

— This is page ii  
— Printer: Opaque this

# Contents

<b>List of Figures</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Motivation . . . . .	1
1.2 Systems theory concepts in finite dimensions . . . . .	4
1.3 Aims of this book . . . . .	10
<b>2 Semigroup Theory</b>	<b>13</b>
2.1 Strongly continuous semigroups . . . . .	13
2.2 Abstract differential equations . . . . .	34
2.3 Contraction and dual semigroups . . . . .	36
2.4 Exercises . . . . .	42
2.5 Notes and references . . . . .	50
<b>3 Classes of Semigroups</b>	<b>51</b>
3.1 Spatially invariant semigroups . . . . .	51
3.2 Riesz-spectral operators . . . . .	59
3.3 Exercises . . . . .	77
3.4 Notes and references . . . . .	89
<b>4 The Cauchy Problem</b>	<b>90</b>
4.1 The abstract Cauchy problem . . . . .	90
4.2 Perturbations and composite systems . . . . .	96
4.3 Exercises . . . . .	107

4.4	Notes and references . . . . .	111
<b>5</b>	<b>State Linear Systems</b>	<b>113</b>
5.1	Input and outputs . . . . .	113
5.2	Boundary control systems . . . . .	116
5.3	Exercises . . . . .	123
5.4	Notes and references . . . . .	124
<b>6</b>	<b>Input-Output Maps</b>	<b>126</b>
6.1	Impulse response . . . . .	126
6.2	Transfer functions . . . . .	130
6.3	Transfer functions for boundary control systems . . . . .	139
6.4	Transfer functions and the Laplace transform of the impulse response . . . . .	141
6.5	Exercises . . . . .	146
6.6	Notes and references . . . . .	162
<b>7</b>	<b>Stability</b>	<b>163</b>
7.1	Exponential stability . . . . .	163
7.2	Weak and strong stability . . . . .	176
7.3	Exercises . . . . .	182
7.4	Notes and references . . . . .	187
<b>8</b>	<b>Stabilizability and Detectability</b>	<b>189</b>
8.1	Exponential stabilizability and detectability . . . . .	189
8.2	Tests for exponential stabilizability and detectability . . . . .	199
8.3	Compensator design . . . . .	209
8.4	Stabilization of colocated systems . . . . .	215
8.5	Exercises . . . . .	217
8.6	Notes and references . . . . .	225
<b>A</b>	<b>Mathematical Background</b>	<b>228</b>
A.1	Complex analysis . . . . .	228
A.2	Normed linear spaces . . . . .	235
	A.2.1 General theory . . . . .	235
	A.2.2 Hilbert spaces . . . . .	241
A.3	Operators on normed linear spaces . . . . .	246
	A.3.1 General theory . . . . .	246
	A.3.2 Operators on Hilbert spaces . . . . .	262
A.4	Spectral theory . . . . .	274
	A.4.1 General spectral theory . . . . .	274
	A.4.2 Spectral theory for compact normal operators . . . . .	281
A.5	Integration and differentiation theory . . . . .	287
	A.5.1 Measure theory . . . . .	287
	A.5.2 Integration theory . . . . .	288

	A.5.3	Differentiation theory . . . . .	296
A.6		Frequency-domain spaces . . . . .	302
	A.6.1	Laplace and Fourier transforms . . . . .	302
	A.6.2	Frequency-domain spaces . . . . .	305
	A.6.3	The Hardy spaces . . . . .	309
	A.6.4	Frequency domain spaces on the unit disc . . . . .	314
A.7		Algebraic concepts . . . . .	321
	A.7.1	General definitions . . . . .	321
	A.7.2	Coprime factorizations over principal ideal domains . . . . .	326
	A.7.3	Coprime factorizations over commutative integral domains . . . . .	331
	A.7.4	The convolution algebras $\mathcal{A}(\beta)$ . . . . .	333
		<b>References</b>	<b>340</b>
		<b>Index</b>	<b>351</b>

— This is page vi  
— Printer: Opaque this

## List of Figures

3.1	Left: Spectrum of $\check{A}$ for Example 3.1.7 Right: Spectrum of $\check{A}$ for Example 3.1.8 . . . . .	58
5.1	A one-dimensional heated bar with boundary control . . . . .	116
6.1	Series connection . . . . .	146
6.2	Parallel connection . . . . .	147
6.3	Feedback connection . . . . .	147
6.4	A one-dimensional heated rod . . . . .	153
6.5	A one-dimensional heated rod . . . . .	161
8.1	General closed-loop system . . . . .	210
8.2	$\Sigma(A, B, C, -)$ with compensator (5.58) . . . . .	212
8.3	Closed-loop system of Exercise 5.21 . . . . .	222
A.1	The relationship between $T^*$ and $T'$ . . . . .	266

— This is page viii  
— Printer: Opaque this

# 4

## The Cauchy Problem

### 4.1 The abstract Cauchy problem

In Lemma 2.2.2 we saw that if  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $T(t)$  and  $z_0 \in \mathbf{D}(A)$ , then the unique classical solution of the abstract homogeneous Cauchy initial value problem

$$\dot{z}(t) = Az(t), \quad t \geq 0, \quad z(0) = z_0$$

is given by

$$z(t) = T(t)z_0,$$

which coincides with the mild solution (see Definition 2.2.3). If  $z_0 \notin \mathbf{D}(A)$ , then  $z(t) = T(t)z_0$  is by definition still the mild solution.

In this section we extend these concepts to the abstract inhomogeneous Cauchy problem

$$\dot{z}(t) = Az(t) + f(t), \quad t \geq 0, \quad z(0) = z_0, \quad (4.1)$$

where for the moment we shall assume that  $f \in \mathbf{C}([0, \tau]; Z)$ . (4.1) is also called an *abstract evolution equation* or *abstract differential equation*. First we have to define what we mean by a solution of (4.1), and we begin with the notion of a classical solution.  $\mathbf{C}^1([0, \tau]; Z)$  will denote the class of continuous functions on  $[0, \tau]$  whose derivative is again continuous on  $[0, \tau]$ .



**Definition 4.1.1** Consider equation (4.1) on the Hilbert space  $Z$ . The function  $z(t)$  is a *classical solution* of (4.1) on  $[0, \tau]$  if  $z(t) \in C^1([0, \tau]; Z)$ ,  $z(t) \in D(A)$  for all  $t \in [0, \tau]$  and  $z(t)$  satisfies (4.1) for all  $t \in [0, \tau]$ .

The function  $z(t)$  is a *classical solution on  $[0, \infty)$*  if  $z(t)$  is a classical solution on  $[0, \tau]$  for every  $\tau \geq 0$ . ■

We remark that when  $f = 0$  and  $\tau = \infty$  this definition reduces to Definition 2.2.1 for a classical solution for the homogeneous case.

**Lemma 4.1.2** Assume that  $f \in C([0, \tau]; Z)$  and that  $z$  is a classical solution of (4.1) on  $[0, \tau]$ . Then  $Az(\cdot)$  is an element of  $C([0, \tau]; Z)$ , and

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)ds. \quad (4.2)$$

**Proof** From (4.1), we have that  $Az(t) = \dot{z}(t) - f(t)$  and  $\dot{z} \in C([0, \tau]; Z)$  shows that  $Az(\cdot) \in C([0, \tau]; Z)$ .

We now prove (4.2). Let  $t$  be an arbitrary, but fixed, element of  $(0, \tau]$  and consider the function  $T(t-s)z(s)$  for  $s \in [0, t]$ . We shall show that this function is differentiable in  $s$ . Let  $h$  be sufficiently small and consider

$$\begin{aligned} & \frac{T(t-s-h)z(s+h) - T(t-s)z(s)}{h} \\ &= \frac{T(t-s-h)z(s+h) - T(t-s-h)z(s)}{h} + \\ & \quad \frac{T(t-s-h)z(s) - T(t-s)z(s)}{h}. \end{aligned}$$

If  $h$  converges to zero, then the last term converges to  $-AT(t-s)z(s)$ , since  $z(s) \in D(A)$ . Thus it remains to show that the first term converges. We have the following equality

$$\begin{aligned} & \frac{T(t-s-h)z(s+h) - T(t-s-h)z(s)}{h} - T(t-s)\dot{z}(s) \\ &= T(t-s-h)\frac{z(s+h) - z(s)}{h} - T(t-s-h)\dot{z}(s) + \\ & \quad T(t-s-h)\dot{z}(s) - T(t-s)\dot{z}(s). \end{aligned}$$

The uniform boundedness of  $T(t)$  on any compact interval and the strong continuity allow us to conclude from the last equality that

$$\lim_{h \rightarrow 0} \|T(t-s-h)\frac{z(s+h) - z(s)}{h} - T(t-s)\dot{z}(s)\| = 0.$$

So we have proved that

$$\begin{aligned} \frac{d}{ds}[T(t-s)z(s)] &= -AT(t-s)z(s) + T(t-s)[Az(s) + f(s)] \\ &= T(t-s)f(s). \end{aligned}$$

Thus a classical solution to (4.1) necessarily has the form (4.2). ■

Equation (4.2) is reminiscent of the variation of constants formula for ordinary differential equations. It may be thought that (4.2) is always a classical solution of (4.1), but this is not generally true. However, we are able to prove the following partial converse.

**Theorem 4.1.3** *If  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $T(t)$  on a Hilbert space  $Z$ ,  $f \in C^1([0, \tau]; Z)$  and  $z_0 \in \mathbf{D}(A)$ , then (4.2) is continuously differentiable on  $[0, \tau]$  and it is the unique classical solution of (4.1).*

**Proof Uniqueness:** If  $z_1$  and  $z_2$  are two different solutions, then their difference  $\Delta(t) = z_1(t) - z_2(t)$  satisfies the differential equation

$$\frac{d\Delta}{dt} = A\Delta, \quad \Delta(0) = 0$$

and so we need to show that its only solution is  $\Delta(t) \equiv 0$ . To do this, define  $y(s) = T(t-s)\Delta(s)$  for a fixed  $t$  and  $0 \leq s \leq t$ . Clearly,  $\frac{dy}{ds} = 0$  and so  $y(s) = \text{constant} = T(t)\Delta(0) = 0$ . However,  $y(t) = \Delta(t)$  shows that  $\Delta(t) = 0$ .

**Existence:** Clearly, all we need to show now is that  $v(t) = \int_0^t T(t-s)f(s)ds$  is an element of  $C^1([0, \tau]; Z)$ , takes values in  $\mathbf{D}(A)$ , and satisfies differential equation (4.1). Now

$$\begin{aligned} v(t) &= \int_0^t T(t-s)[f(0) + \int_0^s \dot{f}(\alpha)d\alpha]ds \\ &= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(t-s)\dot{f}(\alpha)dsd\alpha, \end{aligned}$$

where we have used Fubini's Theorem A.5.27. From Theorem 2.1.13.e, it follows that

$$T(t-\alpha)z - z = A \int_\alpha^t T(t-s)zds \quad \text{for all } z \in Z.$$

Hence  $v(t) \in \mathbf{D}(A)$ , and  $\int_0^t \|A \int_\alpha^t T(t-s)\dot{f}(\alpha)ds\|d\alpha = \int_0^t \|T(t-\alpha)\dot{f}(\alpha) - \dot{f}(\alpha)\|d\alpha < \infty$ . Thus, since  $A$  is closed, by Theorem A.5.28 we have that

$$\begin{aligned} Av(t) &= [T(t) - I]f(0) + \int_0^t [T(t-\alpha) - I]\dot{f}(\alpha)d\alpha \\ &= T(t)f(0) + \int_0^t T(t-\alpha)\dot{f}(\alpha)d\alpha - f(t). \end{aligned}$$

Now, since the convolution product is commutative, i.e.,  $\int_0^t g(t-s)h(s)ds = \int_0^t g(s)h(t-s)ds$ , we have that

$$v(t) = \int_0^t T(s)f(t-s)ds$$

and so

$$\begin{aligned}\frac{dv}{dt}(t) &= T(t)f(0) + \int_0^t T(s)\dot{f}(t-s)ds \\ &= T(t)f(0) + \int_0^t T(t-s)\dot{f}(s)ds,\end{aligned}$$

once again using commutativity of the convolution product. It follows that  $\frac{dv}{dt}$  is continuous and

$$\frac{dv}{dt}(t) = Av(t) + f(t).$$

■

The conditions of Theorem 4.1.3 are too strong for control applications, where in general we do not wish to assume that  $f \in \mathbf{C}^1([0, \tau]; Z)$ . So we introduce the following weaker concept of a solution of (4.1).

