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The Cauchy Problem

4.1 The abstract Cauchy problem

In Lemma 2.2.2 we saw that if A is the infinitesimal generator of a C_0-semigroup $T(t)$ and $z_0 \in \boldsymbol{D}(A)$, then the unique classical solution of the abstract homogeneous Cauchy initial value problem

$$\dot{z}(t) = Az(t),\ t \geq 0, \qquad z(0) = z_0$$

is given by

$$z(t) = T(t)z_0,$$

which coincides with the mild solution (see Definition 2.2.3). If $z_0 \notin \boldsymbol{D}(A)$, then $z(t) = T(t)z_0$ is by definition still the mild solution.

In this section we extend these concepts to the abstract inhomogeneous Cauchy problem

$$\dot{z}(t) = Az(t) + f(t),\ t \geq 0, \qquad z(0) = z_0, \tag{4.1}$$

where for the moment we shall assume that $f \in \boldsymbol{C}([0, \tau]; Z)$. (4.1) is also called an *abstract evolution equation* or *abstract differential equation*. First we have to define what we mean by a solution of (4.1), and we begin with the notion of a classical solution. $\boldsymbol{C}^1([0, \tau]; Z)$ will denote the class of continuous functions on $[0, \tau]$ whose derivative is again continuous on $[0, \tau]$.

Definition 4.1.1 Consider equation (4.1) on the Hilbert space Z. The function $z(t)$ is a *classical solution* of (4.1) *on* $[0, \tau]$ if $z(t) \in \boldsymbol{C}^1([0, \tau]; Z)$, $z(t) \in \boldsymbol{D}(A)$ for all $t \in [0, \tau]$ and $z(t)$ satisfies (4.1) for all $t \in [0, \tau]$.

The function $z(t)$ is a *classical solution on* $[0, \infty)$ if $z(t)$ is a classical solution on $[0, \tau]$ for every $\tau \geq 0$. ■

We remark that when $f = 0$ and $\tau = \infty$ this definition reduces to Definition 2.2.1 for a classical solution for the homogeneous case.

Lemma 4.1.2 *Assume that* $f \in \boldsymbol{C}([0, \tau]; Z)$ *and that* z *is a classical solution of (4.1) on* $[0, \tau]$. *Then* $Az(\cdot)$ *is an element of* $\boldsymbol{C}([0, \tau]; Z)$, *and*

$$z(t) = T(t)z_0 + \int_0^t T(t-s) f(s) ds. \tag{4.2}$$

Proof From (4.1), we have that $Az(t) = \dot{z}(t) - f(t)$ and $\dot{z} \in \boldsymbol{C}([0, \tau]; Z)$ shows that $Az(\cdot) \in \boldsymbol{C}([0, \tau]; Z)$.

We now prove (4.2). Let t be an arbitrary, but fixed, element of $(0, \tau]$ and consider the function $T(t-s)z(s)$ for $s \in [0, t)$. We shall show that this function is differentiable in s. Let h be sufficiently small and consider

$$\begin{aligned}&\frac{T(t-s-h)z(s+h) - T(t-s)z(s)}{h} \\ &\quad = \frac{T(t-s-h)z(s+h) - T(t-s-h)z(s)}{h} + \\ &\qquad \frac{T(t-s-h)z(s) - T(t-s)z(s)}{h}.\end{aligned}$$

If h converges to zero, then the last term converges to $-AT(t-s)z(s)$, since $z(s) \in \boldsymbol{D}(A)$. Thus it remains to show that the first term converges. We have the following equality

$$\begin{aligned}&\frac{T(t-s-h)z(s+h) - T(t-s-h)z(s)}{h} - T(t-s)\dot{z}(s) \\ &\quad = T(t-s-h)\frac{z(s+h) - z(s)}{h} - T(t-s-h)\dot{z}(s) + \\ &\qquad T(t-s-h)\dot{z}(s) - T(t-s)\dot{z}(s).\end{aligned}$$

The uniform boundedness of $T(t)$ on any compact interval and the strong continuity allow us to conclude from the last equality that

$$\lim_{h \to 0} \| T(t-s-h)\frac{z(s+h) - z(s)}{h} - T(t-s)\dot{z}(s) \| = 0.$$

So we have proved that

$$\begin{aligned}\frac{d}{ds}[T(t-s)z(s)] &= -AT(t-s)z(s) + T(t-s)[Az(s) + f(s)] \\ &= T(t-s) f(s).\end{aligned}$$

Thus a classical solution to (4.1) necessarily has the form (4.2). ■

Equation (4.2) is reminiscent of the variation of constants formula for ordinary differential equations. It may be thought that (4.2) is always a classical solution of (4.1), but this is not generally true. However, we are able to prove the following partial converse.

Theorem 4.1.3 *If A is the infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space Z, $f \in \boldsymbol{C}^1([0,\tau]; Z)$ and $z_0 \in \boldsymbol{D}(A)$, then* (4.2) *is continuously differentiable on* $[0,\tau]$ *and it is the unique classical solution of (4.1).*

Proof *Uniqueness*: If z_1 and z_2 are two different solutions, then their difference $\Delta(t) = z_1(t) - z_2(t)$ satisfies the differential equation

$$\frac{d\Delta}{dt} = A\Delta, \qquad \Delta(0) = 0$$

and so we need to show that its only solution is $\Delta(t) \equiv 0$. To do this, define $y(s) = T(t-s)\Delta(s)$ for a fixed t and $0 \leq s \leq t$. Clearly, $\frac{dy}{ds} = 0$ and so $y(s) =$ constant $= T(t)\Delta(0) = 0$. However, $y(t) = \Delta(t)$ shows that $\Delta(t) = 0$.

Existence: Clearly, all we need to show now is that $\upsilon(t) = \int_0^t T(t-s)f(s)ds$ is an element of $\boldsymbol{C}^1([0,\tau]; Z)$, takes values in $\boldsymbol{D}(A)$, and satisfies differential equation (4.1). Now

$$\begin{aligned} \upsilon(t) &= \int_0^t T(t-s)[f(0) + \int_0^s \dot{f}(\alpha)d\alpha]ds \\ &= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(t-s)\dot{f}(\alpha)dsd\alpha, \end{aligned}$$

where we have used Fubini's Theorem A.5.27. From Theorem 2.1.13.e, it follows that

$$T(t-\alpha)z - z = A\int_\alpha^t T(t-s)zds \qquad \text{for all } z \in Z.$$

Hence $\upsilon(t) \in \boldsymbol{D}(A)$, and $\int_0^t \|A\int_\alpha^t T(t-s)\dot{f}(\alpha)ds\|d\alpha = \int_0^t \|T(t-\alpha)\dot{f}(\alpha) - \dot{f}(\alpha)\|d\alpha < \infty$. Thus, since A is closed, by Theorem A.5.28 we have that

$$\begin{aligned} A\upsilon(t) &= [T(t) - I]f(0) + \int_0^t [T(t-\alpha) - I]\dot{f}(\alpha)d\alpha \\ &= T(t)f(0) + \int_0^t T(t-\alpha)\dot{f}(\alpha)d\alpha - f(t). \end{aligned}$$

Now, since the convolution product is commutative, i.e., $\int_0^t g(t-s)h(s)ds = \int_0^t g(s)h(t-s)ds$, we have that

$$\upsilon(t) = \int_0^t T(s)f(t-s)ds$$

and so

$$\begin{aligned}\frac{dv}{dt}(t) &= T(t)f(0) + \int_0^t T(s)\dot{f}(t-s)ds \\ &= T(t)f(0) + \int_0^t T(t-s)\dot{f}(s)ds,\end{aligned}$$

once again using commutativity of the convolution product. It follows that $\frac{dv}{dt}$ is continuous and

$$\frac{dv}{dt}(t) = Av(t) + f(t).$$

■

The conditions of Theorem 4.1.3 are too strong for control applications, where in general we do not wish to assume that $f \in \boldsymbol{C}^1([0,\tau];Z)$. So we introduce the following weaker concept of a solution of (4.1).

Definition 4.1.4 If $f \in \boldsymbol{L}_p([0,\tau];Z)$ for a $p \geq 1$ and $z_0 \in Z$, then we call (4.2) a *mild solution* of (4.1) on $[0,\tau]$. ■

We note that (4.2) is a well defined integral in the sense of Bochner or Pettis, (see Lemma A.5.10 and Example A.5.20). Of course, if $f \in \boldsymbol{L}_p([0,\tau];Z)$ for some $p \geq 1$, then necessarily $f \in \boldsymbol{L}_1([0,\tau];Z)$. In our applications, we usually consider mild solutions for f in $\boldsymbol{L}_2([0,\tau];Z)$.

In Exercise 4.2 we show that the mild solution satisfies an integral equation, which for bounded A is the differential equation (4.1) once integrated.

Lemma 4.1.5 *Assume that $f \in \boldsymbol{L}_p([0,\tau];Z)$ for a $p \geq 1$ and $z_0 \in Z$. The mild solution $z(t)$ defined by (4.2) is continuous on* $[0,\tau]$.

Proof Since $T(t)z_0$ is continuous, we can assume without loss of generality that $z_0 = 0$. For $\delta > 0$, consider

$$\begin{aligned}z(t+\delta) - z(t) &= \int_0^t [(T(t+\delta-s) - T(t-s)]f(s)ds + \\ &\quad \int_t^{t+\delta} T(t+\delta-s)f(s)ds.\end{aligned}$$

Then with $\frac{1}{p} + \frac{1}{q} = 1$ we estimate

$$\begin{aligned}\|z(t+\delta) - z(t)\| &\leq \|[T(\delta) - I]z(t)\| + \\ &\quad (\int_t^{t+\delta} \|T(t+\delta-s)\|^q ds)^{\frac{1}{q}} (\int_t^{t+\delta} \|f(s)\|^p ds)^{\frac{1}{p}}\end{aligned}$$

and the right-hand side converges to 0 as $\delta \to 0^+$ by (2.13) and Theorem 2.1.7.a. Now consider

$$\begin{aligned}&z(t-\delta) - z(t) \\ &\quad = \int_0^{t-\delta} [T(t-\delta-s) - T(t-s)]f(s)ds - \int_{t-\delta}^t T(t-s)f(s)ds,\end{aligned}$$

noting that $[T(t-\delta-s)-T(t-s)]f(s)$ is integrable, since $f \in \boldsymbol{L}_p([0,\tau];Z)$ and using the properties of $T(t)$ from Theorem 2.1.7.a and Lemma A.5.10 (see Example A.5.20). Estimating the integral above yields

$$\begin{aligned}\|z(t-\delta)-z(t)\| \quad \le \quad & \int_0^{t-\delta} \|[T(t-\delta-s)-T(t-s)]f(s)\|ds + \\ & \int_{t-\delta}^{t} \|T(t-s)f(s)\|ds.\end{aligned}$$

Now $[T(t-\delta-s)-T(t-s)]f(s) \to 0$ as $\delta \to 0$, and from Theorem 2.1.7 there exists a constant M_t, depending only on t, such that $\|[T(t-\delta-s)-T(t-s)]f(s)\| \le M_t\|f(s)\|$. So the first term converges to zero $\delta \to 0$ by the Lebesgue Dominated Convergence Theorem A.5.26, and the second term also tends to zero by similar arguments. ■

In fact, this mild solution is the same as the concept of a *weak solution* used in the study of partial differential equations.

