) = RI(b)$, $R \geq 0$, then

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & -R & 0 & 0 \end{bmatrix}.$$

This matrix has rank $n = 2$.

Furthermore, using these boundary conditions, we have that

$$V(a)I(a) - V(b)I(b) \leq 0.$$

Hence the operator A generates a contraction semigroup on Z. $\square$

8 Well-posedness for PHS

For $t \geq 0$ and $\zeta \in [a, b]$ we consider the system:

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial z}(\mathcal{H}x)(\zeta, t), \quad x(0, z) = x_0(\zeta) \tag{3}$$

$$0 = M_{11}(\mathcal{H}x)(b, t) + M_{12}(\mathcal{H}x)(a, t) \tag{4}$$

$$u(t) = M_{21}(\mathcal{H}x)(b, t) + M_{22}(\mathcal{H}x)(a, t) \tag{5}$$

$$y(t) = C_1(\mathcal{H}x)(b, t) + C_2(\mathcal{H}x)(a, t). \tag{6}$$

We assume that x takes values in $\mathbb{C}^n$, $P_1^T = P_1, \det(P_1) \neq 0$, $\mathcal{H} > 0$, and that $\operatorname{rank} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ C_1 & C_2 \end{bmatrix} = n + \operatorname{rank} \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

Definition

Consider the system (3)–(6). This system is *well-posed* if there exists a $t_f > 0$ and m_f such that the following holds:

1. **The homogeneous p.d.e., i.e., $u \equiv 0$ has for any initial condition, x_0, a unique (weak) solution.**

2. **The following inequality holds for all smooth initial conditions, and all smooth inputs**

$$\|x(t_f)\|^2 + \int_0^{t_f} \|y(t)\|^2 dt \leq m_f \left[\|x_0\|^2 + \int_0^{t_f} \|u(t)\|^2 dt \right],$$

where $\| \cdot \|^2$ is the energy/Hamiltonian: $\frac{1}{2} \int_a^b x^T \mathcal{H} \overline{x} d\zeta$.

□

Remark

- The semigroup need not to be a contraction.
- So well-posedness gives you that you have a unique solution for every square integrable input function and every initial condition. Furthermore, the output signal is always square integrable.

8.1 Well-posedness for simple PHS

Consider the (very) simple port-Hamiltonian system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= v\frac{\partial x}{\partial \zeta}(\zeta,t) \\ vx(b,t) &= u(t) \\ y(t) &= vx(a,t), \end{aligned}$$

with v a positive constant.

The norm (energy/Hamiltonian) is given by $\frac{1}{2}\int_a^b v\|f(\zeta)\|^2 d\zeta$.

We know that for smooth inputs and smooth initial conditions, there exists a solution. For this (classical) solution the balance equation reads as

$$\begin{aligned}\frac{d}{dt}\|x(t)\|^2 &= \frac{1}{2}\left[|vx(\zeta,t)|^2\right]_a^b \\ &= \frac{1}{2}\left[|u(t)|^2 - |y(t)|^2\right]\end{aligned}$$

Hence, we have that for all $t_f > 0$

$$\|x(t_f)\|^2 + \int_0^{t_f} \|y(t)\|^2 dt = \left[\|x_0\|^2 + \int_0^{t_f} \|u(t)\|^2 dt\right].$$

Since the smooth inputs and smooth initial conditions are dense in X **and** $L^2(0, t_f)$**, respectively, we find that this system is well-posed.**

A similar result holds for the other (very) simple port-Hamiltonian system

$$
\begin{aligned}
\frac{\partial x}{\partial t}(\zeta, t) &= -v\frac{\partial x}{\partial \zeta}(\zeta, t) \\
vx(a, t) &= u(t) \\
y(t) &= vx(b, t),
\end{aligned}
$$

with v a positive constant.

The energy is the same as for the previous p.d.e.

8.2 Characterization of well-posedness for PHS

Theorem

Under the conditions we have imposed, the following holds:

- **If condition 1 holds, then automatically condition 2 holds.**
- **That is: If the homogeneous p.d.e., i.e., $u \equiv 0$, has a weak solution, then the system (3)–(6) is well-posed.**
- **There is a matrix condition for checking condition 1.**
- **Same theorem holds for**
$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x)(\zeta, t) + P_0 (\mathcal{H}x)(\zeta, t).$$

We want to give an idea of the proof. Therefor we do it for our transmission line system with the following choice of boundary input and output

$$u(t) = \begin{pmatrix} V(a,t) \\ V(b,t) \end{pmatrix}, \qquad y(t) = \begin{pmatrix} I(a,t) \\ I(b,t) \end{pmatrix}. \tag{7}$$

Now we preform the following steps:

1. **Since** $V = Q/C = x_1/C$ **and** $I = \phi/L = x_2/L$**, we have that**

$$[M_{21}, M_{22}] = \left(\begin{array}{cc|cc} 0 & 0 & \frac{1}{C} & 0 \\ \frac{1}{C} & 0 & 0 & 0 \end{array}\right), [C_1, C_2] = \left(\begin{array}{cc|cc} 0 & 0 & 0 & \frac{1}{L} \\ 0 & \frac{1}{L} & 0 & 0 \end{array}\right).$$