**Definition 4.1.4** If  $f \in L_p([0, \tau]; Z)$  for a  $p \geq 1$  and  $z_0 \in Z$ , then we call (4.2) a *mild solution* of (4.1) on  $[0, \tau]$ . ■

We note that (4.2) is a well defined integral in the sense of Bochner or Pettis, (see Lemma A.5.10 and Example A.5.20). Of course, if  $f \in L_p([0, \tau]; Z)$  for some  $p \geq 1$ , then necessarily  $f \in L_1([0, \tau]; Z)$ . In our applications, we usually consider mild solutions for  $f$  in  $L_2([0, \tau]; Z)$ .

In Exercise 4.2 we show that the mild solution satisfies an integral equation, which for bounded  $A$  is the differential equation (4.1) once integrated.

**Lemma 4.1.5** Assume that  $f \in L_p([0, \tau]; Z)$  for a  $p \geq 1$  and  $z_0 \in Z$ . The mild solution  $z(t)$  defined by (4.2) is continuous on  $[0, \tau]$ .

**Proof** Since  $T(t)z_0$  is continuous, we can assume without loss of generality that  $z_0 = 0$ . For  $\delta > 0$ , consider

$$\begin{aligned}z(t+\delta) - z(t) &= \int_0^t [(T(t+\delta-s) - T(t-s))f(s)]ds + \\ &\quad \int_t^{t+\delta} T(t+\delta-s)f(s)ds.\end{aligned}$$

Then with  $\frac{1}{p} + \frac{1}{q} = 1$  we estimate

$$\begin{aligned}\|z(t+\delta) - z(t)\| &\leq \| [T(\delta) - I]z(t) \| + \\ &\quad \left( \int_t^{t+\delta} \|T(t+\delta-s)\|^q ds \right)^{\frac{1}{q}} \left( \int_t^{t+\delta} \|f(s)\|^p ds \right)^{\frac{1}{p}}\end{aligned}$$

and the right-hand side converges to 0 as  $\delta \rightarrow 0^+$  by (2.13) and Theorem 2.1.7.a. Now consider

$$\begin{aligned}z(t-\delta) - z(t) &= \int_0^{t-\delta} [T(t-\delta-s) - T(t-s)]f(s)ds - \int_{t-\delta}^t T(t-s)f(s)ds,\end{aligned}$$

noting that  $[T(t - \delta - s) - T(t - s)]f(s)$  is integrable, since  $f \in L_p([0, \tau]; Z)$  and using the properties of  $T(t)$  from Theorem 2.1.7.a and Lemma A.5.10 (see Example A.5.20). Estimating the integral above yields

$$\|z(t - \delta) - z(t)\| \leq \int_0^{t-\delta} \| [T(t - \delta - s) - T(t - s)]f(s) \| ds + \int_{t-\delta}^t \| T(t - s)f(s) \| ds.$$

Now  $[T(t - \delta - s) - T(t - s)]f(s) \rightarrow 0$  as  $\delta \rightarrow 0$ , and from Theorem 2.1.7 there exists a constant  $M_t$ , depending only on  $t$ , such that  $\| [T(t - \delta - s) - T(t - s)]f(s) \| \leq M_t \| f(s) \|$ . So the first term converges to zero  $\delta \rightarrow 0$  by the Lebesgue Dominated Convergence Theorem A.5.26, and the second term also tends to zero by similar arguments. ■

In fact, this mild solution is the same as the concept of a *weak solution* used in the study of partial differential equations.

**Definition 4.1.6** Let  $f \in L_p([0, \tau]; Z)$  for a  $p \geq 1$ . We call  $z$  a *weak solution* of (4.1) on  $[0, \tau]$  if for every  $z_1 \in \mathcal{D}(A^*)$  the following holds:

- a.  $z(t)$  is continuous on  $[0, \tau]$  and  $z(0) = z_0$ ;
- b.  $\langle z(t), z_1 \rangle$  is absolutely continuous on  $[0, \tau]$ ;
- c. For almost all  $t \in [0, \tau]$

$$\frac{d}{dt} \langle z(t), z_1 \rangle = \langle z(t), A^* z_1 \rangle + \langle f(t), z_1 \rangle.$$

We call  $z$  a *weak solution* of (4.1) on  $[0, \infty)$  if it is a weak solution on  $[0, \tau]$  for every  $\tau \geq 0$ . ■

**Theorem 4.1.7** For every  $z_0 \in Z$  and every  $f \in L_p([0, \tau]; Z)$  there exists a unique weak solution of (4.1) that is the mild solution of (4.1).

**Proof** a. First we prove that  $z(t) = T(t)z_0$  is a weak solution when  $f = 0$ . Theorem 2.1.7.b. implies that  $T(t)z_0$  is continuous on  $[0, \tau]$ . Recall from Theorem 2.3.6 that  $T^*(t)$  is a  $C_0$  semigroup with generator  $A^*$ . Thus using Theorem 2.1.13.b. we have

$$\frac{d}{dt} T^*(t)z_1 = A^* T^*(t)z_1 = T^*(t)A^* z_1 \text{ for } z_1 \in \mathcal{D}(A^*).$$

Hence

$$\frac{d}{dt} \langle T(t)z_0, z_1 \rangle = \frac{d}{dt} \langle z_0, T^*(t)z_1 \rangle = \langle z_0, T^*(t)A^* z_1 \rangle = \langle z(t), A^* z_1 \rangle,$$

b. Next we prove that  $z(t) = \int_0^t T(t-s)f(s)ds$  is a weak solution when  $z_0 = 0$ . In Lemma 4.1.5 we showed that it is continuous on  $[0, \tau]$ . Now

$$\begin{aligned} \frac{d}{dt} \langle \int_0^t T(t-s)f(s)ds, z_1 \rangle &= \frac{d}{dt} \int_0^t \langle f(s), T^*(t-s)z_1 \rangle ds \\ &= \langle f(t), z_1 \rangle + \int_0^t \frac{d}{dt} \langle f(s), T^*(t-s)z_1 \rangle ds \\ &= \langle f(t), z_1 \rangle + \int_0^t \langle f(s), T^*(t-s)A^*z_1 \rangle ds \\ &= \langle f(t), z_1 \rangle + \langle z(t), A^*z_1 \rangle, \end{aligned}$$

and  $z(t) = \int_0^t T(t-s)f(s)ds$  is a weak solution when  $z_0 = 0$ .

c. Combining part a and b shows that the mild solution is a weak solution.

d. It remains to prove the uniqueness of the weak solution. Suppose that  $\bar{z}(t)$  is a second weak solution and  $z_1 \in \mathbf{D}(A^*)$ . Then  $\Delta(t) = z(t) - \bar{z}(t)$  satisfies  $\Delta(0) = 0$  and

$$\frac{d}{dt} \langle \Delta(t), z_1 \rangle = \langle \Delta(t), A^*z_1 \rangle \quad \text{for almost all } t \in [0, \tau].$$

For fixed  $t > 0$  and  $0 \leq s \leq t$  differentiating  $\langle T(t-s)\Delta(s), z_1 \rangle$  yields

$$\begin{aligned} \frac{d}{ds} \langle T(t-s)\Delta(s), z_1 \rangle &= \frac{d}{ds} \langle \Delta(s), T(t-s)^*z_1 \rangle \\ &= \langle \Delta(s), A^*T(t-s)^*z_1 \rangle - \langle \Delta(s), T(t-s)A^*z_1 \rangle = 0. \end{aligned}$$

Thus  $\langle T(t-s)\Delta(s), z_1 \rangle$  is constant on  $[0, t]$ . Hence the value  $\langle \Delta(t), z_1 \rangle$  at  $s = 0$  equals 0, the value at  $s = 0$ . Since  $\mathbf{D}(A^*)$  is dense in  $Z$ , it follows that  $\Delta(t) = 0$  for  $t \geq 0$ . ■

From the above it might be considered more logical to use (4.2) as the definition of the dynamical system. However, we follow the custom of using equation (4.1), even when  $z(t)$  is not differentiable in the usual sense. It is the natural generalization of the finite-dimensional differential equation. When we write (4.1) we use it symbolically and what we actually mean is the mild solution (4.2).

**Example 4.1.8** In this example, we again consider the heat equation of Example 2.1.1. The model of the heated bar was given by

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + u(x, t), \quad z(x, 0) = z_0(x),$$

$$\frac{\partial z}{\partial x}(0, t) = 0 = \frac{\partial z}{\partial x}(1, t).$$

We saw in Example 3.2.11 that with  $u = 0$ , this can be formulated as an abstract differential equation on  $Z = L_2(0, 1)$  of the form

$$\dot{z}(t) = Az(t), \quad t \geq 0, \quad z(0) = z_0,$$

where

$$\begin{aligned} Ah &= \frac{d^2h}{dx^2} \text{ with} \\ D(A) &= \left\{ h \in L_2(0, 1) \mid h, \frac{dh}{dx} \text{ are absolutely continuous,} \right. \\ &\quad \left. \frac{d^2h}{dx^2} \in L_2(0, 1) \text{ and } \frac{dh}{dx}(0) = 0 = \frac{dh}{dx}(1) \right\}. \end{aligned}$$

In Example 3.2.11 we showed that this operator generates the  $C_0$ -semigroup from Example 2.1.5.

We can include the control term in this formulation as follows:

$$\dot{z}(t) = Az(t) + u(t), \quad t \geq 0, \quad z(0) = z_0,$$

provided that  $u(x, t) \in L_p([0, \tau]; L_2(0, 1))$  for some  $p \geq 1$ . The solution is given by (4.2), which, by Example 3.2.11, we can write as

$$\begin{aligned} z(x, t) &= \sum_{n=0}^{\infty} e^{\lambda_n t} \langle z_0, \phi_n \rangle \phi_n(x) + \int_0^t \sum_{n=0}^{\infty} e^{\lambda_n(t-s)} \langle u(\cdot, s), \phi_n(\cdot) \rangle \phi_n(x) ds \\ &= \int_0^1 z_0(y) dy + \sum_{n=1}^{\infty} 2e^{\lambda_n t} \int_0^1 z_0(y) \cos(n\pi y) dy \cos(n\pi x) + \\ &\quad \int_0^t \int_0^1 u(y, s) dy ds + \\ &\quad \int_0^t \sum_{n=1}^{\infty} e^{-n^2\pi^2(t-s)} 2 \int_0^1 u(y, s) \cos(n\pi y) dy \cos(n\pi x) ds, \quad (4.3) \end{aligned}$$

since  $\lambda_n = -n^2\pi^2$ ,  $n \geq 0$ ,  $\phi_n(x) = \sqrt{2} \cos(n\pi x)$ ,  $n \geq 1$  and  $\phi_0(x) = 1$ . We see that (4.3) equals (2.4). ■

The above example is typical for the Riesz-spectral class of operators discussed in Section 2.3 for which the mild solution has the explicit form

$$\sum_{n=1}^{\infty} \left[ e^{\lambda_n t} \langle z_0, \psi_n \rangle \phi_n + \int_0^t e^{\lambda_n(t-s)} \langle f(s), \psi_n \rangle \phi_n ds \right]. \quad (4.4)$$

In general, we do not have an explicit solution for (4.1).