Definition 4.1.6 Let $f \in \boldsymbol{L}_p([0,\tau];Z)$ for a $p \ge 1$. We call z a *weak solution* of (4.1) *on* $[0,\tau]$ if for every $z_1 \in \boldsymbol{D}(A^*)$ the following holds:

a. $z(t)$ is continuous on $[0,\tau]$ and $z(0) = z_0$;

b. $\langle z(t), z_1\rangle$ is absolutely continuous on $[0,\tau]$;

c. For almost all $t \in [0,\tau]$

$$\frac{d}{dt}\langle z(t), z_1\rangle = \langle z(t), A^* z_1\rangle + \langle f(t), z_1\rangle.$$

We call z a *weak solution* of (4.1) *on* $[0,\infty)$ if it is a weak solution on $[0,\tau]$ for every $\tau \ge 0$. ■

Theorem 4.1.7 *For every $z_0 \in Z$ and every $f \in \boldsymbol{L}_p([0,\tau];Z)$ there exists a unique weak solution of (4.1) that is the mild solution of (4.1).*

Proof *a.* First we prove that $z(t) = T(t)z_0$ is a weak solution when $f = 0$. Theorem 2.1.7.b. implies that $T(t)z_0$ is continuous on $[0,\tau]$. Recall from Theorem 2.3.6 that $T^*(t)$ is a C_0 semigroup with generator A^*. Thus using Theorem 2.1.13.b. we have

$$\frac{d}{dt}T^*(t)z_1 = A^*T^*(t)z_1 = T^*(t)A^*z_1 \text{ for } z_1 \in \boldsymbol{D}(A^*).$$

Hence

$$\frac{d}{dt}\langle T(t)z_0, z_1\rangle = \frac{d}{dt}\langle z_0, T^*(t)z_1\rangle = \langle z_0, T^*(t)A^*z_1\rangle = \langle z(t), A^*z_1\rangle,$$

b. Next we prove that $z(t) = \int_0^t T(t-s) f(s)ds$ is a weak solution when $z_0 = 0$. In Lemma 4.1.5 we showed that it is continuous on $[0, \tau]$. Now

$$\begin{aligned} \frac{d}{dt}\langle \int_0^t T(t-s) f(s)ds, z_1\rangle &= \frac{d}{dt}\int_0^t \langle f(s), T^*(t-s)z_1\rangle ds \\ &= \langle f(t), z_1\rangle + \int_0^t \frac{d}{dt}\langle f(s), T^*(t-s)z_1\rangle ds \\ &= \langle f(t), z_1\rangle + \int_0^t \langle f(s), T^*(t-s)A^* z_1\rangle ds \\ &= \langle f(t), z_1\rangle + \langle z(t), A^* z_1\rangle, \end{aligned}$$

and $z(t) = \int_0^t T(t-s) f(s)ds$ is a weak solution when $z_0 = 0$.

c. Combining part a and b shows that the mild solution is a weak solution.

d. It remains to prove the uniqueness of the weak solution. Suppose that $\bar{z}(t)$ is a second weak solution and $z_1 \in \boldsymbol{D}(A^*)$. Then $\Delta(t) = z(t) - \bar{z}(t)$ satisfies $\Delta(0) = 0$ and

$$\frac{d}{dt}\langle \Delta(t), z_1\rangle = \langle \Delta(t), A^* z_1\rangle \quad \text{for almost all } t \in [0, \tau].$$

For fixed $t > 0$ and $0 \le s \le t$ differentiating $\langle T(t-s)\Delta(s), z_1\rangle$ yields

$$\begin{aligned} \frac{d}{ds}\langle T(t-s)\Delta(s), z_1\rangle &= \frac{d}{ds}\langle \Delta(s), T(t-s)^* z_1\rangle \\ &= \langle \Delta(s), A^* T(t-s)^* z_1\rangle - \langle \Delta(s), T(t-s)A^* z_1\rangle = 0. \end{aligned}$$

Thus $\langle T(t-s)\Delta(s), z_1\rangle$ is constant on $[0, t]$. Hence the value $\langle \Delta(t), z_1\rangle$ at $s = 0$ equals 0, the value at $s = 0$. Since $\boldsymbol{D}(A^*)$ is dense in Z, it follows that $\Delta(t) = 0$ for $t \ge 0$. ■

From the above it might be considered more logical to use (4.2) as the definition of the dynamical system. However, we follow the custom of using equation (4.1), even when $z(t)$ is not differentiable in the usual sense. It is the natural generalization of the finite-dimensional differential equation. When we write (4.1) we use it symbolically and what we actually mean is the mild solution (4.2).

Example 4.1.8 In this example, we again consider the heat equation of Example 2.1.1. The model of the heated bar was given by

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + u(x,t), \quad z(x,0) = z_0(x),$$

$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t).$$

We saw in Example 3.2.11 that with $u = 0$, this can be formulated as an abstract differential equation on $Z = \boldsymbol{L}_2(0,1)$ of the form

$$\dot{z}(t) = Az(t), \ t \ge 0, \qquad z(0) = z_0,$$

where

$$\begin{aligned} Ah &= \frac{d^2h}{dx^2} \text{ with} \\ \boldsymbol{D}(A) &= \Big\{h \in \boldsymbol{L}_2(0,1) \mid h, \tfrac{dh}{dx} \text{ are absolutely continuous,} \\ &\quad \tfrac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1) \text{ and } \tfrac{dh}{dx}(0) = 0 = \tfrac{dh}{dx}(1)\Big\}. \end{aligned}$$

In Example 3.2.11 we showed that this operator generates the C_0-semigroup from Example 2.1.5.

We can include the control term in this formulation as follows:

$$\dot{z}(t) = Az(t) + u(t), \ t \geq 0, \qquad z(0) = z_0,$$

provided that $u(x,t) \in \boldsymbol{L}_p([0,\tau]; \boldsymbol{L}_2(0,1))$ for some $p \geq 1$. The solution is given by (4.2), which, by Example 3.2.11, we can write as

$$\begin{aligned} &z(x,t) \\ &= \sum_{n=0}^{\infty} e^{\lambda_n t}\langle z_0, \phi_n\rangle \phi_n(x) + \int_0^t \sum_{n=0}^{\infty} e^{\lambda_n(t-s)}\langle u(\cdot,s), \phi_n(\cdot)\rangle \phi_n(x) ds \\ &= \int_0^1 z_0(y)dy + \sum_{n=1}^{\infty} 2e^{\lambda_n t} \int_0^1 z_0(y)\cos(n\pi y)dy \cos(n\pi x) + \\ &\quad \int_0^t \int_0^1 u(y,s) dy ds + \\ &\quad \int_0^t \sum_{n=1}^{\infty} e^{-n^2\pi^2(t-s)} 2 \int_0^1 u(y,s)\cos(n\pi y)dy \cos(n\pi x) ds, \qquad (4.3) \end{aligned}$$

since $\lambda_n = -n^2\pi^2, n \geq 0$, $\phi_n(x) = \sqrt{2}\cos(n\pi x), n \geq 1$ and $\phi_0(x) = 1$. We see that (4.3) equals (2.4). ■

The above example is typical for the Riesz-spectral class of operators discussed in Section 2.3 for which the mild solution has the explicit form

$$\sum_{n=1}^{\infty} \left[e^{\lambda_n t}\langle z_0, \psi_n\rangle \phi_n + \int_0^t e^{\lambda_n(t-s)}\langle f(s), \psi_n\rangle \phi_n ds \right]. \qquad (4.4)$$

In general, we do not have an explicit solution for (4.1).

4.2 Perturbations and composite systems

In applications to control problems, the inhomogeneous term f in (4.1) is often determined by a control input of feedback type, namely,

$$f(t) = Dz(t),$$

where $D \in \mathcal{L}(Z)$. This leads to the new Cauchy problem

$$\dot{z}(t) = (A+D)z(t), \ t \geq 0, \qquad z(0) = z_0, \qquad (4.5)$$

or in its integrated form

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Dz(s)ds. \tag{4.6}$$

We expect that the perturbed system operator, $A+D$, is the infinitesimal generator of another C_0-semigroup $T_D(t)$ so that the solution of (4.5) is given by $z(t) = T_D(t)z_0$. To prove this, we must study the operator integral equation

$$S(t)z_0 = T(t)z_0 + \int_0^t T(t-s)DS(s)z_0 ds \qquad \text{where } z_0 \in Z, \tag{4.7}$$

which is obtained by substituting $z(t) = S(t)z_0$ in (4.6).

Theorem 4.2.1 *Suppose that A is the infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space Z and that $D \in \mathcal{L}(Z)$. Then $A+D$ is the infinitesimal generator of a C_0-semigroup $T_D(t)$ which is the unique solution of* (4.7) *in the class of strongly continuous operators on Z. Moreover, if $\|T(t)\| \le Me^{\omega t}$, then*

$$\|T_D(t)\| \le Me^{(\omega + M\|D\|)t}. \tag{4.8}$$

This C_0-semigroup satisfies the following equations for every $z_0 \in Z$

$$T_D(t)z_0 = T(t)z_0 + \int_0^t T(t-s)DT_D(s)z_0 ds \tag{4.9}$$

and

$$T_D(t)z_0 = T(t)z_0 + \int_0^t T_D(t-s)DT(s)z_0 ds. \tag{4.10}$$

Proof The proof is divided into three parts. In part a, we shall show that there exists a unique solution $S(t)$ of (4.7). This solution satisfies (4.8) and (4.9) and is strongly continuous at zero. In part b, we shall prove that $T_D(t)$ is a C_0-semigroup. In the last part, we shall show that the infinitesimal generator of $T_D(t)$ is $A+D$ and that $T_D(t)$ satisfies the equation (4.10).

a. First we show that (4.7) has the unique solution given by

$$T_D(t) = \sum_{n=0}^{\infty} S_n(t), \tag{4.11}$$

where

$$S_n(t)x = \int_0^t T(t-s)DS_{n-1}(s)x ds, \qquad S_0(t) = T(t). \tag{4.12}$$

It is easy to verify the following estimate by induction

$$\|S_n(t)\| \le M^{n+1}\|D\|^n e^{\omega t}\frac{t^n}{n!}, \tag{4.13}$$

and so the series (4.11) is majorized by

$$Me^{\omega t}\sum_{n=0}^{\infty}\frac{(M\|D\|t)^n}{n!} = Me^{(\omega+M\|D\|)t}.$$

So the series (4.11) converges absolutely in the uniform topology of $\mathcal{L}(Z)$ on any compact interval, $[0, \tau]$, and $T_D(t)$ satisfies the estimate (4.8). Furthermore,

$$\begin{aligned} T_D(t)z_0 = \sum_{n=0}^{\infty} S_n(t)z_0 &= S_0(t)z_0 + \sum_{n=1}^{\infty} S_n(t)z_0 \\ &= T(t)z_0 + \sum_{n=1}^{\infty} \int_0^t T(t-s)DS_{n-1}(s)z_0 ds \\ &= T(t)z_0 + \int_0^t T(t-s)DT_D(s)z_0 ds \end{aligned}$$

by the absolute convergence of (4.11) and the estimate (4.8). So $T_D(t)$ satisfies (4.9).

To prove uniqueness we assume that $S(t)$ is also a solution, and by subtracting the equations for $T_D(t)$ and $S(t)$ we obtain

$$[T_D(t) - S(t)]z_0 = \int_0^t T(t-s)D[T_D(s) - S(s)]z_0 ds.$$

Hence

$$\|[T_D(t) - S(t)]z_0\| \le \int_0^t Me^{\omega(t-s)}\|D\|\|[T_D(s) - S(s)]z_0\|ds.$$

Setting $e^{-\omega t}\|[T_D(t) - S(t)]z_0\| = g(t)$ yields

$$0 \le g(t) \le M\|D\| \int_0^t g(s)ds$$

and Gronwall's Lemma A.6.7 shows that $g(t) \le g(0)e^{M\|D\|t} = 0$.

To prove that $S(t)$ is strongly continuous at zero, for $h > 0$ we deduce the following estimate:

$$\|T_D(h)z_0 - z_0\| \le \|T(h)z_0 - z_0\| + \int_0^h \|T(h-s)DT_D(s)z_0\|ds.$$

So using the strong continuity of $T(t)$ and the bounds for $T(t)$ and $T_D(t)$, we see that $T_D(t)z_0$ is strongly continuous at zero.

b. From *a* we already know that $T_D(t)$ is strongly continuous at zero, and $T_D(0) = I$ follows from (4.11) and (4.12).