2. **Write the system in the "characteristic". That is diagonalize** $P_1\mathcal{H} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{-1}{L} & 0 \end{pmatrix}$. **Hence**

$$P_1\mathcal{H} = \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{L}{C}} & \sqrt{\frac{L}{C}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{LC}} & 0 \\ 0 & -\sqrt{\frac{1}{LC}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{\frac{C}{L}} \\ 1 & \sqrt{\frac{C}{L}} \end{pmatrix} \frac{1}{2}.$$

In the new variables (characteristics)

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} := \begin{pmatrix} 1 & -\sqrt{\frac{C}{L}} \\ 1 & \sqrt{\frac{C}{L}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the transmission line becomes

$$\frac{\partial \xi_1}{\partial t} = \sqrt{\frac{1}{LC}} \frac{\partial \xi_1}{\partial \zeta}, \qquad \frac{\partial \xi_2}{\partial t} = -\sqrt{\frac{1}{LC}} \frac{\partial \xi_2}{\partial \zeta}.$$

The “perfect” input and output for this system is

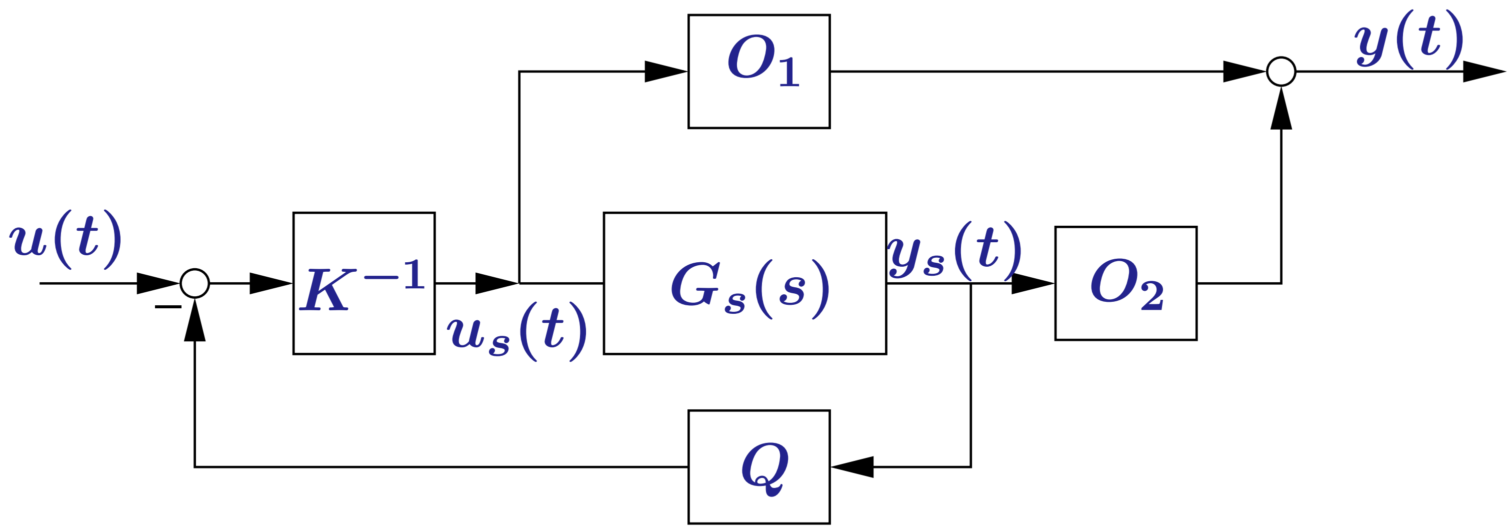
$$u_s(t) = \begin{pmatrix} \xi_1(b,t) \\ \xi_2(a,t) \end{pmatrix}, \qquad y_s(t) = \begin{pmatrix} \xi_1(a,t) \\ \xi_2(b,t) \end{pmatrix}. \tag{8}$$

3. Write our input-output pair in this input-output pair.

$$\begin{aligned} u(t) &= \begin{pmatrix} 0 & \frac{1}{2C} \\ \frac{1}{2C} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{1}{2C} & 0 \\ 0 & \frac{1}{2C} \end{pmatrix} y_s(t) \\ &= K u_s(t) + Q y_s(t), \end{aligned} \tag{9}$$

$$
\begin{aligned}
y(t) &= \begin{pmatrix} 0 & \frac{1}{2\sqrt{LC}} \\ \frac{-1}{2\sqrt{LC}} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{-1}{2\sqrt{LC}} & 0 \\ 0 & \frac{1}{2\sqrt{LC}} \end{pmatrix} y_s(t) \\
&= O_1 u_s(t) + O_2 y_s(t). \qquad (10)
\end{aligned}
$$

We regard the system with input/output u, y as a feedback of the system with input/output u_s, y_s, i.e.,



Now G_s has feed-through zero, and so any feedback is allowed.

Only condition: K^{-1} exists.

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