## 4.2 Perturbations and composite systems

In applications to control problems, the inhomogeneous term  $f$  in (4.1) is often determined by a control input of feedback type, namely,

$$f(t) = Dz(t),$$

where  $D \in \mathcal{L}(Z)$ . This leads to the new Cauchy problem

$$\dot{z}(t) = (A + D)z(t), \quad t \geq 0, \quad z(0) = z_0, \quad (4.5)$$

or in its integrated form

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Dz(s)ds. \quad (4.6)$$

We expect that the perturbed system operator,  $A + D$ , is the infinitesimal generator of another  $C_0$ -semigroup  $T_D(t)$  so that the solution of (4.5) is given by  $z(t) = T_D(t)z_0$ . To prove this, we must study the operator integral equation

$$S(t)z_0 = T(t)z_0 + \int_0^t T(t-s)DS(s)z_0ds \quad \text{where } z_0 \in Z, \quad (4.7)$$

which is obtained by substituting  $z(t) = S(t)z_0$  in (4.6).

**Theorem 4.2.1** *Suppose that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $T(t)$  on a Hilbert space  $Z$  and that  $D \in \mathcal{L}(Z)$ . Then  $A + D$  is the infinitesimal generator of a  $C_0$ -semigroup  $T_D(t)$  which is the unique solution of (4.7) in the class of strongly continuous operators on  $Z$ . Moreover, if  $\|T(t)\| \leq Me^{\omega t}$ , then*

$$\|T_D(t)\| \leq Me^{(\omega+M\|D\|)t}. \quad (4.8)$$

*This  $C_0$ -semigroup satisfies the following equations for every  $z_0 \in Z$*

$$T_D(t)z_0 = T(t)z_0 + \int_0^t T(t-s)DT_D(s)z_0ds \quad (4.9)$$

and

$$T_D(t)z_0 = T(t)z_0 + \int_0^t T_D(t-s)DT(s)z_0ds. \quad (4.10)$$

**Proof** The proof is divided into three parts. In part *a*, we shall show that there exists a unique solution  $S(t)$  of (4.7). This solution satisfies (4.8) and (4.9) and is strongly continuous at zero. In part *b*, we shall prove that  $T_D(t)$  is a  $C_0$ -semigroup. In the last part, we shall show that the infinitesimal generator of  $T_D(t)$  is  $A + D$  and that  $T_D(t)$  satisfies the equation (4.10).

*a.* First we show that (4.7) has the unique solution given by

$$T_D(t) = \sum_{n=0}^{\infty} S_n(t), \quad (4.11)$$

where

$$S_n(t)x = \int_0^t T(t-s)DS_{n-1}(s)xds, \quad S_0(t) = T(t). \quad (4.12)$$

It is easy to verify the following estimate by induction

$$\|S_n(t)\| \leq M^{n+1} \|D\|^n e^{\omega t} \frac{t^n}{n!}, \quad (4.13)$$

and so the series (4.11) is majorized by

$$Me^{\omega t} \sum_{n=0}^{\infty} \frac{(M\|D\|t)^n}{n!} = Me^{(\omega+M\|D\|)t}.$$

So the series (4.11) converges absolutely in the uniform topology of  $\mathcal{L}(Z)$  on any compact interval,  $[0, \tau]$ , and  $T_D(t)$  satisfies the estimate (4.8). Furthermore,

$$\begin{aligned} T_D(t)z_0 &= \sum_{n=0}^{\infty} S_n(t)z_0 = S_0(t)z_0 + \sum_{n=1}^{\infty} S_n(t)z_0 \\ &= T(t)z_0 + \sum_{n=1}^{\infty} \int_0^t T(t-s)DS_{n-1}(s)z_0 ds \\ &= T(t)z_0 + \int_0^t T(t-s)DT_D(s)z_0 ds \end{aligned}$$

by the absolute convergence of (4.11) and the estimate (4.8). So  $T_D(t)$  satisfies (4.9).

To prove uniqueness we assume that  $S(t)$  is also a solution, and by subtracting the equations for  $T_D(t)$  and  $S(t)$  we obtain

$$[T_D(t) - S(t)]z_0 = \int_0^t T(t-s)D[T_D(s) - S(s)]z_0 ds.$$

Hence

$$\|[T_D(t) - S(t)]z_0\| \leq \int_0^t M e^{\omega(t-s)} \|D\| \|[T_D(s) - S(s)]z_0\| ds.$$

Setting  $e^{-\omega t} \|[T_D(t) - S(t)]z_0\| = g(t)$  yields

$$0 \leq g(t) \leq M \|D\| \int_0^t g(s) ds$$

and Gronwall's Lemma A.6.7 shows that  $g(t) \leq g(0)e^{M\|D\|t} = 0$ .

To prove that  $S(t)$  is strongly continuous at zero, for  $h > 0$  we deduce the following estimate:

$$\|T_D(h)z_0 - z_0\| \leq \|T(h)z_0 - z_0\| + \int_0^h \|T(h-s)DT_D(s)z_0\| ds.$$

So using the strong continuity of  $T(t)$  and the bounds for  $T(t)$  and  $T_D(t)$ , we see that  $T_D(t)z_0$  is strongly continuous at zero.

*b.* From *a* we already know that  $T_D(t)$  is strongly continuous at zero, and  $T_D(0) = I$  follows from (4.11) and (4.12).



In order to prove the semigroup property  $T_D(t+s) = T_D(t)T_D(s)$ , we use (4.7) to obtain

$$\begin{aligned}
& T_D(t+s)z_0 - T_D(t)T_D(s)z_0 \\
&= T(t+s)z_0 + \int_0^{t+s} T(t+s-\alpha)DT_D(\alpha)z_0d\alpha - \\
&\quad [T(t) + \int_0^t T(t-\alpha)DT_D(\alpha)d\alpha][T(s)z_0 + \\
&\quad \int_0^s T(s-\beta)DT_D(\beta)z_0d\beta] \\
&= \int_0^{t+s} T(t+s-\alpha)DT_D(\alpha)z_0d\alpha - \\
&\quad \int_0^s T(t+s-\beta)DT_D(\beta)z_0d\beta - \int_0^t T(t-\alpha)DT_D(\alpha)T_D(s)z_0d\alpha \\
&= \int_s^{t+s} T(t+s-\alpha)DT_D(\alpha)z_0d\alpha - \int_0^t T(t-\alpha)DT_D(\alpha)T_D(s)z_0d\alpha \\
&= \int_0^t T(t-\alpha)D[T_D(s+\alpha) - T_D(\alpha)T_D(s)]z_0d\alpha.
\end{aligned}$$

Hence

$$\begin{aligned}
& \|T_D(t+s)z_0 - T_D(t)T_D(s)z_0\| \\
&\leq M\|D\| \int_0^t e^{\omega(t-s)} \| [T_D(s+\alpha) - T_D(\alpha)T_D(s)]z_0 \| d\alpha.
\end{aligned}$$

Letting  $g(t) = e^{-\omega t} \|T_D(t+s)z_0 - T_D(t)T_D(s)z_0\|$ , we obtain

$$0 \leq g(t) \leq M\|D\| \int_0^t g(\alpha)d\alpha.$$

So, since  $g(0) = \|T_D(s)z_0 - T_D(s)z_0\| = 0$ , applying Gronwall's Lemma A.6.7, we have

$$0 \leq g(t) \leq g(0)e^{M\|D\|t} = 0.$$

Thus  $T_D(t+s) = T_D(t)T_D(s)$  and so  $T_D(t)$  is a  $C_0$ -semigroup.

c. We now prove that its generator is  $A + D$ . First we show that

$$\lim_{h \rightarrow 0^+} \left\| \frac{T_D(h)z - z}{h} - \frac{T(h)z - z}{h} - Dz \right\| = 0 \quad \text{for } z \in Z. \quad (4.14)$$

From (4.7), we have that for any  $z \in Z$ ,

$$\begin{aligned}
& \left\| \frac{T_D(h)z - z}{h} - \frac{T(h)z - z}{h} - Dz \right\| \\
&= \left\| \frac{1}{h} \int_0^h T(h-s)DT_D(s)zds - Dz \right\|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{h} \left\| \int_0^h T(h-s) D[T_D(s)z - z] ds \right\| + \\ &\quad \left\| \frac{1}{h} \int_0^h T(h-s) Dz ds - Dz \right\|. \end{aligned} \quad (4.15)$$

The following inequality holds for the first term of (4.15):

$$\begin{aligned} &\frac{1}{h} \left\| \int_0^h T(h-s) D[T_D(s)z - z] ds \right\| \\ &\leq \frac{1}{h} \int_0^h \|T(h-s)\| \|D\| \|T_D(s)z - z\| ds. \end{aligned}$$

Let  $\varepsilon > 0$  be a given number. Then by the strong continuity of  $T_D(t)$ , there exists an  $h$  such that  $\|T_D(s)z - z\| \leq \varepsilon$  for  $s \in [0, h]$ . Without loss of generality, we may assume that  $h \leq 1$ . From Theorem 2.1.7 we have the existence of a constant  $M_1$  such that  $\|T(h-s)\| \leq M_1$  for all  $s \in [0, h]$ . Applying this in the above inequality gives

$$\frac{1}{h} \left\| \int_0^h T(h-s) D[T_D(s)z - z] ds \right\| \leq \frac{1}{h} \int_0^h M_1 \|D\| \varepsilon ds = M_1 \|D\| \varepsilon.$$

Thus

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \left\| \int_0^h T(h-s) D[T_D(s)z - z] ds \right\| = 0.$$

Next consider the second term in (4.15). It is easily seen that

$$\int_0^h T(h-s) Dz ds = \int_0^h T(s) Dz ds,$$

and since  $T(t)$  is a  $C_0$ -semigroup, we can apply Theorem 2.1.7 to obtain

$$\lim_{h \rightarrow 0^+} \left\| \frac{1}{h} \int_0^h T(h-s) Dz ds - Dz \right\| = 0.$$

Thus we have proved equation (4.14), and this shows that the domain of the generator of  $T_D(t)$  is  $\mathbf{D}(A)$  and on this domain it equals  $A + D$ .

Equation (4.10) is easily proved by the observation that  $A$  is the perturbation of  $A + D$  by  $-D$ . So, using (4.9) we have

$$T(t)z_0 = [T_D]_{-D}(t)z_0 = T_D(t)z_0 + \int_0^t T_D(t-s)[-D]T(s)z_0 ds,$$

which is equal to (4.10). ■

Another way of generating new  $C_0$ -semigroups is given in the following lemma.

**Lemma 4.2.2** *Let  $T_1(t)$  and  $T_2(t)$  be  $C_0$ -semigroups on their respective Hilbert spaces  $Z_1$  and  $Z_2$  and with the infinitesimal generators  $A_1$  and  $A_2$ , respectively.*

Suppose that

$$\|T_i(t)\| \leq M_i e^{\omega_i t}, \quad i = 1, 2, \quad (4.16)$$

and  $D \in \mathcal{L}(Z_1, Z_2)$ . Then the operator  $A = \begin{pmatrix} A_1 & 0 \\ D & A_2 \end{pmatrix}$  with  $D(A) = D(A_1) \oplus D(A_2)$  is the infinitesimal generator of the  $C_0$ -semigroup  $T(t)$  on  $Z = Z_1 \oplus Z_2$  given by

$$T(t) = \begin{pmatrix} T_1(t) & 0 \\ S(t) & T_2(t) \end{pmatrix}, \quad S(t)x = \int_0^t T_2(t-s)DT_1(s)x ds. \quad (4.17)$$

Furthermore, there exists a positive constant  $M$  such that

$$\|T(t)\| \leq M e^{\omega t}, \quad (4.18)$$

where  $\omega = \max(\omega_1, \omega_2)$  if  $\omega_1 \neq \omega_2$  and  $\omega > \omega_1$  if  $\omega_1 = \omega_2$ .

**Proof** It is clear that  $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  is the infinitesimal generator of the  $C_0$ -semigroup  $\begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}$  on  $Z = Z_1 \oplus Z_2$ . Since  $A$  is the sum of this operator and the bounded perturbation  $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$ , we know from Theorem 4.2.1 that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on  $Z$ . Now  $T(t)$  given by (4.17) satisfies (4.9) and hence it is the  $C_0$ -semigroup generated by  $A$ .