In order to prove the semigroup property $T_D(t+s) = T_D(t)T_D(s)$, we use (4.7) to obtain

$$\begin{aligned}
&T_D(t+s)z_0 - T_D(t)T_D(s)z_0 \\
&\quad = T(t+s)z_0 + \int_0^{t+s} T(t+s-\alpha)DT_D(\alpha)z_0 d\alpha - \\
&\qquad [T(t) + \int_0^t T(t-\alpha)DT_D(\alpha)d\alpha][T(s)z_0 + \\
&\qquad \int_0^s T(s-\beta)DT_D(\beta)z_0 d\beta] \\
&\quad = \int_0^{t+s} T(t+s-\alpha)DT_D(\alpha)z_0 d\alpha - \\
&\qquad \int_0^s T(t+s-\beta)DT_D(\beta)z_0 d\beta - \int_0^t T(t-\alpha)DT_D(\alpha)T_D(s)z_0 d\alpha \\
&\quad = \int_s^{t+s} T(t+s-\alpha)DT_D(\alpha)z_0 d\alpha - \int_0^t T(t-\alpha)DT_D(\alpha)T_D(s)z_0 d\alpha \\
&\quad = \int_0^t T(t-\alpha)D[T_D(s+\alpha) - T_D(\alpha)T_D(s)]z_0 d\alpha.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|T_D(t+s)z_0 - T_D(t)T_D(s)z_0\| \\
&\quad \leq M\|D\| \int_0^t e^{\omega(t-s)} \|[T_D(s+\alpha) - T_D(\alpha)T_D(s)]z_0\| d\alpha.
\end{aligned}$$

Letting $g(t) = e^{-\omega t}\|T_D(t+s)z_0 - T_D(t)T_D(s)z_0\|$, we obtain

$$0 \leq g(t) \leq M\|D\| \int_0^t g(\alpha)d\alpha.$$

So, since $g(0) = \|T_D(s)z_0 - T_D(s)z_0\| = 0$, applying Gronwall's Lemma A.6.7, we have

$$0 \leq g(t) \leq g(0)e^{M\|D\|t} = 0.$$

Thus $T_D(t+s) = T_D(t)T_D(s)$ and so $T_D(t)$ is a C_0-semigroup.

c. We now prove that its generator is $A + D$. First we show that

$$\lim_{h\to 0^+} \|\frac{T_D(h)z - z}{h} - \frac{T(h)z - z}{h} - Dz\| = 0 \qquad \text{for } z \in Z. \tag{4.14}$$

From (4.7), we have that for any $z \in Z$,

$$\begin{aligned}
&\|\frac{T_D(h)z - z}{h} - \frac{T(h)z - z}{h} - Dz\| \\
&\quad = \|\frac{1}{h}\int_0^h T(h-s)DT_D(s)zds - Dz\|
\end{aligned}$$

$$\leq \frac{1}{h}\|\int_0^h T(h-s)D[T_D(s)z - z]ds\| + \|\frac{1}{h}\int_0^h T(h-s)Dzds - Dz\|. \qquad (4.15)$$

The following inequality holds for the first term of (4.15):

$$\frac{1}{h}\|\int_0^h T(h-s)D[T_D(s)z - z]ds\| \leq \frac{1}{h}\int_0^h \|T(h-s)\|\|D\|\|T_D(s)z - z\|ds.$$

Let $\varepsilon > 0$ be a given number. Then by the strong continuity of $T_D(t)$, there exists an h such that $\|T_D(s)z - z\| \leq \varepsilon$ for $s \in [0, h]$. Without loss of generality, we may assume that $h \leq 1$. From Theorem 2.1.7 we have the existence of a constant M_1 such that $\|T(h-s)\| \leq M_1$ for all $s \in [0, h]$. Applying this in the above inequality gives

$$\frac{1}{h}\|\int_0^h T(h-s)D[T_D(s)z - z]ds\| \leq \frac{1}{h}\int_0^h M_1\|D\|\varepsilon ds = M_1\|D\|\varepsilon.$$

Thus

$$\lim_{h\to 0^+} \frac{1}{h}\|\int_0^h T(h-s)D[T_D(s)z - z]ds\| = 0.$$

Next consider the second term in (4.15). It is easily seen that

$$\int_0^h T(h-s)Dzds = \int_0^h T(s)Dzds,$$

and since $T(t)$ is a C_0-semigroup, we can apply Theorem 2.1.7 to obtain

$$\lim_{h\to 0^+} \|\frac{1}{h}\int_0^h T(h-s)Dzds - Dz\| = 0.$$

Thus we have proved equation (4.14), and this shows that the domain of the generator of $T_D(t)$ is $\boldsymbol{D}(A)$ and on this domain it equals $A + D$.

Equation (4.10) is easily proved by the observation that A is the perturbation of $A + D$ by $-D$. So, using (4.9) we have

$$T(t)z_0 = [T_D]_{-D}(t)z_0 = T_D(t)z_0 + \int_0^t T_D(t-s)[-D]T(s)z_0 ds,$$

which is equal to (4.10). ■

Another way of generating new C_0-semigroups is given in the following lemma.

Lemma 4.2.2 *Let $T_1(t)$ and $T_2(t)$ be C_0-semigroups on their respective Hilbert spaces Z_1 and Z_2 and with the infinitesimal generators A_1 and A_2, respectively.*

Suppose that

$$\|T_i(t)\| \leq M_i e^{\omega_i t}, \qquad i = 1, 2, \tag{4.16}$$

and $D \in \mathcal{L}(Z_1, Z_2)$*. Then the operator* $A = \begin{pmatrix} A_1 & 0 \\ D & A_2 \end{pmatrix}$ *with* $\boldsymbol{D}(A) = \boldsymbol{D}(A_1) \oplus \boldsymbol{D}(A_2)$ *is the infinitesimal generator of the* C_0*-semigroup* $T(t)$ *on* $Z = Z_1 \oplus Z_2$ *given by*

$$T(t) = \begin{pmatrix} T_1(t) & 0 \\ S(t) & T_2(t) \end{pmatrix}, \quad S(t)x = \int_0^t T_2(t-s) D T_1(s) x ds. \tag{4.17}$$

Furthermore, there exists a positive constant M *such that*

$$\|T(t)\| \leq M e^{\omega t}, \tag{4.18}$$

where $\omega = \max(\omega_1, \omega_2)$ *if* $\omega_1 \neq \omega_2$ *and* $\omega > \omega_1$ *if* $\omega_1 = \omega_2$*.*

Proof It is clear that $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is the infinitesimal generator of the C_0-semigroup $\begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}$ on $Z = Z_1 \oplus Z_2$. Since A is the sum of this operator and the bounded perturbation $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$, we know from Theorem 4.2.1 that A is the infinitesimal generator of a C_0-semigroup on Z. Now $T(t)$ given by (4.17) satisfies (4.9) and hence it is the C_0-semigroup generated by A.

For the estimates, consider the following for $\omega_1 \neq \omega_2$:

$$T(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} T_1(t)x_1 \\ \int_0^t T_2(t-s) D T_1(s) x_1 ds + T_2(t) x_2 \end{pmatrix}.$$

Then

$$\begin{aligned} \|T(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_Z &\leq M_1 e^{\omega_1 t}\|x_1\| + M_1 M_2 \|D\| \frac{e^{\omega_1 t} - e^{\omega_2 t}}{w_1 - w_2} \|x_1\| + \\ &\quad M_2 e^{\omega_2 t}\|x_2\| \\ &\leq M e^{\omega t} \|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_Z \end{aligned}$$

for some positive constant M and $\omega = \max(\omega_1, \omega_2)$. For $\omega_1 = \omega_2$, we obtain

$$\|T(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_Z \leq M(1+t)e^{\omega_1 t}\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_Z.$$

■

In Theorem 4.2.1, we assumed that $D \in \mathcal{L}(Z)$. However, we shall also need to consider time-dependent perturbation operators $D \in \boldsymbol{P}_\infty([0,\tau]; \mathcal{L}(Z))$, where

$$\begin{aligned}&\boldsymbol{P}_\infty([0,\tau]; \mathcal{L}(Z))\\ &\qquad := \Big\{ D \mid \langle z_1, D(\cdot)z_2\rangle \text{ is measurable} \\ &\qquad\qquad \text{for every } z_1, z_2 \in Z \text{ and } \operatorname*{ess\,sup}_{0\le t\le\tau} \|D(t)\|_{\mathcal{L}(Z)} < \infty \Big\}\end{aligned} \tag{4.19}$$

(see also Definition A.6.8).

These perturbations arise as the result of time-dependent feedbacks $f(t) = D(t)z(t)$, which leads us to the following version of (4.7) on $[0, \tau]$:

$$z(t) = U(t,0)z_0 = T(t)z_0 + \int_0^t T(t-\alpha)D(\alpha)U(\alpha,0)z_0 d\alpha.$$

Since the perturbed operator is not necessarily time-invariant, we have to specify the initial time too. If we denote by $U(t,s)z_0$ the (mild) solution of (4.1) with $f(\cdot) = D(\cdot)z(\cdot)$ and initial condition $z(s) = z_0$, then we obtain the following time-dependent version of (4.7) on $[0,\tau]$:

$$z(t) = U(t,s)z_0 = T(t-s)z_0 + \int_s^t T(t-\alpha)D(\alpha)U(\alpha,s)z_0 d\alpha. \tag{4.20}$$

Following the method of the proof in Theorem 4.2.1, we shall show that (4.20) has a unique solution. Furthermore, one can show that $U(t,s)$ given by (4.20) is strongly continuous in t on $[0,\tau]$ for each fixed s, $U(t,t) = I$ and it satisfies a semigroup property. In fact, it is a mild evolution operator.

Definition 4.2.3 Let $\Delta(\tau) = \{(t,s); 0 \le s \le t \le \tau\}$. $U(t,s) : \Delta(\tau) \to \mathcal{L}(Z)$ is a *mild evolution operator* if it has the following properties:

a. $U(s,s) = I$, $s \in [0,\tau]$;

b. $U(t,r)U(r,s) = U(t,s)$, $0 \le s \le t \le \tau$;

c. $U(\cdot,s)$ is strongly continuous on $[s,\tau]$ and $U(t,\cdot)$ is strongly continuous on $[0,t]$. ■

Theorem 4.2.4 *If $T(t)$ is a C_0-semigroup on Z and $D \in \boldsymbol{P}_\infty([0,\tau]; \mathcal{L}(Z))$, then* (4.20) *has a solution in the class of mild evolution operators on Z. This solution is unique in the larger class of operators $Q(t,s)$ that satisfy $Q(\cdot,s) \in \boldsymbol{P}_\infty([s,\tau]; \mathcal{L}(Z))$ for all $s \in [0,\tau]$.*

Proof First we note that it is possible to apply a similar argument as in Theorem 4.2.1 to show that (4.20) has a unique solution and that it is given by

$$U(t,s) = \sum_{n=0}^{\infty} U^n(t,s), \tag{4.21}$$

where

$$U^n(t,s)x = \int_s^t T(t-\alpha)D(\alpha)U^{n-1}(\alpha,s)xd\alpha,$$
$$U^0(t,s) = T(t-s).$$

However, in this proof we shall apply the Contraction Mapping Theorem A.3.1.