For the estimates, consider the following for  $\omega_1 \neq \omega_2$ :

$$T(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} T_1(t)x_1 \\ \int_0^t T_2(t-s)DT_1(s)x_1 ds + T_2(t)x_2 \end{pmatrix}.$$

Then

$$\begin{aligned} \|T(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_Z &\leq M_1 e^{\omega_1 t} \|x_1\| + M_1 M_2 \|D\| \frac{e^{\omega_1 t} - e^{\omega_2 t}}{\omega_1 - \omega_2} \|x_1\| + \\ &\quad M_2 e^{\omega_2 t} \|x_2\| \\ &\leq M e^{\omega t} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_Z \end{aligned}$$

for some positive constant  $M$  and  $\omega = \max(\omega_1, \omega_2)$ . For  $\omega_1 = \omega_2$ , we obtain

$$\|T(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_Z \leq M(1+t)e^{\omega_1 t} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_Z.$$

■

In Theorem 4.2.1, we assumed that  $D \in \mathcal{L}(Z)$ . However, we shall also need to consider time-dependent perturbation operators  $D \in \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$ , where

$$\begin{aligned} \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z)) \\ := \left\{ D \mid \langle z_1, D(\cdot)z_2 \rangle \text{ is measurable} \right. \\ \left. \text{for every } z_1, z_2 \in Z \text{ and } \operatorname{ess\,sup}_{0 \leq t \leq \tau} \|D(t)\|_{\mathcal{L}(Z)} < \infty \right\} \end{aligned} \quad (4.19)$$

(see also Definition A.6.8).

These perturbations arise as the result of time-dependent feedbacks  $f(t) = D(t)z(t)$ , which leads us to the following version of (4.7) on  $[0, \tau]$ :

$$z(t) = U(t, 0)z_0 = T(t)z_0 + \int_0^t T(t-\alpha)D(\alpha)U(\alpha, 0)z_0 d\alpha.$$

Since the perturbed operator is not necessarily time-invariant, we have to specify the initial time too. If we denote by  $U(t, s)z_0$  the (mild) solution of (4.1) with  $f(\cdot) = D(\cdot)z(\cdot)$  and initial condition  $z(s) = z_0$ , then we obtain the following time-dependent version of (4.7) on  $[0, \tau]$ :

$$z(t) = U(t, s)z_0 = T(t-s)z_0 + \int_s^t T(t-\alpha)D(\alpha)U(\alpha, s)z_0 d\alpha. \quad (4.20)$$

Following the method of the proof in Theorem 4.2.1, we shall show that (4.20) has a unique solution. Furthermore, one can show that  $U(t, s)$  given by (4.20) is strongly continuous in  $t$  on  $[0, \tau]$  for each fixed  $s$ ,  $U(t, t) = I$  and it satisfies a semigroup property. In fact, it is a mild evolution operator.

**Definition 4.2.3** Let  $\Delta(\tau) = \{(t, s); 0 \leq s \leq t \leq \tau\}$ .  $U(t, s) : \Delta(\tau) \rightarrow \mathcal{L}(Z)$  is a *mild evolution operator* if it has the following properties:

- a.  $U(s, s) = I, s \in [0, \tau]$ ;
- b.  $U(t, r)U(r, s) = U(t, s), 0 \leq s \leq t \leq \tau$ ;
- c.  $U(\cdot, s)$  is strongly continuous on  $[s, \tau]$  and  $U(t, \cdot)$  is strongly continuous on  $[0, t]$ . ■

**Theorem 4.2.4** *If  $T(t)$  is a  $C_0$ -semigroup on  $Z$  and  $D \in \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$ , then (4.20) has a solution in the class of mild evolution operators on  $Z$ . This solution is unique in the larger class of operators  $Q(t, s)$  that satisfy  $Q(\cdot, s) \in \mathbf{P}_\infty([s, \tau]; \mathcal{L}(Z))$  for all  $s \in [0, \tau]$ .*

**Proof** First we note that it is possible to apply a similar argument as in Theorem 4.2.1 to show that (4.20) has a unique solution and that it is given by

$$U(t, s) = \sum_{n=0}^{\infty} U^n(t, s), \quad (4.21)$$

where

$$\begin{aligned} U^n(t, s)x &= \int_s^t T(t-\alpha)D(\alpha)U^{n-1}(\alpha, s)x d\alpha, \\ U^0(t, s) &= T(t-s). \end{aligned}$$

However, in this proof we shall apply the Contraction Mapping Theorem A.3.1.

For fixed  $s \in [0, \tau]$ , consider the operator  $G_s : \mathbf{P}_\infty([s, \tau]; \mathcal{L}(Z)) \rightarrow \mathbf{P}_\infty([s, \tau]; \mathcal{L}(Z))$ , defined by

$$G_s(U)(t)z := T(t-s)z + \int_s^t T(t-\alpha)D(\alpha)U(\alpha)z d\alpha. \quad (4.22)$$

Let  $\gamma := \sup_{0 \leq t \leq \tau} \|T(t)\|$ .

$$\|G_s(U_1)(t) - G_s(U_2)(t)\|_{\mathcal{L}(Z)} \leq (t-s)\gamma \|D\| \|U_1 - U_2\|_{\mathbf{P}_\infty}.$$

Thus

$$\|G_s(U_1)(\cdot) - G_s(U_2)(\cdot)\|_{\mathbf{P}_\infty} \leq (\tau-s)\gamma \|D\| \|U_1 - U_2\|_{\mathbf{P}_\infty}, \quad (4.23)$$

and by induction it follows that

$$\|G_s^k(U_1) - G_s^k(U_2)\|_{\mathbf{P}_\infty} \leq \frac{(\tau-s)^k}{k!} \gamma^k \|D\|_{\mathbf{P}_\infty}^k \|U_1 - U_2\|_{\mathbf{P}_\infty}. \quad (4.24)$$

If we choose  $k$  such that  $\frac{(\tau-s)^k \gamma^k \|D\|_{\mathbf{P}_\infty}^k}{k!} < 1$ , then we see from equation (4.24) that  $G_s^k$  is a contraction. So there exists a unique fixed point of (4.22). It is easily verified that  $U(\cdot, s)$  given by (4.21) is this fixed point. We shall show that this function is a mild evolution operator. Property a of Definition 4.2.3 is trivial. The proof of property b is very similar to the proof of property b in Theorem 4.2.1. We have that

$$\begin{aligned} U(t, r)U(r, s) - U(t, s) &= \int_r^t T(t-\alpha)D(\alpha)[U(\alpha, r)U(r, s) - U(\alpha, s)] d\alpha. \end{aligned}$$

Thus

$$\begin{aligned} \|U(t, r)U(r, s) - U(t, s)\| &\leq \int_r^t M e^{\omega(t-\alpha)} \|D(\alpha)\| \|U(\alpha, r)U(r, s) - U(\alpha, s)\| d\alpha. \end{aligned}$$

Applying Gronwall's Lemma A.6.7 once again, we have that  $U(t, r)U(r, s) = U(t, s)$ . From (4.20), we have that

$$\|U(t, s)\| \leq \|T(t-s)\| + \int_s^t \|T(t-\alpha)\| \|D(\alpha)\| \|U(\alpha, s)\| d\alpha.$$

Using Gronwall's Lemma, this implies that  $U(\cdot, \cdot)$  is uniformly bounded, with bound  $\gamma e^{\gamma \|D\|_{\mathbf{P}_\infty} \tau}$ . Using this together with the strong continuity of  $T(t)$ , it is not

hard to see that  $U(\cdot, s)$  is strongly continuous on  $[s, \tau]$ . Thus it remains to show the strong continuity of  $U(t, \cdot)$  on  $[0, t]$ . Let  $s \in [0, t)$ , and  $h > 0$  such that  $s + h \leq t$ , and consider  $U(t, s + h)z_0 - U(t, s)z_0$ . Using (4.20) we see that

$$\begin{aligned} & [U(t, s + h) - U(t, s)]z_0 \\ &= T(t - s - h)[z_0 - T(h)z_0] - \int_s^{s+h} T(t - \alpha)D(\alpha)U(\alpha, s)z_0 d\alpha + \\ & \quad \int_{s+h}^t T(t - \alpha)D(\alpha)[U(\alpha, s + h) - U(\alpha, s)]z_0 d\alpha. \end{aligned}$$

Hence we see that

$$\begin{aligned} & \|[U(t, s + h) - U(t, s)]z_0\| \\ & \leq \gamma \|T(h)z_0 - z_0\| + \int_{s+h}^t \gamma \|D\|_{\mathbf{P}_\infty} \|[U(\alpha, s + h) - U(\alpha, s)]z_0\| d\alpha + \\ & \quad \int_s^{s+h} \gamma \|D\|_{\mathbf{P}_\infty} \|U(\alpha, s)z_0\| d\alpha \\ & \leq \varepsilon(h) + \int_{s+h}^t \gamma \|D\|_{\mathbf{P}_\infty} \|[U(\alpha, s + h) - U(\alpha, s)]z_0\| d\alpha, \end{aligned}$$

where  $\varepsilon(h) := \gamma \|T(h)z_0 - z_0\| + \int_s^{s+h} \gamma \|D\|_{\mathbf{P}_\infty} \|U(\alpha, s)z_0\| d\alpha$  converges to zero for  $h$  converging to zero. Since the above inequality holds for all  $t \in [s + h, \tau]$ , we conclude from Gronwall's Lemma that

$$\|[U(t, s + h) - U(t, s)]z_0\| \leq \varepsilon(h)e^{\gamma \|D\|_{\mathbf{P}_\infty} (t-s-h)}.$$

From this we see that  $\|[U(t, s + h) - U(t, s)]z_0\|$  converges to zero for  $h$  converging to zero. Similarly, one can show that  $\|[U(t, s) - U(t, s - h)]z_0\|$  converges to zero as  $h$  converges to zero. Thus  $U(t, \cdot)$  is strongly continuous on  $[0, t]$ . ■

This theorem motivates the following definition.

**Definition 4.2.5** Suppose that  $A$  is the infinitesimal generator of the  $C_0$ -semigroup  $T(t)$  on the Hilbert space  $Z$ . For  $D(\cdot) \in \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$  we call the unique solution of (4.20) the *mild evolution operator generated by  $A + D(\cdot)$* . ■

Of course,  $U(t, s)z_0$  may be regarded as the mild solution of

$$\dot{z}(t) = (A + D(t))z(t), \quad z(s) = z_0, \quad (4.25)$$

since it satisfies the integral equation

$$z(t) = T(t)z_0 + \int_s^t T(t - \alpha)D(\alpha)z(\alpha) d\alpha.$$

For smooth initial conditions, we can prove differentiability in the second variable.