For fixed $s \in [0,\tau]$, consider the operator $G_s : \boldsymbol{P}_\infty([s,\tau];\mathcal{L}(Z)) \to \boldsymbol{P}_\infty([s,\tau];\mathcal{L}(Z))$, defined by

$$G_s(U)(t)z := T(t-s)z + \int_s^t T(t-\alpha)D(\alpha)U(\alpha)zd\alpha. \tag{4.22}$$

Let $\gamma := \sup_{0\le t\le\tau} \|T(t)\|$.

$$\|G_s(U_1)(t) - G_s(U_2)(t)\|_{\mathcal{L}(Z)} \le (t-s)\gamma\|D\|\|U_1-U_2\|_{\boldsymbol{P}_\infty}.$$

Thus

$$\|G_s(U_1)(\cdot) - G_s(U_2)(\cdot)\|_{\boldsymbol{P}_\infty} \le (\tau-s)\gamma\|D\|\|U_1-U_2\|_{\boldsymbol{P}_\infty}, \tag{4.23}$$

and by induction it follows that

$$\|G_s^k(U_1) - G_s^k(U_2)\|_{\boldsymbol{P}_\infty} \le \frac{(\tau-s)^k}{k!}\gamma^k\|D\|^k_{\boldsymbol{P}_\infty}\|U_1-U_2\|_{\boldsymbol{P}_\infty}. \tag{4.24}$$

If we choose k such that $\dfrac{(\tau-s)^k\gamma^k\|D\|^k_{\boldsymbol{P}_\infty}}{k!} < 1$, then we see from equation (4.24) that G_s^k is a contraction. So there exists a unique fixed point of (4.22). It is easily verified that $U(\cdot,s)$ given by (4.21) is this fixed point. We shall show that this function is a mild evolution operator. Property a of Definition 4.2.3 is trivial. The proof of property b is very similar to the proof of property b in Theorem 4.2.1. We have that

$$U(t,r)U(r,s) - U(t,s) = \int_r^t T(t-\alpha)D(\alpha)[U(\alpha,r)U(r,s) - U(\alpha,s)]d\alpha.$$

Thus

$$\|U(t,r)U(r,s) - U(t,s)\| \le \int_r^t Me^{\omega(t-\alpha)}\|D(\alpha)\|\|U(\alpha,r)U(r,s) - U(\alpha,s)\|d\alpha.$$

Applying Gronwall's Lemma A.6.7 once again, we have that $U(t,r)U(r,s) = U(t,s)$. From (4.20), we have that

$$\|U(t,s)\| \le \|T(t-s)\| + \int_s^t \|T(t-\alpha)\|\|D(\alpha)\|\|U(\alpha,s)\|d\alpha.$$

Using Gronwall's Lemma, this implies that $U(\cdot,\cdot)$ is uniformly bounded, with bound $\gamma e^{\gamma\|D\|_{\boldsymbol{P}_\infty}\tau}$. Using this together with the strong continuity of $T(t)$, it is not hard to see that $U(\cdot, s)$ is strongly continuous on $[s, \tau]$. Thus it remains to show the strong continuity of $U(t, \cdot)$ on $[0, t]$. Let $s \in [0, t)$, and $h > 0$ such that $s + h \leq t$, and consider $U(t, s+h)z_0 - U(t, s)z_0$. Using (4.20) we see that

$$\begin{aligned}
&[U(t, s+h) - U(t, s)]z_0 \\
&\quad = T(t-s-h)[z_0 - T(h)z_0] - \int_s^{s+h} T(t-\alpha)D(\alpha)U(\alpha, s)z_0 d\alpha + \\
&\qquad \int_{s+h}^t T(t-\alpha)D(\alpha)[U(\alpha, s+h) - U(\alpha, s)]z_0 d\alpha.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
&\|[U(t, s+h) - U(t, s)]z_0\| \\
&\quad \leq \gamma\|T(h)z_0 - z_0\| + \int_{s+h}^t \gamma\|D\|_{\boldsymbol{P}_\infty}\|[U(\alpha, s+h) - U(\alpha, s)]z_0\|d\alpha + \\
&\qquad \int_s^{s+h} \gamma\|D\|_{\boldsymbol{P}_\infty}\|U(\alpha, s)z_0\|d\alpha \\
&\quad \leq \varepsilon(h) + \int_{s+h}^t \gamma\|D\|_{\boldsymbol{P}_\infty}\|[U(\alpha, s+h) - U(\alpha, s)]z_0\|d\alpha,
\end{aligned}$$

where $\varepsilon(h) := \gamma\|T(h)z_0 - z_0\| + \int_s^{s+h} \gamma\|D\|_{\boldsymbol{P}_\infty}\|U(\alpha, s)z_0\|d\alpha$ converges to zero for h converging to zero. Since the above inequality holds for all $t \in [s+h, \tau]$, we conclude from Gronwall's Lemma that

$$\|[U(t, s+h) - U(t, s)]z_0\| \leq \varepsilon(h)e^{\gamma\|D\|_{\boldsymbol{P}_\infty}(t-s-h)}.$$

From this we see that $\|[U(t, s+h) - U(t, s)]z_0\|$ converges to zero for h converging to zero. Similarly, one can show that $\|[U(t, s) - U(t, s-h)]z_0\|$ converges to zero as h converges to zero. Thus $U(t, \cdot)$ is strongly continuous on $[0, t]$. ■

This theorem motivates the following definition.

Definition 4.2.5 Suppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z. For $D(\cdot) \in \boldsymbol{P}_\infty([0, \tau]; \mathcal{L}(Z))$ we call the unique solution of (4.20) the *mild evolution operator generated by* $A + D(\cdot)$. ■

Of course, $U(t, s)z_0$ may be regarded as the mild solution of

$$\dot{z}(t) = (A + D(t))z(t), \qquad z(s) = z_0, \tag{4.25}$$

since it satisfies the integral equation

$$z(t) = T(t)z_0 + \int_s^t T(t-\alpha)D(\alpha)z(\alpha)d\alpha.$$

For smooth initial conditions, we can prove differentiability in the second variable.

Theorem 4.2.6 *Assume that $T(t)$ is a C_0-semigroup on Z, $D \in \boldsymbol{P}_\infty([0,\tau];\mathcal{L}(Z))$ and $U(t,s)$ is the unique solution of (4.20). Then for all $z_0 \in \boldsymbol{D}(A)$ the following hold:*

$$\int_s^t U(t,\alpha)(A+D(\alpha))z_0 d\alpha = U(t,s)z_0 - z_0; \tag{4.26}$$

$$\frac{\partial}{\partial s}U(t,s)z_0 = -U(t,s)(A+D(s))z_0 \qquad \text{a.e. in } [0,t). \tag{4.27}$$

Proof Consider the sequence

$$\begin{aligned} \tilde{U}_0(t,s) &= T(t-s), \\ \tilde{U}_n(t,s)z_0 &= \int_s^t \tilde{U}_{n-1}(t,\alpha)D(\alpha)T(\alpha-s)z_0 d\alpha. \end{aligned}$$

Then, as in Theorem 4.2.4, we can show that

$$\tilde{U}(t,s) = \sum_{n=0}^{\infty} \tilde{U}_n(t,s)$$

is the unique solution of

$$\tilde{U}(t,s)z_0 = T(t-s)z_0 + \int_s^t \tilde{U}(t,\alpha)D(\alpha)T(\alpha-s)z_0 d\alpha. \tag{4.28}$$

We shall show that $U(t,s) = \tilde{U}(t,s)$ using an induction argument. By definition $U^0(t,s) = \tilde{U}_0(t,s)$ and $U^1(t,s) = \tilde{U}_1(t,s)$. Now supposing that $U^{n-1}(t,s) = \tilde{U}_{n-1}(t,s)$ and $U^{n-2}(t,s) = \tilde{U}_{n-2}(t,s)$, we shall prove that $U^n(t,s) = \tilde{U}_n(t,s)$. By definition and since $U^{n-1}(t,s) = \tilde{U}_{n-1}(t,s)$, we have that

$$\begin{aligned} \tilde{U}_n(t,s)z_0 &= \int_s^t U^{n-1}(t,\alpha)D(\alpha)T(\alpha-s)z_0 d\alpha \\ &= \int_s^t \int_\alpha^t T(t-\beta)D(\beta)U^{n-2}(\beta,\alpha)D(\alpha)T(\alpha-s)z_0 d\beta d\alpha \\ &= \int_s^t \int_s^\beta T(t-\beta)D(\beta)U^{n-2}(\beta,\alpha)D(\alpha)T(\alpha-s)z_0 d\alpha d\beta \\ &\qquad \text{by Fubini's Theorem A.5.27} \\ &= \int_s^t T(t-\beta)D(\beta)\tilde{U}_{n-1}(\beta,s)z_0 d\beta \\ &\qquad \text{since } \tilde{U}_{n-2} = U^{n-2} \\ &= U^n(t,s)z_0, \qquad \text{since } \tilde{U}_{n-1} = U^{n-1}. \end{aligned}$$

Thus by induction we have proved that

$$U(t,s) = \tilde{U}(t,s).$$

Hence for $z_0 \in \boldsymbol{D}(A)$, we have

$$U(t,\alpha)Az_0 = T(t-\alpha)Az_0 + \int_\alpha^t U(t,\beta)D(\beta)T(\beta-\alpha)Az_0 d\beta,$$

and both terms on the right are integrable by Theorem 2.1.13, Definition 4.2.3, and Lemma A.5.10. Thus

$$\begin{aligned}
&\int_s^t U(t,\alpha)Az_0 d\alpha \\
&\quad= \int_s^t T(t-\alpha)Az_0 d\alpha + \int_s^t \int_\alpha^t U(t,\beta)D(\beta)T(\beta-\alpha)Az_0 d\beta d\alpha \\
&\quad= \int_s^t T(t-\alpha)Az_0 d\alpha + \int_s^t \int_s^\beta U(t,\beta)D(\beta)T(\beta-\alpha)Az_0 d\alpha d\beta \\
&\qquad\qquad \text{by Fubini's Theorem A.5.27} \\
&\quad= (T(t-s)-I)z_0 + \int_s^t U(t,\beta)D(\beta)(T(\beta-s)-I)z_0 d\beta \\
&\qquad\qquad \text{by Theorem 2.1.13.d.}
\end{aligned}$$

Since $U(t,s)$ is the solution to (4.28), we have proved that $\int_s^t U(t,\alpha)(A + D(\alpha))z_0 d\alpha = U(t,s)z_0 - z_0$, which is (4.26), and this implies (4.27) by Theorem A.5.35. ■

Notice that we have also proved the following useful corollary.

Corollary 4.2.7 *If $T(t)$ is a C_0-semigroup on Z and $D \in \boldsymbol{P}_\infty([0,\tau]; \mathcal{L}(Z))$, then* (4.28) *and* (4.20) *have the same unique solution in the class of mild evolution operators on Z.*

In the sequel, we encounter abstract evolution equations of the type

$$\dot{z}(t) = (A + D(t))z(t) + f(t), \qquad z(0) = z_0. \tag{4.29}$$

Following the previous discussion it is natural to define solutions for these equations.

Definition 4.2.8 Consider equation (4.29), where A is the infinitesimal generator of a C_0-semigroup on the Hilbert space Z, $z_0 \in Z$, $D(\cdot) \in \boldsymbol{P}_\infty([0,\tau]; \mathcal{L}(Z))$ and $f \in \boldsymbol{L}_p([0,\tau]; Z)$, $p \geq 1$.

The function $z(t)$ is a *classical solution* of (4.29) on $[0,\tau]$ if $z(\cdot) \in \boldsymbol{C}^1([0,\tau]; Z)$, $z(t) \in \boldsymbol{D}(A)$ for all $t \in [0,\tau]$, and $z(t)$ satisfies (4.29) for all $t \in [0,\tau]$.

If $f \in \boldsymbol{L}_p([0,\tau]; Z)$ for a $p \geq 1$ and $z_0 \in Z$, then we define the *mild solution* of (4.29) to be

$$z(t) = U(t,0)z_0 + \int_0^t U(t,s)f(s)ds.$$

■

The mild solutions to these abstract evolution equations have properties similar to those with $D(\cdot) = 0$; this is discussed further in Exercise 4.13.

In general, $U(t,s)$ is not differentiable in t; this means that (4.29) rarely has a classical solution.

4.3 Exercises

4.1. Prove the existence of a unique solution to the operator integral equation (4.7) by applying the Contraction Mapping Theorem A.3.1.

4.2. Show that for any $f \in \boldsymbol{L}_1([0, \tau]; Z)$ the mild solution of Definition 4.1.4 as given by (4.2) satisfies

(i) z is continuous,
(ii) For all $\tau \in [0, \tau]$ we have $\int_0^\tau z(t)dt \in \boldsymbol{D}(A)$, and
(iii) For all $\tau \in [0, \tau]$ there holds

$$z(\tau) - z(0) = A\int_0^\tau z(t)dt + \int_0^\tau f(t)dt.$$

Show that the mild solution is the only function satisfying these three conditions.

4.3. Let $T(t)$ be an exponentially stable semigroup on the Hilbert space Z.

(a) Show that if $f \in \boldsymbol{L}_p([0, \infty); Z)$ for $1 \leq p < \infty$, then the mild solution to (4.1) is asymptotically stable.
(b) Show that for $f \in \boldsymbol{L}_\infty([0, \infty); Z)$ the mild solution to (4.1) is not necessarily asymptotically stable.

4.4. In Theorem 4.2.1, we derived the estimate $\|S(t)\| \leq Me^{(\omega+M\|D\|)t}$ for the perturbed semigroup. In this exercise, we investigate this bound further. Let A be the self-adjoint operator on the Hilbert space Z given by

$$\mathrm{A} \quad = \quad \sum_{n=1}^{\infty} -n\langle\cdot, \phi_n\rangle\phi_n$$

with domain

$$\boldsymbol{D}(A) \quad = \quad \{z \in Z \mid \sum_{n=1}^{\infty} n^2|\langle z, \phi_n\rangle|^2 < \infty\},$$

where $\{\phi_n, n \geq 1\}$ is an orthonormal basis of Z. For $k \geq 1$, define the operator

$$D_k = -\sum_{n=1}^{k} n\langle\cdot, \phi_n\rangle\phi_n$$

and denote by $T(t)$, $S_k^+(t)$, $S_k^-(t)$, the semigroups generated by A, $A+D_k$, and $A-D_k$, respectively.

a. Calculate the growth bound of $T(t)$.
b. Calculate the growth bound of $S_k^+(t)$ and $S_k^-(t)$.
c. What is the norm of D_k? Using parts a and b show that the estimate in Theorem 4.2.1 can be very conservative, but that it cannot be improved upon.

4.5. Let A be an infinitesimal generator of a C_0-semigroup on the Hilbert space Z, and let $B \in \mathcal{L}(U, Z)$. Prove that for any $F \in \mathcal{L}(Z, U)$ the mild solution of

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0$$

equals the mild solution of

$$\dot{z}(t) = (A + BF)z(t) + B(u(t) - Fz(t)), \qquad z(0) = z_0.$$

4.6. In this exercise we show that a rank one perturbation can change the growth bound drastically.
Consider on $\boldsymbol{L}_2(0, 1)$ the left-shift semigroup from Exercise 2.6,

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{for } 0 \le x \le 1 \quad \text{and } t \le 1 - x, \\ 0 & \text{for } 0 \le x \le 1 \quad \text{and } t > 1 - x. \end{cases}$$

The growth bound of this semigroup is minus infinity and its infinitesimal generator is given by $Af = \dfrac{df}{dx}$ with $\boldsymbol{D}(A) = \{ f \in \boldsymbol{L}_2(0, 1) \mid f$ is absolutely continuous, $\frac{df}{dx}$ is an element of $\boldsymbol{L}_2(0, 1)$ and $f(1) = 0\}$

a. Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. Define the operator D_λ by

$$(D_\lambda f)(x) := \frac{\lambda^2 e^\lambda}{e^\lambda + \lambda e^\lambda - 1} \int_0^1 f(\theta) d\theta \, \mathbb{1}_{[0,1]}(x).$$

Show that D_λ is a bounded linear operator on $\boldsymbol{L}_2(0, 1)$ with one-dimensional range.

b. Show that $A + D_\lambda$ has an eigenvalue at λ.
Hint: Solve $Af + \alpha \mathbb{1} = \lambda f$, where α is a constant. Then integrate to find α.

c. What you say about the growth bound of the C_0-semigroup generated by $A + D_\lambda$?

4.7. Let $T_1(t)$ and $T_2(t)$ be C_0-semigroups on their respective Hilbert spaces Z_1 and Z_2 and with the infinitesimal generators A_1 and A_2, respectively, and suppose that $D \in \mathcal{L}(Z_2, Z_1)$.

a. Show that the operator A defined by

$$A = \begin{pmatrix} A_1 & D \\ 0 & A_2 \end{pmatrix},$$

$$\boldsymbol{D}(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in \boldsymbol{D}(A_1), y \in \boldsymbol{D}(A_2) \right\}$$

is the infinitesimal generator of a C_0-semigroup $T(t)$ on $Z = Z_1 \oplus Z_2$.

b. Find the expression for $T(t)$.

c. Let ω_1 and ω_2 be the growth bounds of $T_1(t)$ and $T_2(t)$, respectively. Prove that the growth bound of $T(t)$ is the maximum of ω_1 and ω_2.

d. Give the expression for $(\lambda I - A)^{-1}$ for $\lambda \in \rho(A_1) \cap \rho(A_2)$.

4.8. Let A be the infinitesimal generator of a C_0-group $T(t)$ on the Hilbert space Z, and let $D \in \mathcal{L}(Z)$.

a. Show that $A + D$ is the infinitesimal generator of a C_0-group on Z.
b. If $T(t)$ is a unitary group, give necessary and sufficient conditions on D such that $A + D$ is the infinitesimal generator of a unitary group. Hint: See Exercise 2.24.

4.9. Consider the following system

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \beta \frac{\partial w}{\partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) + \alpha \frac{\partial w}{\partial x}(x,t) = 0,$$

$$w(x,0) = w_0(x), \ \frac{\partial w}{\partial t}(x,0) = w_1(x), \tag{4.30}$$

$$w(0,t) = 0 = w(1,t),$$

where α and β are real numbers. As in Exercise 3.11, we define the operator

$$A_0 = -\frac{d^2 h}{dx^2} + \alpha \frac{dh}{dx}$$

with domain

$$\begin{aligned} \boldsymbol{D}(A_0) \quad = \quad & \{h \in \boldsymbol{L}_2(0,1) \mid h, \tfrac{dh}{dx} \text{ are absolutely continuous}, \\ & \tfrac{dh}{dx} \in \boldsymbol{L}_2(0,1) \text{ and } h(0) = h(1) = 0\} \end{aligned}$$

on the Hilbert space $Z_\alpha = \boldsymbol{L}_2(0,1)$ with inner product

$$\langle h, k \rangle_\alpha = \int_0^1 e^{-\alpha x} h(x) \overline{k(x)} dx.$$

a. Formulate the partial differential equation (4.30) as an abstract differential equation $\dot{z}(t) = Az(t)$ on the Hilbert space $Z = \boldsymbol{D}(A_0^{\frac{1}{2}}) \oplus Z_\alpha$. Find A and show that it is the infinitesimal generator of a C_0-semigroup.
b. Show that, for pairs (α, β) such that $\beta^2 - \alpha^2 \neq 4n^2\pi^2$, A is a Riesz-spectral operator.

4.10. Consider the following abstract Cauchy problem on the Hilbert space Z

$$\dot{z}(t) = Az(t) + Bu(t), \ t \geq 0, \qquad z(0) = z_0, \tag{4.31}$$

where A is the infinitesimal generator of a C_0-semigroup $T(t)$ on Z, the input space U is a Hilbert space, $B \in \mathcal{L}(U, Z)$, and $z_0 \in Z$.

a. Suppose that we have the digital controller: $u(t) = u(k)$ for $k \leq t < k+1$, $k \geq 0$. Show that the values of the mild solution of (4.31) at the discrete-time instants, $t = k$, $k \geq 0$, satisfy the equation

$$z(k+1) = \hat{A}z(k) + \hat{B}u(k), \qquad z(0) = z_0$$

for certain operators $\hat{A}$ and $\hat{B}$.

b. Let $u \in \boldsymbol{L}_2([0, \tau]; U)$ and show that the values of the mild solution of (4.31) at the discrete-time instants satisfy

$$z(k+1) = \tilde{A}z(k) + \tilde{B}\tilde{u}(k), \qquad z(0) = z_0,$$

for certain operators $\tilde{A}$, $\tilde{B}$, and $\tilde{u}(k) \in \boldsymbol{L}_2([0, 1]; Z)$.

4.11. Assume that A is the infinitesimal generator of a C_0-semigroup on the Hilbert space Z, and let $Q \in \mathcal{L}(Z)$. Define the operator-valued function $D(t) = Q$, $t \in [0, \tau]$. Prove that $A + D(t)$ generates a mild evolution operator, $U(t, s)$, and show that $U(t, s)z_0 = T_Q(t - s)z_0$.

4.12. Consider the partial differential equation

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t) + a(t)z(x, t), \\ \frac{\partial z}{\partial x}(0, t) &= 0, \ \frac{\partial z}{\partial x}(1, t) = 0, \\ z(x, 0) &= z_0(x). \end{aligned} \tag{4.32}$$

a. Assume that $a(\cdot) \equiv a \in \mathbb{C}$. Formulate the partial differential equation (4.32) as an abstract differential equation

$$\dot{z}(t) = (A + D)z(t), \qquad z(0) = z_0, \tag{4.33}$$

where A is given by (2.3). Show that $D \in \mathcal{L}(Z)$, and hence that (4.33) is well posed.

b. If $a(\cdot)$ is a continuous function, show that (4.32) can be formulated as an abstract evolution equation

$$\dot{z}(t) = [A + D(t)]z(t), \qquad z(0) = z_0,$$

where $A + D(t)$ generates a mild evolution operator.

4.13. Consider an inhomogeneous abstract evolution equation of the type

$$\dot{z}(t) = [A + D(t)]z(t) + f(t), \qquad z(s) = z_0, \tag{4.34}$$

where A is the infinitesimal generator of the C_0-semigroup, $T(t)$, and the operator-valued function D is in $\boldsymbol{P}_\infty([0, \tau]; \mathcal{L}(Z))$. We recall from Definition 4.2.8 that the mild solution of (4.34) is given by

$$z(t) = U(t, s)z_0 + \int_s^t U(t, \alpha) f(\alpha)d\alpha, \tag{4.35}$$

where $U(t, s)$ is the mild evolution operator generated by $A + D(\cdot)$.

a. Prove that if $f \in \boldsymbol{L}_p([0, \tau]; Z)$, $p \geq 1$, then (4.35) is continuous on $[0, \tau]$.
Hint: See Lemma 4.1.5 and use the fact that $U(t, s)$ is uniformly bounded on $\Delta(\tau)$ (see the proof of Theorem 4.2.4).

b. Show that if (4.34) does have a classical solution, then it is equal to (4.35).

4.14. Let A be the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z.

a. For $f \in \boldsymbol{L}_p([t_0, t_e]; Z)$, $p \geq 1$, prove that the solution of

$$\dot{z}(t) = -A^* z(t) + f(t), \qquad z(t_e) = z_e \tag{4.36}$$

on $[t_0, t_e]$ is equivalent to the solution of

$$\dot{\tilde{z}}(t) = A^* \tilde{z}(t) + \tilde{f}(t), \qquad \tilde{z}(t_0) = z_e \tag{4.37}$$

on $[t_0, t_e]$. Hence prove that

$$z(t) = T^*(t_e - t) z_e + \int_t^{t_e} T^*(s - t) f(s) ds \tag{4.38}$$

is the unique classical solution of (4.36) when $z_e \in \boldsymbol{D}(A^*)$ and $f \in \boldsymbol{C}^1([t_0, t_e]); Z)$.