**Theorem 4.2.6** Assume that  $T(t)$  is a  $C_0$ -semigroup on  $Z$ ,  $D \in \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$  and  $U(t, s)$  is the unique solution of (4.20). Then for all  $z_0 \in \mathbf{D}(A)$  the following hold:

$$\int_s^t U(t, \alpha)(A + D(\alpha))z_0 d\alpha = U(t, s)z_0 - z_0; \quad (4.26)$$

$$\frac{\partial}{\partial s} U(t, s)z_0 = -U(t, s)(A + D(s))z_0 \quad \text{a.e. in } [0, t]. \quad (4.27)$$

**Proof** Consider the sequence

$$\begin{aligned} \tilde{U}_0(t, s) &= T(t-s), \\ \tilde{U}_n(t, s)z_0 &= \int_s^t \tilde{U}_{n-1}(t, \alpha)D(\alpha)T(\alpha-s)z_0 d\alpha. \end{aligned}$$

Then, as in Theorem 4.2.4, we can show that

$$\tilde{U}(t, s) = \sum_{n=0}^{\infty} \tilde{U}_n(t, s)$$

is the unique solution of

$$\tilde{U}(t, s)z_0 = T(t-s)z_0 + \int_s^t \tilde{U}(t, \alpha)D(\alpha)T(\alpha-s)z_0 d\alpha. \quad (4.28)$$

We shall show that  $U(t, s) = \tilde{U}(t, s)$  using an induction argument. By definition  $U^0(t, s) = \tilde{U}_0(t, s)$  and  $U^1(t, s) = \tilde{U}_1(t, s)$ . Now supposing that  $U^{n-1}(t, s) = \tilde{U}_{n-1}(t, s)$  and  $U^{n-2}(t, s) = \tilde{U}_{n-2}(t, s)$ , we shall prove that  $U^n(t, s) = \tilde{U}_n(t, s)$ . By definition and since  $U^{n-1}(t, s) = \tilde{U}_{n-1}(t, s)$ , we have that

$$\begin{aligned} \tilde{U}_n(t, s)z_0 &= \int_s^t U^{n-1}(t, \alpha)D(\alpha)T(\alpha-s)z_0 d\alpha \\ &= \int_s^t \int_\alpha^t T(t-\beta)D(\beta)U^{n-2}(\beta, \alpha)D(\alpha)T(\alpha-s)z_0 d\beta d\alpha \\ &= \int_s^t \int_s^\beta T(t-\beta)D(\beta)U^{n-2}(\beta, \alpha)D(\alpha)T(\alpha-s)z_0 d\alpha d\beta \\ &\quad \text{by Fubini's Theorem A.5.27} \\ &= \int_s^t T(t-\beta)D(\beta)\tilde{U}_{n-1}(\beta, s)z_0 d\beta \\ &\quad \text{since } \tilde{U}_{n-2} = U^{n-2} \\ &= U^n(t, s)z_0, \quad \text{since } \tilde{U}_{n-1} = U^{n-1}. \end{aligned}$$

Thus by induction we have proved that

$$U(t, s) = \tilde{U}(t, s).$$

Hence for  $z_0 \in \mathbf{D}(A)$ , we have

$$U(t, \alpha)Az_0 = T(t-\alpha)Az_0 + \int_\alpha^t U(t, \beta)D(\beta)T(\beta-\alpha)Az_0 d\beta,$$

and both terms on the right are integrable by Theorem 2.1.13, Definition 4.2.3, and Lemma A.5.10. Thus

$$\begin{aligned}
& \int_s^t U(t, \alpha) A z_0 d\alpha \\
&= \int_s^t T(t - \alpha) A z_0 d\alpha + \int_s^t \int_\alpha^t U(t, \beta) D(\beta) T(\beta - \alpha) A z_0 d\beta d\alpha \\
&= \int_s^t T(t - \alpha) A z_0 d\alpha + \int_s^t \int_s^\beta U(t, \beta) D(\beta) T(\beta - \alpha) A z_0 d\alpha d\beta \\
&\quad \text{by Fubini's Theorem A.5.27} \\
&= (T(t - s) - I) z_0 + \int_s^t U(t, \beta) D(\beta) (T(\beta - s) - I) z_0 d\beta \\
&\quad \text{by Theorem 2.1.13.d.}
\end{aligned}$$

Since  $U(t, s)$  is the solution to (4.28), we have proved that  $\int_s^t U(t, \alpha) (A + D(\alpha)) z_0 d\alpha = U(t, s) z_0 - z_0$ , which is (4.26), and this implies (4.27) by Theorem A.5.35. ■

Notice that we have also proved the following useful corollary.

**Corollary 4.2.7** *If  $T(t)$  is a  $C_0$ -semigroup on  $Z$  and  $D \in \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$ , then (4.28) and (4.20) have the same unique solution in the class of mild evolution operators on  $Z$ .*

In the sequel, we encounter abstract evolution equations of the type

$$\dot{z}(t) = (A + D(t))z(t) + f(t), \quad z(0) = z_0. \quad (4.29)$$

Following the previous discussion it is natural to define solutions for these equations.

**Definition 4.2.8** Consider equation (4.29), where  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on the Hilbert space  $Z$ ,  $z_0 \in Z$ ,  $D(\cdot) \in \mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$  and  $f \in \mathbf{L}_p([0, \tau]; Z)$ ,  $p \geq 1$ .

The function  $z(t)$  is a *classical solution* of (4.29) on  $[0, \tau]$  if  $z(\cdot) \in \mathbf{C}^1([0, \tau]; Z)$ ,  $z(t) \in \mathbf{D}(A)$  for all  $t \in [0, \tau]$ , and  $z(t)$  satisfies (4.29) for all  $t \in [0, \tau]$ .

If  $f \in \mathbf{L}_p([0, \tau]; Z)$  for a  $p \geq 1$  and  $z_0 \in Z$ , then we define the *mild solution* of (4.29) to be

$$z(t) = U(t, 0)z_0 + \int_0^t U(t, s)f(s)ds.$$

■

The mild solutions to these abstract evolution equations have properties similar to those with  $D(\cdot) = 0$ ; this is discussed further in Exercise 4.13.

In general,  $U(t, s)$  is not differentiable in  $t$ ; this means that (4.29) rarely has a classical solution.



### 4.3 Exercises

- 4.1. Prove the existence of a unique solution to the operator integral equation (4.7) by applying the Contraction Mapping Theorem A.3.1.
- 4.2. Show that for any  $f \in L_1([0, \tau]; Z)$  the mild solution of Definition 4.1.4 as given by (4.2) satisfies
- $z$  is continuous,
  - For all  $\tau \in [0, \tau]$  we have  $\int_0^\tau z(t)dt \in D(A)$ , and
  - For all  $\tau \in [0, \tau]$  there holds

$$z(\tau) - z(0) = A \int_0^\tau z(t)dt + \int_0^\tau f(t)dt.$$

Show that the mild solution is the only function satisfying these three conditions.

- 4.3. Let  $T(t)$  be an exponentially stable semigroup on the Hilbert space  $Z$ .
- Show that if  $f \in L_p([0, \infty); Z)$  for  $1 \leq p < \infty$ , then the mild solution to (4.1) is asymptotically stable.
  - Show that for  $f \in L_\infty([0, \infty); Z)$  the mild solution to (4.1) is not necessarily asymptotically stable.
- 4.4. In Theorem 4.2.1, we derived the estimate  $\|S(t)\| \leq Me^{(w+M\|D\|)t}$  for the perturbed semigroup. In this exercise, we investigate this bound further. Let  $A$  be the self-adjoint operator on the Hilbert space  $Z$  given by

$$A = \sum_{n=1}^{\infty} -n \langle \cdot, \phi_n \rangle \phi_n$$

with domain

$$D(A) = \{z \in Z \mid \sum_{n=1}^{\infty} n^2 |\langle z, \phi_n \rangle|^2 < \infty\},$$

where  $\{\phi_n, n \geq 1\}$  is an orthonormal basis of  $Z$ . For  $k \geq 1$ , define the operator

$$D_k = -\sum_{n=1}^k n \langle \cdot, \phi_n \rangle \phi_n$$

and denote by  $T(t)$ ,  $S_k^+(t)$ ,  $S_k^-(t)$ , the semigroups generated by  $A$ ,  $A + D_k$ , and  $A - D_k$ , respectively.

- Calculate the growth bound of  $T(t)$ .
- Calculate the growth bound of  $S_k^+(t)$  and  $S_k^-(t)$ .
- What is the norm of  $D_k$ ? Using parts a and b show that the estimate in Theorem 4.2.1 can be very conservative, but that it cannot be improved upon.

- 4.5. Let  $A$  be an infinitesimal generator of a  $C_0$ -semigroup on the Hilbert space  $Z$ , and let  $B \in \mathcal{L}(U, Z)$ . Prove that for any  $F \in \mathcal{L}(Z, U)$  the mild solution of

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0$$

equals the mild solution of

$$\dot{z}(t) = (A + BF)z(t) + B(u(t) - Fz(t)), \quad z(0) = z_0.$$

- 4.6. In this exercise we show that a rank one perturbation can change the growth bound drastically.

Consider on  $L_2(0, 1)$  the left-shift semigroup from Exercise 2.6,

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{for } 0 \leq x \leq 1 \text{ and } t \leq 1-x, \\ 0 & \text{for } 0 \leq x \leq 1 \text{ and } t > 1-x. \end{cases}$$

The growth bound of this semigroup is minus infinity and its infinitesimal generator is given by  $Af = \frac{df}{dx}$  with  $D(A) = \{f \in L_2(0, 1) \mid f \text{ is absolutely continuous, } \frac{df}{dx} \text{ is an element of } L_2(0, 1) \text{ and } f(1) = 0\}$

- a. Let  $\lambda \in \mathbb{R}$  with  $\lambda \neq 0$ . Define the operator  $D_\lambda$  by

$$(D_\lambda f)(x) := \frac{\lambda^2 e^\lambda}{e^\lambda + \lambda e^\lambda - 1} \int_0^1 f(\theta) d\theta \mathbb{1}_{[0,1]}(x).$$

Show that  $D_\lambda$  is a bounded linear operator on  $L_2(0, 1)$  with one-dimensional range.

- b. Show that  $A + D_\lambda$  has an eigenvalue at  $\lambda$ .  
Hint: Solve  $Af + \alpha \mathbb{1} = \lambda f$ , where  $\alpha$  is a constant. Then integrate to find  $\alpha$ .
- c. What you say about the growth bound of the  $C_0$ -semigroup generated by  $A + D_\lambda$ ?

- 4.7. Let  $T_1(t)$  and  $T_2(t)$  be  $C_0$ -semigroups on their respective Hilbert spaces  $Z_1$  and  $Z_2$  and with the infinitesimal generators  $A_1$  and  $A_2$ , respectively, and suppose that  $D \in \mathcal{L}(Z_2, Z_1)$ .

- a. Show that the operator  $A$  defined by

$$A = \begin{pmatrix} A_1 & D \\ 0 & A_2 \end{pmatrix},$$

$$D(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in D(A_1), y \in D(A_2) \right\}$$

is the infinitesimal generator of a  $C_0$ -semigroup  $T(t)$  on  $Z = Z_1 \oplus Z_2$ .

- b. Find the expression for  $T(t)$ .
- c. Let  $\omega_1$  and  $\omega_2$  be the growth bounds of  $T_1(t)$  and  $T_2(t)$ , respectively. Prove that the growth bound of  $T(t)$  is the maximum of  $\omega_1$  and  $\omega_2$ .

d. Give the expression for  $(\lambda I - A)^{-1}$  for  $\lambda \in \rho(A_1) \cap \rho(A_2)$ .

4.8. Let  $A$  be the infinitesimal generator of a  $C_0$ -group  $T(t)$  on the Hilbert space  $Z$ , and let  $D \in \mathcal{L}(Z)$ .

- a. Show that  $A + D$  is the infinitesimal generator of a  $C_0$ -group on  $Z$ .
- b. If  $T(t)$  is a unitary group, give necessary and sufficient conditions on  $D$  such that  $A + D$  is the infinitesimal generator of a unitary group.  
Hint: See Exercise 2.24.

4.9. Consider the following system

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2}(x, t) + \beta \frac{\partial w}{\partial t}(x, t) - \frac{\partial^2 w}{\partial x^2}(x, t) + \alpha \frac{\partial w}{\partial x}(x, t) &= 0, \\ w(x, 0) = w_0(x), \quad \frac{\partial w}{\partial t}(x, 0) &= w_1(x), \end{aligned} \tag{4.30}$$

$$w(0, t) = 0 = w(1, t),$$

where  $\alpha$  and  $\beta$  are real numbers. As in Exercise 3.11, we define the operator

$$A_0 = -\frac{d^2 h}{dx^2} + \alpha \frac{dh}{dx}$$

with domain

$$\begin{aligned} \mathbf{D}(A_0) &= \{h \in L_2(0, 1) \mid h, \frac{dh}{dx} \text{ are absolutely continuous,} \\ &\quad \frac{dh}{dx} \in L_2(0, 1) \text{ and } h(0) = h(1) = 0\} \end{aligned}$$

on the Hilbert space  $Z_\alpha = L_2(0, 1)$  with inner product

$$\langle h, k \rangle_\alpha = \int_0^1 e^{-\alpha x} h(x) \overline{k(x)} dx.$$

- a. Formulate the partial differential equation (4.30) as an abstract differential equation  $\dot{z}(t) = Az(t)$  on the Hilbert space  $Z = \mathbf{D}(A_0^{\frac{1}{2}}) \oplus Z_\alpha$ . Find  $A$  and show that it is the infinitesimal generator of a  $C_0$ -semigroup.
- b. Show that, for pairs  $(\alpha, \beta)$  such that  $\beta^2 - \alpha^2 \neq 4n^2\pi^2$ ,  $A$  is a Riesz-spectral operator.

4.10. Consider the following abstract Cauchy problem on the Hilbert space  $Z$

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0, \tag{4.31}$$

where  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $T(t)$  on  $Z$ , the input space  $U$  is a Hilbert space,  $B \in \mathcal{L}(U, Z)$ , and  $z_0 \in Z$ .

- a. Suppose that we have the digital controller:  $u(t) = u(k)$  for  $k \leq t < k + 1, k \geq 0$ . Show that the values of the mild solution of (4.31) at the discrete-time instants,  $t = k, k \geq 0$ , satisfy the equation

$$z(k + 1) = \hat{A}z(k) + \hat{B}u(k), \quad z(0) = z_0$$

for certain operators  $\hat{A}$  and  $\hat{B}$ .

- b. Let  $u \in \mathbf{L}_2([0, \tau]; U)$  and show that the values of the mild solution of (4.31) at the discrete-time instants satisfy

$$z(k+1) = \tilde{A}z(k) + \tilde{B}\tilde{u}(k), \quad z(0) = z_0,$$

for certain operators  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{u}(k) \in \mathbf{L}_2([0, 1]; Z)$ .

- 4.11. Assume that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on the Hilbert space  $Z$ , and let  $Q \in \mathcal{L}(Z)$ . Define the operator-valued function  $D(t) = Q$ ,  $t \in [0, \tau]$ . Prove that  $A + D(t)$  generates a mild evolution operator,  $U(t, s)$ , and show that  $U(t, s)z_0 = T_Q(t-s)z_0$ .

- 4.12. Consider the partial differential equation

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t) + a(t)z(x, t), \\ \frac{\partial z}{\partial x}(0, t) &= 0, \quad \frac{\partial z}{\partial x}(1, t) = 0, \\ z(x, 0) &= z_0(x). \end{aligned} \tag{4.32}$$

- a. Assume that  $a(\cdot) \equiv a \in \mathbb{C}$ . Formulate the partial differential equation (4.32) as an abstract differential equation

$$\dot{z}(t) = (A + D)z(t), \quad z(0) = z_0, \tag{4.33}$$

where  $A$  is given by (2.3). Show that  $D \in \mathcal{L}(Z)$ , and hence that (4.33) is well posed.

- b. If  $a(\cdot)$  is a continuous function, show that (4.32) can be formulated as an abstract evolution equation

$$\dot{z}(t) = [A + D(t)]z(t), \quad z(0) = z_0,$$

where  $A + D(t)$  generates a mild evolution operator.

- 4.13. Consider an inhomogeneous abstract evolution equation of the type

$$\dot{z}(t) = [A + D(t)]z(t) + f(t), \quad z(s) = z_0, \tag{4.34}$$

where  $A$  is the infinitesimal generator of the  $C_0$ -semigroup,  $T(t)$ , and the operator-valued function  $D$  is in  $\mathbf{P}_\infty([0, \tau]; \mathcal{L}(Z))$ . We recall from Definition 4.2.8 that the mild solution of (4.34) is given by

$$z(t) = U(t, s)z_0 + \int_s^t U(t, \alpha)f(\alpha)d\alpha, \tag{4.35}$$

where  $U(t, s)$  is the mild evolution operator generated by  $A + D(\cdot)$ .

- a. Prove that if  $f \in \mathbf{L}_p([0, \tau]; Z)$ ,  $p \geq 1$ , then (4.35) is continuous on  $[0, \tau]$ .

Hint: See Lemma 4.1.5 and use the fact that  $U(t, s)$  is uniformly bounded on  $\Delta(\tau)$  (see the proof of Theorem 4.2.4).

- b. Show that if (4.34) does have a classical solution, then it is equal to (4.35).

4.14. Let  $A$  be the infinitesimal generator of the  $C_0$ -semigroup  $T(t)$  on the Hilbert space  $Z$ .

- a. For  $f \in L_p([t_0, t_e]; Z)$ ,  $p \geq 1$ , prove that the solution of

$$\dot{z}(t) = -A^*z(t) + f(t), \quad z(t_e) = z_e \quad (4.36)$$

on  $[t_0, t_e]$  is equivalent to the solution of

$$\dot{\tilde{z}}(t) = A^*\tilde{z}(t) + \tilde{f}(t), \quad \tilde{z}(t_0) = z_e \quad (4.37)$$

on  $[t_0, t_e]$ . Hence prove that

$$z(t) = T^*(t_e - t)z_e + \int_t^{t_e} T^*(s - t)f(s)ds \quad (4.38)$$

is the unique classical solution of (4.36) when  $z_e \in \mathbf{D}(A^*)$  and  $f \in C^1([t_0, t_e]; Z)$ .

For general  $z_e \in Z$  and  $f \in L_p([t_0, t_e]; Z)$ ,  $p \geq 1$  (4.38) is the *mild solution* of (4.36).

- b. Suppose that  $U(t, s)$  is the mild evolution operator generated by  $A + D(\cdot)$ , where  $D \in P_\infty([t_0, t_e]; \mathcal{L}(Z))$ .

- i. Prove that the mild solution of

$$\dot{z}(t) = -A^*z(t) - D^*(t)z(t), \quad z(t_e) = z_e \quad (4.39)$$

on  $[t_0, t_e]$  is given by

$$z(t) = U^*(t_e, t)z_e.$$

Hint: Use Corollary 4.2.7.

- ii. Show that the solution to

$$\dot{z}(t) = -A^*z(t) - D^*(t)z(t) + f(t), \quad z(t_e) = z_e$$

is equivalent to the solution of the standard evolution equation

$$\dot{\tilde{z}}(t) = \tilde{A}\tilde{z}(t) - \tilde{D}(t)\tilde{z}(t) + \tilde{f}(t), \quad \tilde{z}(t_0) = \tilde{z}_0$$

for certain  $\tilde{A}$ ,  $\tilde{D}$ ,  $\tilde{f}$ , and  $\tilde{z}_0$ .

- iii. Will the mild solution of i and ii be continuous on  $[t_0, t_e]$ ?

Hint: Show that  $U^*(t, s)$  is a mild evolution operator.

## 4.4 Notes and references

The analysis of the abstract Cauchy problem is now standard and can be found in many texts on semigroup theory, for example, in Bensoussan et al. [19, chapter 1.3] and Pazy [125, chapter 4]. For the case where  $Z$  is a Banach space, there

are many different definitions of types of solutions to the inhomogeneous Cauchy problem (4.1); our definitions of classical and mild solutions follow Pazy [125, chapter 4], and the concept of a weak solution was originally due to Ball [8] (see also Curtain and Pritchard [40]). Our concepts of weak and mild solution coincide, since we consider the Hilbert space case. However, it is also important to know that the mild solution of the inhomogeneous abstract Cauchy problem on a Hilbert space agrees with the usual weak solution in the case of partial differential equations. Since we could not find a clarification of this in the literature, we have included a justification in the appendix in Example A.5.34. Sharper regularity results of the mild solutions can be found in Bensoussan et al. [19, chapter 1.3] and Pazy [125].

The treatment of perturbed semigroups, mild evolution operators, and abstract evolution equations follows that in Curtain and Pritchard [40, chapter 2]. More detailed results on special types of evolution operators can be found in Pazy [125] and Tanabe [156], for example.

# 5

## State Linear Systems

### 5.1 Input and outputs

In this chapter, we consider the following class of infinite-dimensional systems with input  $u$  and output  $y$ :

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0, \quad (5.1)$$

$$y(t) = Cz(t) + Du(t). \quad (5.2)$$

**Definition 5.1.1**  $\Sigma(A, B, C, D)$  denotes the *state linear system* (5.1)–(5.2), where  $A$  is the infinitesimal generator of the strongly continuous semigroup  $T(t)$  on a Hilbert space  $Z$ , the *state space*.  $B$  is a bounded linear operator from the *input space*  $U$  to  $Z$ ,  $C$  is a bounded linear operator from  $Z$  to the *output space*  $Y$ , and  $D$  is a bounded operator from  $U$  to  $Y$ . Both  $U$  and  $Y$  are Hilbert spaces. We consider  $\Sigma(A, B, C, D)$  for all initial states  $z_0 \in Z$  and all *inputs*  $u \in L_2([0, \tau]; U)$ . The *state* is the mild solution (see Definition 4.1.4) of (5.1)

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad 0 \leq t \leq \tau \quad (5.3)$$

and the *output*  $y(\cdot)$  is defined by (5.2). ■

When  $A, B, C, D$  are all spatially invariant operators, we call  $\Sigma(A, B, C, D)$  a *spatially invariant system*. When  $A$  is a Riesz-spectral operator, we call  $\Sigma(A, B, C, D)$  a *Riesz-spectral system*.

We recall from Lemma 4.1.5 that  $z(\cdot)$  is in  $C([0, \tau]; Z)$  and so the output  $y$  defined by (5.2) is always in  $L_2([0, \tau]; Y)$ . To avoid clutter, we shall also use the notation  $\Sigma(A, B, C, -)$  if the value  $D$  is unimportant for the definition. Similarly,  $\Sigma(A, B, -, -)$  when the operators  $C$  and  $D$  are not needed, and  $\Sigma(A, -, C, -)$  when the operators  $B$  and  $D$  are not needed.

Depending upon the context, inputs may have the physical interpretation as controls (which one may choose) or as disturbances (an act of nature) and outputs may be interpreted as observations (which one can measure) or as parts of the system whose behaviour we wish to influence.

As our first example we consider a spatially invariant system.

**Example 5.1.2** Consider the spatially invariant system

$$\begin{aligned}\dot{z}(t) &= A_{cv}z(t) + B_{cv}u(t), \quad t \geq 0, \quad z(0) = z_0, \\ y(t) &= C_{cv}z(t) + D_{cv}u(t),\end{aligned}$$

where  $A_{cv}, B_{cv}, D_{cv}, C_{cv}$  are bounded convolution operators on the state space  $Z = \ell_2(\mathbb{Z}; \mathbb{C}^n)$ , the input space  $U = \ell_2(\mathbb{Z}; \mathbb{C}^m)$  and the output space  $Y = \ell_2(\mathbb{Z}; \mathbb{C}^p)$ , respectively. As we saw in Lemma 3.1.3 if  $A_{cv}$  is bounded, then there exists  $\check{A} \in L_\infty(\partial\mathbb{D}; \mathbb{C}^{n \times n})$  such that  $A_{cv} = \mathfrak{F}^{-1}\Lambda_{\check{A}}\mathfrak{F}$ . Thus the spatially invariant system  $\Sigma(A_{cv}, B_{cv}, C_{cv}, D_{cv})$  is isometrically isomorphic to the system  $\Sigma(\mathfrak{F}^{-1}A_{cv}\mathfrak{F}, \mathfrak{F}^{-1}B_{cv}\mathfrak{F}, \mathfrak{F}^{-1}C_{cv}\mathfrak{F}, \mathfrak{F}^{-1}D_{cv}\mathfrak{F}) = \Sigma(\Lambda_{\check{A}}, \Lambda_{\check{B}}, \Lambda_{\check{C}}, \Lambda_{\check{D}})$ , which is defined on the state space  $L_2(\partial\mathbb{D}; \mathbb{C}^n)$ , the input space  $L_2(\partial\mathbb{D}; \mathbb{C}^m)$ , and the output space  $L_2(\partial\mathbb{D}; \mathbb{C}^p)$ . As we saw in Section 3.1, it is more convenient to work with the Fourier transformed system with parameter  $\phi \in \partial\mathbb{D}$

$$\begin{aligned}\frac{\partial \check{z}}{\partial t}(\phi, t) &= \check{A}(\phi)z(\phi, t) + \check{B}(\phi)\check{u}(\phi, t), \quad t \geq 0, \quad z(\phi, 0) = z_0(\phi); \\ y(\phi, t) &= \check{C}(\phi)\check{z}(\phi, t) + \check{D}(\phi)u(\phi, t),\end{aligned}$$

■

Next we return to our canonical heat equation example.