For general $z_e \in Z$ and $f \in \boldsymbol{L}_p([t_0, t_e]; Z)$, $p \geq 1$ (4.38) is the *mild solution* of (4.36).

b. Suppose that $U(t, s)$ is the mild evolution operator generated by $A + D(\cdot)$, where $D \in \boldsymbol{P}_\infty([t_0, t_e]; \mathcal{L}(Z))$.

i. Prove that the mild solution of

$$\dot{z}(t) = -A^* z(t) - D^*(t) z(t), \qquad z(t_e) = z_e \tag{4.39}$$

on $[t_0, t_e]$ is given by

$$z(t) = U^*(t_e, t) z_e.$$

Hint: Use Corollary 4.2.7.

ii. Show that the solution to

$$\dot{z}(t) = -A^* z(t) - D^*(t) z(t) + f(t), \qquad z(t_e) = z_e$$

is equivalent to the solution of the standard evolution equation

$$\dot{\tilde{z}}(t) = \tilde{A} \tilde{z}(t) - \tilde{D}(t) \tilde{z}(t) + \tilde{f}(t), \qquad \tilde{z}(t_0) = \tilde{z}_0$$

for certain $\tilde{A}$, $\tilde{D}$, $\tilde{f}$, and $\tilde{z}_0$.

iii. Will the mild solution of i and ii be continuous on $[t_0, t_e]$?
Hint: Show that $U^*(t, s)$ is a mild evolution operator.

4.4 Notes and references

The analysis of the abstract Cauchy problem is now standard and can be found in many texts on semigroup theory, for example, in Bensoussan et al. [19, chapter 1.3] and Pazy [125, chapter 4]. For the case where Z is a Banach space, there are many different definitions of types of solutions to the inhomogeneous Cauchy problem (4.1); our definitions of classical and mild solutions follow Pazy [125, chapter 4], and the concept of a weak solution was originally due to Ball [8] (see also Curtain and Pritchard [40]). Our concepts of weak and mild solution coincide, since we consider the Hilbert space case. However, it is also important to know that the mild solution of the inhomogeneous abstract Cauchy problem on a Hilbert space agrees with the usual weak solution in the case of partial differential equations. Since we could not find a clarification of this in the literature, we have included a justification in the appendix in Example A.5.34. Sharper regularity results of the mild solutions can be found in Bensoussan et al. [19, chapter 1.3] and Pazy [125].

The treatment of perturbed semigroups, mild evolution operators, and abstract evolution equations follows that in Curtain and Pritchard [40, chapter 2]. More detailed results on special types of evolution operators can be found in Pazy [125] and Tanabe [156], for example.

5
State Linear Systems

5.1 Input and outputs

In this chapter, we consider the following class of infinite-dimensional systems with input u and output y:

$$\dot{z}(t) = Az(t) + Bu(t), \ t \geq 0, \qquad z(0) = z_0, \tag{5.1}$$

$$y(t) = Cz(t) + Du(t). \tag{5.2}$$

Definition 5.1.1 $\Sigma(A, B, C, D)$ denotes the *state linear system* (5.1)–(5.2), where A is the infinitesimal generator of the strongly continuous semigroup $T(t)$ on a Hilbert space Z, the *state space*. B is a bounded linear operator from the *input space* U to Z, C is a bounded linear operator from Z to the *output space* Y, and D is a bounded operator from U to Y. Both U and Y are Hilbert spaces. We consider $\Sigma(A, B, C, D)$ for all initial states $z_0 \in Z$ and all *inputs* $u \in \boldsymbol{L}_2([0, \tau]; U)$. The *state* is the mild solution (see Definition 4.1.4) of (5.1)

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \qquad 0 \leq t \leq \tau \tag{5.3}$$

and the *output* $y(\cdot)$ is defined by (5.2). ■

When A, B, C, D are all spatially invariant operators, we call $\Sigma(A, B, C, D)$ a *spatially invariant system*. When A is a Riesz-spectral operator, we call $\Sigma(A, B, C, D)$ a *Riesz-spectral system*.

We recall from Lemma 4.1.5 that $z(\cdot)$ is in $\boldsymbol{C}([0,\tau];Z)$ and so the output y defined by (5.2) is always in $\boldsymbol{L}_2([0,\tau];Y)$. To avoid clutter, we shall also use the notation $\Sigma(A,B,C,-)$ if the value D is unimportant for the definition. Similarly, $\Sigma(A,B,-,-)$ when the operators C and D are not needed, and $\Sigma(A,-,C,-)$ when the operators B and D are not needed.

Depending upon the context, inputs may have the physical interpretation as controls (which one may choose) or as disturbances (an act of nature) and outputs may be interpreted as observations (which one can measure) or as parts of the system whose behaviour we wish to influence.

As our first example we consider a spatially invariant system.

Example 5.1.2 Consider the spatially invariant system

$$\begin{aligned} \dot{z}(t) &= A_{cv}z(t) + B_{cv}u(t), \quad t \geq 0, \qquad z(0) = z_0, \\ y(t) &= C_{cv}z(t) + D_{cv}u(t), \end{aligned}$$

where $A_{cv}, B_{cv}, D_{cv}, C_{cv}$ are bounded convolution operators on the state space $Z = \ell_2(\mathbb{Z};\mathbb{C}^n)$, the input space $U = \ell_2(\mathbb{Z};\mathbb{C}^m)$ and the output space $Y = \ell_2(\mathbb{Z};\mathbb{C}^p)$, respectively. As we saw in Lemma 3.1.3 if A_{cv} is bounded, then there exists $\check{A} \in \boldsymbol{L}_\infty(\partial\mathbb{D};\mathbb{C}^{n\times n})$ such that $A_{cv} = \mathfrak{F}^{-1}\Lambda_{\check{A}}\mathfrak{F}$. Thus the spatially invariant system $\Sigma(A_{cv}, B_{cv}, C_{cv}, D_{cv})$ is isometrically isomorphic to the system $\Sigma(\mathfrak{F}^{-1}A_{cv}\mathfrak{F}, \mathfrak{F}^{-1}B_{cv}\mathfrak{F}, \mathfrak{F}^{-1}C_{cv}\mathfrak{F}, \mathfrak{F}^{-1}D_{cv}\mathfrak{F}) = \Sigma(\Lambda_{\check{A}}, \Lambda_{\check{B}}, \Lambda_{\check{C}}, \Lambda_{\check{D}})$, which is defined on the state space $\boldsymbol{L}_2(\partial\mathbb{D};\mathbb{C}^n)$, the input space $\boldsymbol{L}_2(\partial\mathbb{D};\mathbb{C}^m)$, and the output space $\boldsymbol{L}_2(\partial\mathbb{D};\mathbb{C}^p)$. As we saw in Section 3.1, it is more convenient to work with the Fourier transformed system with parameter $\phi \in \partial\mathbb{D}$

$$\begin{aligned} \frac{\partial \check{z}}{\partial t}(\phi,t) &= \check{A}(\phi)z(\phi,t) + \check{B}(\phi)\check{u}(\phi,t), \quad t \geq 0,\ z(\phi,0) = z_0(\phi); \\ y(\phi,t) &= \check{C}(\phi)\check{z}(\phi,t) + \check{D}(\phi)u(\phi,t), \end{aligned}$$

■

Next we return to our canonical heat equation example.

Example 5.1.3 Consider the metal rod of Example 2.1.1, but this time we control using a heating element around the point x_0 and we measure its temperature around the point x_1. If we suppose that there is no heating or cooling at either end, then a reasonable mathematical model is

$$\begin{aligned} &\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + b(x)u(t), \quad z(x,0) = z_0(x), \\ &\frac{\partial z}{\partial x}(0,t) = \frac{\partial z}{\partial x}(1,t) = 0, \end{aligned} \tag{5.4}$$

$$y(t) = \int_0^1 c(x)z(x,t)dx, \tag{5.5}$$

where b and c represent the "shaping functions" around the control point x_0 and the sensing point x_1, respectively,

$$\begin{aligned} b(x) &= \frac{1}{2\varepsilon}\mathbb{1}_{[x_0-\varepsilon,x_0+\varepsilon]}(x), \\ c(x) &= \frac{1}{2\nu}\mathbb{1}_{[x_1-\nu,x_1+\nu]}(x), \end{aligned}$$

where $\mathbb{1}_{[\alpha,\beta]}(x) = \begin{cases} 1 & \text{for } \alpha \le x \le \beta, \\ 0 & \text{elsewhere.} \end{cases}$

Notice that b and c in this example are both elements in $\boldsymbol{L}_2(0,1)$ for fixed, small, positive constants ν and ε. In Example 3.2.11, we have that $Z = \boldsymbol{L}_2(0,1)$ is a suitable state space for the heat equation without input. So it is easy to see that (5.4) can be formulated as in (5.1) with $Z = \boldsymbol{L}_2(0,1)$, $U = \mathbb{C}$, and $Bu := b(x)u$. $B \in \mathcal{L}(\mathbb{C}, Z)$ and has norm $1/\sqrt{2\varepsilon}$. Furthermore, with $Y = \mathbb{C}$, (5.5) can be formulated as in (5.2) with $D = 0$ and C given by

$$Cz := \int_0^1 c(x)z(x)dx.$$

$C \in \mathcal{L}(Z, \mathbb{C})$ and has norm $1/\sqrt{2\nu}$. ■

Although it may seem artificial to choose $U = Y = \mathbb{C}$ in a physical example, where inputs and outputs take on real values, we choose to complexify all spaces for mathematical reasons. For example, to discuss the spectrum of an operator one must take the underlying scalar field to be the complex numbers. We remark that operators B and C in Example 5.1.3 approximately model "point actuators" and "point sensors", respectively, and as long as ε and ν are positive we obtain bounded B and C operators. Usually, a point actuator is modelled as a delta distribution in the point, i.e., $\delta_{x_0}(x)$ replaces $b(x)$, and this cannot be represented as a bounded B operator in this example: B maps out of the state space. In Section 5.2, we shall discuss some examples with boundary control and give conditions under which it is possible to reformulate such systems on an extended state space with a new bounded B operator. However, this is not always possible nor desirable. The natural formulation of boundary control action, e.g., $\frac{\partial z}{\partial x}(1,t) = u(t)$, leads to an operator mapping outside the state space. It is customary in the control literature to refer to an *unbounded B operator*, although, strictly speaking, while B is not bounded, it is not unbounded either. We shall follow this abuse of terminology. Similarly, point sensors produce observations of the type $Cz = z(x_1)$ and such operators need not be bounded either. Unbounded B and C operators introduce mathematical technicalities that we prefer to avoid at this stage; for simplicity of exposition we only consider bounded B and C operators for most of this book. However, it is important to realize that the theory presented for bounded B and C has a natural generalization to very general unbounded input and output operators, including point actuators and sensors.

5.2 Boundary control systems

In this section we consider systems described by partial differential equations with the control action on the boundary. A typical example is when the temperature of a bar is controlled by control action at one end instead of in the interior as in Example 2.1.1.

Example 5.2.1 Consider a metal bar of length 1 that is perfectly isolated at the right-hand side, and we inject heat at the left-hand side.

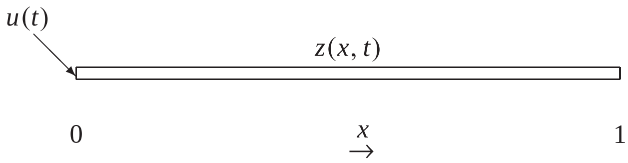

Figure 5.1. A one-dimensional heated bar with boundary control

It is not possible to formulate this in the form

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + u(x,t), \quad z(x,0) = z_0(x),$$
$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t),$$

for an input $u \in \boldsymbol{L}_p([0,\tau];\boldsymbol{L}_2(0,1))$, but we could try to approximate this by the following input

$$u(x,t) := \frac{1}{\varepsilon}1_{[0,\varepsilon]}(x)u(t).$$

The mild solution is then given by (4.4)

$$\begin{aligned} z(x,t) &= \sum_{n=0}^{\infty}\left[e^{\lambda_n t}\langle z_0,\phi_n\rangle\phi_n + \int_0^t e^{\lambda_n(t-s)}\langle u(\cdot,s),\phi_n(\cdot)\rangle\phi_n ds\right] \\ &= \sum_{n=0}^{\infty} e^{\lambda_n t}\langle z_0,\phi_n\rangle\phi_n + \int_0^t u(s)ds + \\ &\quad \sum_{n=1}^{\infty}\frac{2\sin(n\pi\varepsilon)}{n\pi\varepsilon}\cos(n\pi x)\int_0^t e^{-n^2\pi^2(t-s)}u(s)ds, \end{aligned}$$

since $\lambda_n = -n^2\pi^2$, $n \geq 0$, $\phi_n(x) = \sqrt{2}\cos(n\pi x)$, $n \geq 1$, and $\phi_0 = 1$.

However, it is clear the above approximation is not the same as controlling at the boundary. In fact, the correct formulation is

$$\frac{\partial z}{\partial x}(0,t) = z(0,t) - u(t),$$

(see Exercise 6.15). ■

Boundary control problems like the one in the above example occur frequently in the applications, but unfortunately they do not fit into our standard formulation (4.1). However, for sufficiently smooth inputs it is possible to reformulate such problems on an extended state space so that they do lead to an associated system in the standard form (4.1). We shall develop such a theory for the following class of *abstract boundary control problems*:

$$\begin{aligned} \dot{z}(t) &= \mathfrak{A}z(t), \qquad z(0) = z_0, \\ \mathfrak{P}z(t) &= u(t), \end{aligned} \tag{5.6}$$

where $\mathfrak{A} : \boldsymbol{D}(\mathfrak{A}) \subset Z \mapsto Z$, $u(t) \in U$, a separable Hilbert space, and the *boundary operator* $\mathfrak{P} : \boldsymbol{D}(\mathfrak{P}) \subset Z \mapsto U$ satisfies $\boldsymbol{D}(\mathfrak{A}) \subset \boldsymbol{D}(\mathfrak{P})$.

In order to reformulate equation (5.6) into an abstract form (4.1), we need to impose extra conditions on the system.

Definition 5.2.2 The control system (5.6) is a *boundary control system* if the following hold:

a. The operator $A : \boldsymbol{D}(A) \mapsto Z$ with $\boldsymbol{D}(A) = \boldsymbol{D}(\mathfrak{A}) \cap \ker(\mathfrak{P})$ and

$$Az = \mathfrak{A}z \qquad \text{for } z \in \boldsymbol{D}(A) \tag{5.7}$$

is the infinitesimal generator of a C_0-semigroup on Z;

b. There exists a $B \in \mathcal{L}(U, Z)$ such that for all $u \in U$, $Bu \in \boldsymbol{D}(\mathfrak{A})$, the operator $\mathfrak{A}B$ is an element of $\mathcal{L}(U, Z)$ and

$$\mathfrak{P}Bu = u, \qquad u \in U. \tag{5.8}$$

■

Assuming that (5.6) is a boundary control system for $u \in \boldsymbol{C}^2([0, \tau]; U)$, the following abstract differential equation on Z is well posed:

$$\begin{aligned} \dot{v}(t) &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t), \\ v(0) &= v_0. \end{aligned} \tag{5.9}$$

Since A is the infinitesimal generator of a C_0-semigroup and B and $\mathfrak{A}B$ are bounded linear operators, we have from Theorem 4.1.3 that equation (5.9) has a unique classical solution for $v_0 \in \boldsymbol{D}(A)$. Furthermore, we can prove the following relation between the (classical) solutions of (5.6) and (5.9).

Theorem 5.2.3 *Consider the boundary control system (5.6) and the abstract Cauchy equation (5.9). Assume that $u \in \boldsymbol{C}^2([0, \tau]; U)$ for all $\tau > 0$. Then, if*

$v_0 = z_0 - Bu(0) \in \boldsymbol{D}(A)$, *the classical solutions of (5.6) and (5.9) are related by*

$$v(t) = z(t) - Bu(t). \tag{5.10}$$

Furthermore, the classical solution of (5.6) is unique.

Proof Suppose that $v(t)$ is a classical solution of (5.9). Then $v(t) \in \boldsymbol{D}(A) \subset \boldsymbol{D}(\mathfrak{A}) \subset \boldsymbol{D}(\mathfrak{P})$, $Bu(t) \in \boldsymbol{D}(\mathfrak{P})$, and so

$$\mathfrak{P}z(t) = \mathfrak{P}[v(t) + Bu(t)] = \mathfrak{P}v(t) + \mathfrak{P}Bu(t) = u(t),$$

where we have used that $v(t) \in \boldsymbol{D}(A) \subset \ker \mathfrak{P}$ and equation (5.8). Furthermore, from (5.10) we have

$$\begin{aligned} \dot{z}(t) &= \dot{v}(t) + B\dot{u}(t) \\ &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B\dot{u}(t) && \text{by (5.9)} \\ &= Av(t) + \mathfrak{A}Bu(t) \\ &= \mathfrak{A}(v(t) + Bu(t)) && \text{by (5.7)} \\ &= \mathfrak{A}z(t) && \text{by (5.10).} \end{aligned}$$

Thus, if $v(t)$ is a classical solution of (5.9), then $z(t)$ defined by (5.10) is a classical solution of (5.6).

The other implication is proved similarly. The uniqueness of the classical solutions of (5.6) follows from the uniqueness of the classical solutions of (5.9). ■

Although we have reformulated (5.6) as the abstract evolution equation (5.9), it includes a derivative of the control term, which is undesirable. This can be eliminated by reformulating (5.9) on the extended state space $Z^e := U \oplus Z$:

$$\begin{aligned} \dot{z}^e(t) &= \begin{pmatrix} 0 & 0 \\ \mathfrak{A}B & A \end{pmatrix} z^e(t) + \begin{pmatrix} I \\ -B \end{pmatrix} \tilde{u}(t), \\ z^e(0) &= \begin{pmatrix} (z_0^e)_1 \\ (z_0^e)_2 \end{pmatrix}. \end{aligned} \tag{5.11}$$

Theorem 5.2.4 *Consider the abstract differential equation (5.11). If* $v_0 \in \boldsymbol{D}(A)$ *and* $u \in \boldsymbol{C}^2([0,\tau]; U)$, *then (5.11) with* $(z_0^e)_1 = u(0)$, $(z_0^e)_2 = v_0$ *and* $\tilde{u} = \dot{u}$ *has the unique classical solution* $z^e(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, *where* $v(t)$ *is the unique classical solution of (5.9).*

Furthermore, if $z_0 = v_0 + Bu(0)$, *then the classical solution of (5.6) is given by*

$$\begin{aligned} z(t) &= \begin{pmatrix} B & I \end{pmatrix} z^e(t) \\ &= Bu(t) - T(t)Bu(0) + T(t)z_0 - \\ &\quad \int_0^t T(t-s)B\dot{u}(s)ds + \int_0^t T(t-s)\mathfrak{A}Bu(s)ds. \end{aligned} \tag{5.12}$$

Proof From Lemma 4.2.2, we have that the operator $A^e := \begin{pmatrix} 0 & 0 \\ \mathfrak{A}B & A \end{pmatrix}$ with domain $\boldsymbol{D}(A^e) = U \oplus \boldsymbol{D}(A)$ is the infinitesimal generator of a C_0-semigroup on Z^e. Since $\begin{pmatrix} I \\ -B \end{pmatrix} \in \mathcal{L}(U, Z^e)$, (5.11) is well defined, and from Definition 4.1.4 and Lemma 4.2.2 the mild solution of (5.11) is given by

$$\begin{aligned} z^e(t) &= \begin{pmatrix} I & 0 \\ S(t) & T(t) \end{pmatrix} \begin{pmatrix} (z_0^e)_1 \\ (z_0^e)_2 \end{pmatrix} + \\ &\int_0^t \begin{pmatrix} I & 0 \\ S(t-s) & T(t-s) \end{pmatrix} \begin{pmatrix} I \\ -B \end{pmatrix} \tilde{u}(s) ds, \end{aligned}$$

where $S(t)z = \int_0^t T(t-s)\mathfrak{A}Bzds = \int_0^t T(s)\mathfrak{A}Bzds$. Thus

$$(z^e(t))_1 = (z_0^e)_1 + \int_0^t \tilde{u}(s)ds = u(0) + \int_0^t \dot{u}(s)ds = u(t). \tag{5.13}$$

Since $u \in \boldsymbol{C}^2([0, \tau]; U)$, $\tilde{u} \in \boldsymbol{C}^1([0, \tau]; U)$, and $z^e(0) = z_0^e = \begin{pmatrix} u(0) \\ v_0 \end{pmatrix} \in \boldsymbol{D}(A^e)$ by assumption, by Theorem 4.1.3 (5.11) has a unique classical solution that satisfies

$$(\dot{z}^e(t))_1 = \tilde{u}(t) = \dot{u}(t)$$

and

$$\begin{aligned} (\dot{z}^e(t))_2 &= \mathfrak{A}B(z^e(t))_1 + A(z^e(t))_2 - B\tilde{u}(t) \\ &= \mathfrak{A}Bu(t) + A(z^e(t))_2 - B\dot{u}(t) \qquad \text{by (5.13).} \end{aligned}$$

Since $(z_0^e)_2 = v_0$, $(z^e(t))_2 = v(t)$, the unique classical solution of (5.9).

We now suppose that $z_0 = v_0 + Bu(0)$ and calculate

$$\begin{aligned} \begin{pmatrix} B & I \end{pmatrix} z^e(t) &= Bu(t) + (z^e(t))_2 \\ &= Bu(t) + v(t) \\ &= z(t) \qquad \text{by Theorem 5.2.3.} \end{aligned}$$

From Theorem 4.1.3, it follows that

$$v(t) = T(t)v_0 - \int_0^t T(t-s)B\dot{u}(s)ds + \int_0^t T(t-s)\mathfrak{A}Bu(s)ds,$$

which shows (5.12). ■

Summarizing, we have related the classical solutions $z(t)$ of (5.6), $v(t)$ of (5.9) and $z^e(t)$ of (5.11) under the assumptions that $u \in \boldsymbol{C}^2([0, \tau]; U)$ and $z_0 - Bu(0) = \begin{pmatrix} B & I \end{pmatrix} z_0^e - Bu(0) = v_0 \in \boldsymbol{D}(A)$. The abstract equations (5.9) and (5.11) also have well defined mild solutions for $\tilde{u} = \dot{u} \in \boldsymbol{L}_p([0,\tau];U)$ for some $p \geq 1$, $\upsilon_0 \in Z$, and $z_0^e \in Z^e$, respectively. Consequently, under these weaker assumptions, we shall call $z(t)$ defined by (5.12) the *mild solution* of the original boundary control equation (5.6).

To illustrate the mathematical principles, we first consider a rather artificial boundary control problem for a heat equation. The more realistic problem of injecting heat at one end of the metal bar introduced in Example 5.2.1 is a little more complicated and is considered in Exercise 6.15.