**Example 5.1.3** Consider the metal rod of Example 2.1.1, but this time we control using a heating element around the point  $x_0$  and we measure its temperature around the point  $x_1$ . If we suppose that there is no heating or cooling at either end, then a reasonable mathematical model is

$$\begin{aligned}\frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t) + b(x)u(t), \quad z(x, 0) = z_0(x), \\ \frac{\partial z}{\partial x}(0, t) &= \frac{\partial z}{\partial x}(1, t) = 0,\end{aligned}\tag{5.4}$$

$$y(t) = \int_0^1 c(x)z(x, t)dx,\tag{5.5}$$



where  $b$  and  $c$  represent the “shaping functions” around the control point  $x_0$  and the sensing point  $x_1$ , respectively,

$$\begin{aligned} b(x) &= \frac{1}{2\varepsilon} \mathbb{1}_{[x_0-\varepsilon, x_0+\varepsilon]}(x), \\ c(x) &= \frac{1}{2\nu} \mathbb{1}_{[x_1-\nu, x_1+\nu]}(x), \end{aligned}$$

where  $\mathbb{1}_{[a, \beta]}(x) = \begin{cases} 1 & \text{for } a \leq x \leq \beta, \\ 0 & \text{elsewhere.} \end{cases}$

Notice that  $b$  and  $c$  in this example are both elements in  $L_2(0, 1)$  for fixed, small, positive constants  $\nu$  and  $\varepsilon$ . In Example 3.2.11, we have that  $Z = L_2(0, 1)$  is a suitable state space for the heat equation without input. So it is easy to see that (5.4) can be formulated as in (5.1) with  $Z = L_2(0, 1)$ ,  $U = \mathbb{C}$ , and  $Bu := b(x)u$ .  $B \in \mathcal{L}(\mathbb{C}, Z)$  and has norm  $1/\sqrt{2\varepsilon}$ . Furthermore, with  $Y = \mathbb{C}$ , (5.5) can be formulated as in (5.2) with  $D = 0$  and  $C$  given by

$$Cz := \int_0^1 c(x)z(x)dx.$$

$C \in \mathcal{L}(Z, \mathbb{C})$  and has norm  $1/\sqrt{2\nu}$ . ■

Although it may seem artificial to choose  $U = Y = \mathbb{C}$  in a physical example, where inputs and outputs take on real values, we choose to complexify all spaces for mathematical reasons. For example, to discuss the spectrum of an operator one must take the underlying scalar field to be the complex numbers. We remark that operators  $B$  and  $C$  in Example 5.1.3 approximately model “point actuators” and “point sensors”, respectively, and as long as  $\varepsilon$  and  $\nu$  are positive we obtain bounded  $B$  and  $C$  operators. Usually, a point actuator is modelled as a delta distribution in the point, i.e.,  $\delta_{x_0}(x)$  replaces  $b(x)$ , and this cannot be represented as a bounded  $B$  operator in this example:  $B$  maps out of the state space. In Section 5.2, we shall discuss some examples with boundary control and give conditions under which it is possible to reformulate such systems on an extended state space with a new bounded  $B$  operator. However, this is not always possible nor desirable. The natural formulation of boundary control action, e.g.,  $\frac{\partial z}{\partial x}(1, t) = u(t)$ , leads to an operator mapping outside the state space. It is customary in the control literature to refer to an *unbounded  $B$  operator*, although, strictly speaking, while  $B$  is not bounded, it is not unbounded either. We shall follow this abuse of terminology. Similarly, point sensors produce observations of the type  $Cz = z(x_1)$  and such operators need not be bounded either. Unbounded  $B$  and  $C$  operators introduce mathematical technicalities that we prefer to avoid at this stage; for simplicity of exposition we only consider bounded  $B$  and  $C$  operators for most of this book. However, it is important to realize that the theory presented for bounded  $B$  and  $C$  has a natural generalization to very general unbounded input and output operators, including point actuators and sensors.

## 5.2 Boundary control systems

In this section we consider systems described by partial differential equations with the control action on the boundary. A typical example is when the temperature of a bar is controlled by control action at one end instead of in the interior as in Example 2.1.1.

**Example 5.2.1** Consider a metal bar of length 1 that is perfectly isolated at the right-hand side, and we inject heat at the left-hand side.

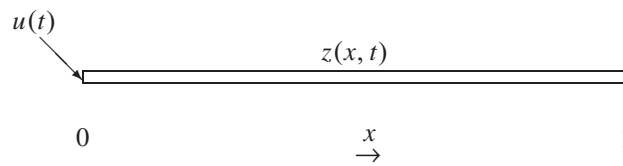


Figure 5.1. A one-dimensional heated bar with boundary control

It is not possible to formulate this in the form

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t) + u(x, t), & z(x, 0) &= z_0(x), \\ \frac{\partial z}{\partial x}(0, t) &= 0 = \frac{\partial z}{\partial x}(1, t), \end{aligned}$$

for an input  $u \in L_p([0, \tau]; L_2(0, 1))$ , but we could try to approximate this by the following input

$$u(x, t) := \frac{1}{\varepsilon} 1_{[0, \varepsilon]}(x) u(t).$$

The mild solution is then given by (4.4)

$$\begin{aligned} z(x, t) &= \sum_{n=0}^{\infty} \left[ e^{\lambda_n t} \langle z_0, \phi_n \rangle \phi_n + \int_0^t e^{\lambda_n(t-s)} \langle u(\cdot, s), \phi_n(\cdot) \rangle \phi_n ds \right] \\ &= \sum_{n=0}^{\infty} e^{\lambda_n t} \langle z_0, \phi_n \rangle \phi_n + \int_0^t u(s) ds + \\ &\quad \sum_{n=1}^{\infty} \frac{2 \sin(n\pi \varepsilon)}{n\pi \varepsilon} \cos(n\pi x) \int_0^t e^{-n^2 \pi^2(t-s)} u(s) ds, \end{aligned}$$

since  $\lambda_n = -n^2 \pi^2$ ,  $n \geq 0$ ,  $\phi_n(x) = \sqrt{2} \cos(n\pi x)$ ,  $n \geq 1$ , and  $\phi_0 = 1$ .

However, it is clear the above approximation is not the same as controlling at the boundary. In fact, the correct formulation is

$$\frac{\partial z}{\partial x}(0, t) = z(0, t) - u(t),$$

(see Exercise 6.15). ■

Boundary control problems like the one in the above example occur frequently in the applications, but unfortunately they do not fit into our standard formulation (4.1). However, for sufficiently smooth inputs it is possible to reformulate such problems on an extended state space so that they do lead to an associated system in the standard form (4.1). We shall develop such a theory for the following class of *abstract boundary control problems*:

$$\begin{aligned} \dot{z}(t) &= \mathfrak{A}z(t), & z(0) &= z_0, \\ \mathfrak{B}z(t) &= u(t), \end{aligned} \tag{5.6}$$

where  $\mathfrak{A} : \mathbf{D}(\mathfrak{A}) \subset Z \mapsto Z$ ,  $u(t) \in U$ , a separable Hilbert space, and the *boundary operator*  $\mathfrak{B} : \mathbf{D}(\mathfrak{B}) \subset Z \mapsto U$  satisfies  $\mathbf{D}(\mathfrak{A}) \subset \mathbf{D}(\mathfrak{B})$ .

In order to reformulate equation (5.6) into an abstract form (4.1), we need to impose extra conditions on the system.

**Definition 5.2.2** The control system (5.6) is a *boundary control system* if the following hold:

- a. The operator  $A : \mathbf{D}(A) \mapsto Z$  with  $\mathbf{D}(A) = \mathbf{D}(\mathfrak{A}) \cap \ker(\mathfrak{B})$  and

$$Az = \mathfrak{A}z \quad \text{for } z \in \mathbf{D}(A) \tag{5.7}$$

is the infinitesimal generator of a  $C_0$ -semigroup on  $Z$ ;

- b. There exists a  $B \in \mathcal{L}(U, Z)$  such that for all  $u \in U$ ,  $Bu \in \mathbf{D}(\mathfrak{A})$ , the operator  $\mathfrak{A}B$  is an element of  $\mathcal{L}(U, Z)$  and

$$\mathfrak{B}Bu = u, \quad u \in U. \tag{5.8}$$

■

Assuming that (5.6) is a boundary control system for  $u \in \mathcal{C}^2([0, \tau]; U)$ , the following abstract differential equation on  $Z$  is well posed:

$$\begin{aligned} \dot{v}(t) &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t), \\ v(0) &= v_0. \end{aligned} \tag{5.9}$$

Since  $A$  is the infinitesimal generator of a  $C_0$ -semigroup and  $B$  and  $\mathfrak{A}B$  are bounded linear operators, we have from Theorem 4.1.3 that equation (5.9) has a unique classical solution for  $v_0 \in \mathbf{D}(A)$ . Furthermore, we can prove the following relation between the (classical) solutions of (5.6) and (5.9).

**Theorem 5.2.3** Consider the boundary control system (5.6) and the abstract Cauchy equation (5.9). Assume that  $u \in \mathcal{C}^2([0, \tau]; U)$  for all  $\tau > 0$ . Then, if

$v_0 = z_0 - Bu(0) \in \mathbf{D}(A)$ , the classical solutions of (5.6) and (5.9) are related by

$$v(t) = z(t) - Bu(t). \quad (5.10)$$

Furthermore, the classical solution of (5.6) is unique.

**Proof** Suppose that  $v(t)$  is a classical solution of (5.9). Then  $v(t) \in \mathbf{D}(A) \subset \mathbf{D}(\mathfrak{A}) \subset \mathbf{D}(\mathfrak{B})$ ,  $Bu(t) \in \mathbf{D}(\mathfrak{B})$ , and so

$$\mathfrak{B}z(t) = \mathfrak{B}[v(t) + Bu(t)] = \mathfrak{B}v(t) + \mathfrak{B}Bu(t) = u(t),$$

where we have used that  $v(t) \in \mathbf{D}(A) \subset \ker \mathfrak{B}$  and equation (5.8). Furthermore, from (5.10) we have

$$\begin{aligned} \dot{z}(t) &= \dot{v}(t) + B\dot{u}(t) \\ &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B\dot{u}(t) && \text{by (5.9)} \\ &= Av(t) + \mathfrak{A}Bu(t) \\ &= \mathfrak{A}(v(t) + Bu(t)) && \text{by (5.7)} \\ &= \mathfrak{A}z(t) && \text{by (5.10)}. \end{aligned}$$

Thus, if  $v(t)$  is a classical solution of (5.9), then  $z(t)$  defined by (5.10) is a classical solution of (5.6).

The other implication is proved similarly. The uniqueness of the classical solutions of (5.6) follows from the uniqueness of the classical solutions of (5.9). ■

Although we have reformulated (5.6) as the abstract evolution equation (5.9), it includes a derivative of the control term, which is undesirable. This can be eliminated by reformulating (5.9) on the extended state space  $Z^e := U \oplus Z$ :

$$\begin{aligned} \dot{z}^e(t) &= \begin{pmatrix} 0 & 0 \\ \mathfrak{A}B & A \end{pmatrix} z^e(t) + \begin{pmatrix} I \\ -B \end{pmatrix} \tilde{u}(t), \\ z^e(0) &= \begin{pmatrix} (z_0^e)_1 \\ (z_0^e)_2 \end{pmatrix}. \end{aligned} \quad (5.11)$$

**Theorem 5.2.4** Consider the abstract differential equation (5.11). If  $v_0 \in \mathbf{D}(A)$  and  $u \in C^2([0, \tau]; U)$ , then (5.11) with  $(z_0^e)_1 = u(0)$ ,  $(z_0^e)_2 = v_0$  and  $\tilde{u} = \dot{u}$  has the unique classical solution  $z^e(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ , where  $v(t)$  is the unique classical solution of (5.9).