Example 5.2.5 Consider the heat equation with boundary control action:

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t), \quad \frac{\partial z}{\partial x}(0,t) = 0, \ \frac{\partial z}{\partial x}(1,t) = u(t), \\ z(x,0) &= z_0(x). \end{aligned} \tag{5.14}$$

(5.14) can be reformulated in the form (5.6) by defining $Z = \boldsymbol{L}_2(0,1)$, $U = \mathbb{C}$,

$$\mathfrak{A} = \frac{d^2}{dx^2} \text{ with } \quad \boldsymbol{D}(\mathfrak{A}) = \{h \in \boldsymbol{L}_2(0,1) \mid h \text{ and } \tfrac{dh}{dx} \text{ are absolutely continuous, } \tfrac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1) \text{ and } \tfrac{dh}{dx}(0) = 0\}$$

and the boundary operator $\mathfrak{P} : \boldsymbol{L}_2(0,1) \to \mathbb{C}$ by

$$\mathfrak{P}h = \frac{dh}{dx}(1) \qquad \text{with } \boldsymbol{D}(\mathfrak{P}) = \boldsymbol{D}(\mathfrak{A}). \tag{5.15}$$

Define $A = \frac{d^2}{dx^2}$ with domain $\boldsymbol{D}(A) = \boldsymbol{D}(\mathfrak{A}) \cap \ker \mathfrak{P} = \{h \in \boldsymbol{L}_2(0,1) \mid h$ and $\frac{dh}{dx}$ are absolutely continuous, $\frac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1)$ and $\frac{dh}{dx}(0) = 0 = \frac{dh}{dx}(1)\}$. As in Example 3.2.11, we see that A is the infinitesimal generator of a C_0-semigroup, and $Bu = b(x)u$, where $b(x) = \frac{1}{2}x^2$ is contained in the domain of $\mathfrak{A}$ with $\mathfrak{P}Bu = u$. Thus (5.14) defines a boundary control system and since $\mathfrak{A}B = I$, the mild solution of (5.14) is given by

$$\begin{aligned} z(t) &= Bu(t) - T(t)Bu(0) + T(t)z_0 - \\ &\int_0^t T(t-s)B\dot{u}(s)ds + \int_0^t T(t-s)1u(s)ds. \end{aligned} \tag{5.16}$$

In Example 2.1.5, we had

$$T(t) = \sum_{n=0}^{\infty} e^{\lambda_n t}\langle\cdot,\phi_n\rangle\phi_n,$$

where $\lambda_n = -n^2\pi^2$, $n \geq 0$, $\phi_0 = 1$ and $\phi_n(\cdot) = \sqrt{2}\cos(n\pi\cdot)$, $n \geq 1$, and so

$$\int_0^t T(t-s)1u(s)ds = \int_0^t T(t-s)\phi_0 u(s)ds = \int_0^t u(s)ds\phi_0, \tag{5.17}$$

and

$$
\begin{aligned}
&\int_0^t T(t-s)B\dot{u}(s)ds \\
&\quad = \int_0^t \sum_{n=0}^{\infty} e^{\lambda_n(t-s)}\langle \frac{1}{2}x^2, \phi_n\rangle \phi_n \dot{u}(s)ds \\
&\quad = \sum_{n=0}^{\infty} \int_0^t e^{\lambda_n(t-s)}\langle \frac{1}{2}x^2, \phi_n\rangle \phi_n \dot{u}(s)ds \\
&\quad = \frac{1}{6}\int_0^t \dot{u}(s)ds\phi_0 + \sum_{n=1}^{\infty} \int_0^t e^{-n^2\pi^2(t-s)}\langle \frac{1}{2}x^2, \phi_n\rangle \dot{u}(s)ds\phi_n \\
&\quad = \frac{1}{6}[u(t)-u(0)]\phi_0 + \sum_{n=1}^{\infty} \int_0^t e^{-n^2\pi^2(t-s)} \frac{(-1)^n\sqrt{2}}{n^2\pi^2}\dot{u}(s)ds\phi_n \\
&\quad = \frac{1}{6}[u(t)-u(0)]\phi_0 + \sum_{n=1}^{\infty} \Bigg[\frac{(-1)^n\sqrt{2}}{n^2\pi^2}[u(t)-u(0)e^{-n^2\pi^2 t}] - \\
&\qquad\qquad (-1)^n\sqrt{2}\int_0^t e^{-n^2\pi^2(t-s)}u(s)ds \Bigg]\phi_n. \qquad (5.18)
\end{aligned}
$$

Furthermore, B has the expansion

$$
Bu = \sum_{n=0}^{\infty}\langle \frac{1}{2}x^2, \phi_n\rangle \phi_n u = \frac{1}{6}\phi_0 u + \sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{2}}{n^2\pi^2}\phi_n u, \qquad (5.19)
$$

and

$$
T(t)Bu = \frac{1}{6}\phi_0 u + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\frac{(-1)^n\sqrt{2}}{n^2\pi^2}\phi_n u. \qquad (5.20)
$$

Combining (5.16), (5.17), (5.18), (5.19), and (5.20), it follows that the mild solution of (5.14) is given by

$$\begin{aligned}
z(t) &= \frac{1}{6}\phi_0 u(t) + \sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{2}}{n^2\pi^2}\phi_n u(t) - \frac{1}{6}\phi_0 u(0) - \\
&\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\frac{(-1)^n\sqrt{2}}{n^2\pi^2}\phi_n u(0) + T(t)z_0 - \frac{1}{6}[u(t)-u_0]\phi_0 - \\
&\sum_{n=1}^{\infty}\left[\frac{(-1)^n\sqrt{2}}{n^2\pi^2}[u(t)-u_0 e^{-n^2\pi^2 t}] - \right. \\
&\left. (-1)^n\sqrt{2}\int_0^t e^{-n^2\pi^2(t-s)}u(s)ds\right]\phi_n + \int_0^t u(s)ds\phi_0 \\
&= T(t)z_0 + \int_0^t u(s)ds\phi_0 + \\
&\sum_{n=1}^{\infty}(-1)^n\sqrt{2}\int_0^t e^{-n^2\pi^2(t-s)}u(s)ds\phi_n. \qquad (5.21)
\end{aligned}$$

The mild solution of (5.14) is defined assuming that $\dot{u} \in \boldsymbol{L}_p([0,\tau];U)$, for some $p \geq 1$, but in fact equation (5.21) is well defined for every $u \in \boldsymbol{L}_2([0,\tau];U)$, since

$$|\int_0^t e^{-n^2\pi^2(t-s)}u(s)ds|^2 \leq \frac{1-e^{-2n^2\pi^2 t}}{2n^2\pi^2}\int_0^t |u(s)|^2 ds.$$

■

Wave equations and beam equations with boundary control can also often be formulated using the same approach.

Example 5.2.6 Consider the wave equation of Example 3.2.13 with boundary control action

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t), \\
&\frac{\partial w}{\partial x}(0,t) = 0, \ \frac{\partial w}{\partial x}(1,t) = u(t), \qquad (5.22) \\
&w(x,0) = w_0(x), \ \frac{\partial w}{\partial t}(x,0) = w_1(x).
\end{aligned}$$

Recall from Example 3.2.13 the following state space $Z = \boldsymbol{D}(A_0^{\frac{1}{2}}) \oplus \boldsymbol{L}_2(0,1)$, where $A_0 h = -\dfrac{d^2h}{dx^2}$ for $h \in \boldsymbol{D}(A_0) = \{h \in \boldsymbol{L}_2(0,1) \mid h$ and $\frac{dh}{dx}$ are absolutely continuous, $\frac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1)$ and $\frac{dh}{dx}(0) = 0 = \frac{dh}{dx}(1)\}$. Then we can reformulate

(5.22) in the form (5.6) by defining $U = \mathbb{C}$,

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}$$

with domain

$$\boldsymbol{D}(\mathfrak{A}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z \mid z_1, \tfrac{dz_1}{dx} \text{ are absolutely continuous,} \right. \\ \left. \tfrac{d^2 z_1}{dx^2} \in \boldsymbol{L}_2(0,1), \quad \tfrac{dz_1}{dx}(0) = 0 \text{ and } z_2 \in \boldsymbol{D}(A_0^{\frac{1}{2}}) \right\}$$

and the boundary operator $\mathfrak{P} : \boldsymbol{D}(\mathfrak{P}) \subset Z \mapsto \mathbb{C}$ by

$$\mathfrak{P}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{dz_1}{dx}(1) \qquad \text{with} \qquad \boldsymbol{D}(\mathfrak{P}) = \boldsymbol{D}(\mathfrak{A}).$$

The system (5.22) is a boundary control system, since

$$A = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}$$

with domain $\boldsymbol{D}(A) = \boldsymbol{D}(\mathfrak{A}) \cap \ker(\mathfrak{P}) = D(A_0) \oplus \boldsymbol{D}(A_0^{\frac{1}{2}})$ is the infinitesimal generator of a C_0-semigroup (see Example 3.2.13), and Bu defined by $Bu = b(x)u$, where $b(x) = \begin{pmatrix} \frac{1}{2}x^2 \\ 0 \end{pmatrix}$, is contained in the domain of $\mathfrak{A}$ with $\mathfrak{P}Bu = u$, and $\mathfrak{A}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. ■

5.3 Exercises

5.1. In this exercise, we formulate some standard examples with inputs and outputs.

a. Show that the following systems can be formulated as linear systems $\Sigma(A, B, C, D)$. Specify A, B, C, and D and the appropriate input, state, and output spaces.

i. The heat equation with Dirichlet boundary conditions:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + b(x)u(t),$$

$$z(0,t) = 0 = z(1,t),$$

$$y(t) = \int_0^1 c(x) z(x,t) dx,$$

where $b(x) = \frac{1}{2\varepsilon}\mathbb{1}_{(x_0-\varepsilon, x_0+\varepsilon)}(x)$, $c(x) = \frac{1}{2\nu}\mathbb{1}_{(x_1-\nu, x_1+\nu)}(x)$.

Hint: See Exercise 3.6.

ii. The perturbed heat equation:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) - \alpha \frac{\partial z}{\partial x}(x,t) + b(x)u(t),$$

$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t),$$

$$y(t) = \int_0^1 c(x)z(x,t)dt,$$

where b and c are chosen as in i.

Hint: See Exercise 2.13.

iii. The wave equation:

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2 w}{\partial x^2}(x,t) = b(x)u(t),$$

$$w(0,t) = 0 = w(1,t),$$

$$y(t) = w(x_1,t), \qquad 0 \le x_1 \le 1,$$

with b chosen as in i.

Hint: See Examples 2.3.5 and 3.2.12 and prove that $C \in \mathcal{L}(Z, \mathbb{C})$ if and only if C is linear and $\sum_n |C\phi_n|^2 < \infty$, where $\{\phi_n\}$ is a Riesz basis of Z.

iv. The simply supported beam:

$$\frac{\partial^2 f}{\partial t^2}(x,t) = -\frac{\partial^4 f}{\partial x^4}(x,t) + b(x)u(t),$$

$$f(0,t) = f(1,t) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0,t) = \frac{\partial^2 f}{\partial x^2}(1,t) = 0,$$

$$f(x,0) = f_1(x), \quad \frac{\partial f}{\partial t}(x,0) = f_2(x),$$

where $f(x,t)$ is the displacement of the beam with respect to the position at rest, b is chosen as in i, and the measurement is taken to be

$$y(t) = f(\frac{1}{3}, t).$$

Hint: See Exercise 3.9 and prove that $C \in \mathcal{L}(Z, \mathbb{C})$ if and only if C is linear and $\sum_n |C\phi_n|^2 < \infty$, where $\{\phi_n\}$ is a Riesz basis of Z.

5.4 Notes and references

The treatment of boundary control systems is very different from that found in other texts such as Bensoussan et al. [19], but it was introduced in Fattorini [59] as an effective means of studying controllability for such systems. Although it is not completely general and has the disadvantage of introducing an artificial state space, it is a mathematically simple way of formulating many partial differential equations with boundary control, which occur in applications. It has been extended to retarded differential equations with delayed control by Pandolfi in [120].

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