Furthermore, if  $z_0 = v_0 + Bu(0)$ , then the classical solution of (5.6) is given by

$$\begin{aligned} z(t) &= \begin{pmatrix} B & I \end{pmatrix} z^e(t) \\ &= Bu(t) - T(t)Bu(0) + T(t)z_0 - \\ &\quad \int_0^t T(t-s)B\dot{u}(s)ds + \int_0^t T(t-s)\mathfrak{A}Bu(s)ds. \end{aligned} \quad (5.12)$$

**Proof** From Lemma 4.2.2, we have that the operator  $A^e := \begin{pmatrix} 0 & 0 \\ \mathfrak{A}B & A \end{pmatrix}$  with domain  $\mathbf{D}(A^e) = U \oplus \mathbf{D}(A)$  is the infinitesimal generator of a  $C_0$ -semigroup on  $Z^e$ . Since  $\begin{pmatrix} I \\ -B \end{pmatrix} \in \mathcal{L}(U, Z^e)$ , (5.11) is well defined, and from Definition 4.1.4 and Lemma 4.2.2 the mild solution of (5.11) is given by

$$z^e(t) = \begin{pmatrix} I & 0 \\ S(t) & T(t) \end{pmatrix} \begin{pmatrix} (z_0^e)_1 \\ (z_0^e)_2 \end{pmatrix} + \int_0^t \begin{pmatrix} I & 0 \\ S(t-s) & T(t-s) \end{pmatrix} \begin{pmatrix} I \\ -B \end{pmatrix} \tilde{u}(s) ds,$$

where  $S(t)z = \int_0^t T(t-s)\mathfrak{A}Bz ds = \int_0^t T(s)\mathfrak{A}Bz ds$ . Thus

$$(z^e(t))_1 = (z_0^e)_1 + \int_0^t \tilde{u}(s) ds = u(0) + \int_0^t \dot{u}(s) ds = u(t). \quad (5.13)$$

Since  $u \in \mathcal{C}^2([0, \tau]; U)$ ,  $\tilde{u} \in \mathcal{C}^1([0, \tau]; U)$ , and  $z^e(0) = z_0^e = \begin{pmatrix} u(0) \\ v_0 \end{pmatrix} \in \mathbf{D}(A^e)$  by assumption, by Theorem 4.1.3 (5.11) has a unique classical solution that satisfies

$$(z^e(t))_1 = \tilde{u}(t) = \dot{u}(t)$$

and

$$\begin{aligned} (z^e(t))_2 &= \mathfrak{A}B(z^e(t))_1 + A(z^e(t))_2 - B\tilde{u}(t) \\ &= \mathfrak{A}Bu(t) + A(z^e(t))_2 - B\dot{u}(t) \quad \text{by (5.13)}. \end{aligned}$$

Since  $(z_0^e)_2 = v_0$ ,  $(z^e(t))_2 = v(t)$ , the unique classical solution of (5.9).

We now suppose that  $z_0 = v_0 + Bu(0)$  and calculate

$$\begin{aligned} \begin{pmatrix} B & I \end{pmatrix} z^e(t) &= Bu(t) + (z^e(t))_2 \\ &= Bu(t) + v(t) \\ &= z(t) \quad \text{by Theorem 5.2.3.} \end{aligned}$$

From Theorem 4.1.3, it follows that

$$v(t) = T(t)v_0 - \int_0^t T(t-s)B\dot{u}(s) ds + \int_0^t T(t-s)\mathfrak{A}Bu(s) ds,$$

which shows (5.12). ■

Summarizing, we have related the classical solutions  $z(t)$  of (5.6),  $v(t)$  of (5.9) and  $z^e(t)$  of (5.11) under the assumptions that  $u \in \mathcal{C}^2([0, \tau]; U)$  and  $z_0 - Bu(0) = \begin{pmatrix} B & I \end{pmatrix} z_0^e - Bu(0) = v_0 \in \mathbf{D}(A)$ . The abstract equations (5.9) and (5.11) also

have well defined mild solutions for  $\tilde{u} = \dot{u} \in L_p([0, \tau]; U)$  for some  $p \geq 1$ ,  $v_0 \in Z$ , and  $z_0^e \in Z^e$ , respectively. Consequently, under these weaker assumptions, we shall call  $z(t)$  defined by (5.12) the *mild solution* of the original boundary control equation (5.6).

To illustrate the mathematical principles, we first consider a rather artificial boundary control problem for a heat equation. The more realistic problem of injecting heat at one end of the metal bar introduced in Example 5.2.1 is a little more complicated and is considered in Exercise 6.15.

**Example 5.2.5** Consider the heat equation with boundary control action:

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t), & \frac{\partial z}{\partial x}(0, t) &= 0, & \frac{\partial z}{\partial x}(1, t) &= u(t), \\ z(x, 0) &= z_0(x). \end{aligned} \quad (5.14)$$

(5.14) can be reformulated in the form (5.6) by defining  $Z = L_2(0, 1)$ ,  $U = \mathbb{C}$ ,

$$\mathfrak{A} = \frac{d^2}{dx^2} \text{ with } \mathbf{D}(\mathfrak{A}) = \{h \in L_2(0, 1) \mid h \text{ and } \frac{dh}{dx} \text{ are absolutely continuous, } \frac{d^2h}{dx^2} \in L_2(0, 1) \text{ and } \frac{dh}{dx}(0) = 0\}$$

and the boundary operator  $\mathfrak{B} : L_2(0, 1) \rightarrow \mathbb{C}$  by

$$\mathfrak{B}h = \frac{dh}{dx}(1) \quad \text{with } \mathbf{D}(\mathfrak{B}) = \mathbf{D}(\mathfrak{A}). \quad (5.15)$$

Define  $A = \frac{d^2}{dx^2}$  with domain  $\mathbf{D}(A) = \mathbf{D}(\mathfrak{A}) \cap \ker \mathfrak{B} = \{h \in L_2(0, 1) \mid h \text{ and } \frac{dh}{dx} \text{ are absolutely continuous, } \frac{d^2h}{dx^2} \in L_2(0, 1) \text{ and } \frac{dh}{dx}(0) = 0 = \frac{dh}{dx}(1)\}$ . As in Example 3.2.11, we see that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup, and  $Bu = b(x)u$ , where  $b(x) = \frac{1}{2}x^2$  is contained in the domain of  $\mathfrak{A}$  with  $\mathfrak{B}Bu = u$ . Thus (5.14) defines a boundary control system and since  $\mathfrak{A}B = I$ , the mild solution of (5.14) is given by

$$\begin{aligned} z(t) &= Bu(t) - T(t)Bu(0) + T(t)z_0 - \\ &\quad \int_0^t T(t-s)B\dot{u}(s)ds + \int_0^t T(t-s)1u(s)ds. \end{aligned} \quad (5.16)$$

In Example 2.1.5, we had

$$T(t) = \sum_{n=0}^{\infty} e^{\lambda_n t} \langle \cdot, \phi_n \rangle \phi_n,$$

where  $\lambda_n = -n^2\pi^2$ ,  $n \geq 0$ ,  $\phi_0 = 1$  and  $\phi_n(\cdot) = \sqrt{2} \cos(n\pi\cdot)$ ,  $n \geq 1$ , and so

$$\int_0^t T(t-s)1u(s)ds = \int_0^t T(t-s)\phi_0 u(s)ds = \int_0^t u(s)ds\phi_0, \quad (5.17)$$

and

$$\begin{aligned}
& \int_0^t T(t-s)B\dot{u}(s)ds \\
&= \int_0^t \sum_{n=0}^{\infty} e^{\lambda_n(t-s)} \langle \frac{1}{2}x^2, \phi_n \rangle \phi_n \dot{u}(s) ds \\
&= \sum_{n=0}^{\infty} \int_0^t e^{\lambda_n(t-s)} \langle \frac{1}{2}x^2, \phi_n \rangle \phi_n \dot{u}(s) ds \\
&= \frac{1}{6} \int_0^t \dot{u}(s) ds \phi_0 + \sum_{n=1}^{\infty} \int_0^t e^{-n^2\pi^2(t-s)} \langle \frac{1}{2}x^2, \phi_n \rangle \dot{u}(s) ds \phi_n \\
&= \frac{1}{6} [u(t) - u(0)] \phi_0 + \sum_{n=1}^{\infty} \int_0^t e^{-n^2\pi^2(t-s)} \frac{(-1)^n \sqrt{2}}{n^2\pi^2} \dot{u}(s) ds \phi_n \\
&= \frac{1}{6} [u(t) - u(0)] \phi_0 + \sum_{n=1}^{\infty} \left[ \frac{(-1)^n \sqrt{2}}{n^2\pi^2} [u(t) - u(0)e^{-n^2\pi^2 t}] - \right. \\
&\quad \left. (-1)^n \sqrt{2} \int_0^t e^{-n^2\pi^2(t-s)} u(s) ds \right] \phi_n. \quad (5.18)
\end{aligned}$$

Furthermore,  $B$  has the expansion

$$Bu = \sum_{n=0}^{\infty} \langle \frac{1}{2}x^2, \phi_n \rangle \phi_n u = \frac{1}{6} \phi_0 u + \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{n^2\pi^2} \phi_n u, \quad (5.19)$$

and

$$T(t)Bu = \frac{1}{6} \phi_0 u + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \frac{(-1)^n \sqrt{2}}{n^2\pi^2} \phi_n u. \quad (5.20)$$

Combining (5.16), (5.17), (5.18), (5.19), and (5.20), it follows that the mild solution of (5.14) is given by

$$\begin{aligned}
z(t) &= \frac{1}{6}\phi_0 u(t) + \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{n^2 \pi^2} \phi_n u(t) - \frac{1}{6}\phi_0 u(0) - \\
&\quad \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \frac{(-1)^n \sqrt{2}}{n^2 \pi^2} \phi_n u(0) + T(t)z_0 - \frac{1}{6}[u(t) - u_0]\phi_0 - \\
&\quad \sum_{n=1}^{\infty} \left[ \frac{(-1)^n \sqrt{2}}{n^2 \pi^2} [u(t) - u_0 e^{-n^2 \pi^2 t}] - \right. \\
&\quad \left. (-1)^n \sqrt{2} \int_0^t e^{-n^2 \pi^2 (t-s)} u(s) ds \right] \phi_n + \int_0^t u(s) ds \phi_0 \\
&= T(t)z_0 + \int_0^t u(s) ds \phi_0 + \\
&\quad \sum_{n=1}^{\infty} (-1)^n \sqrt{2} \int_0^t e^{-n^2 \pi^2 (t-s)} u(s) ds \phi_n. \tag{5.21}
\end{aligned}$$

The mild solution of (5.14) is defined assuming that  $\dot{u} \in \mathbf{L}_p([0, \tau]; U)$ , for some  $p \geq 1$ , but in fact equation (5.21) is well defined for every  $u \in \mathbf{L}_2([0, \tau]; U)$ , since

$$\left| \int_0^t e^{-n^2 \pi^2 (t-s)} u(s) ds \right|^2 \leq \frac{1 - e^{-2n^2 \pi^2 t}}{2n^2 \pi^2} \int_0^t |u(s)|^2 ds.$$

■

Wave equations and beam equations with boundary control can also often be formulated using the same approach.

**Example 5.2.6** Consider the wave equation of Example 3.2.13 with boundary control action

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(x, t) &= \frac{\partial^2 w}{\partial x^2}(x, t), \\
\frac{\partial w}{\partial x}(0, t) &= 0, \quad \frac{\partial w}{\partial x}(1, t) = u(t), \\
w(x, 0) &= w_0(x), \quad \frac{\partial w}{\partial t}(x, 0) = w_1(x).
\end{aligned} \tag{5.22}$$

Recall from Example 3.2.13 the following state space  $Z = \mathbf{D}(A_0^{\frac{1}{2}}) \oplus \mathbf{L}_2(0, 1)$ , where  $A_0 h = -\frac{d^2 h}{dx^2}$  for  $h \in \mathbf{D}(A_0) = \{h \in \mathbf{L}_2(0, 1) \mid h \text{ and } \frac{dh}{dx} \text{ are absolutely continuous, } \frac{d^2 h}{dx^2} \in \mathbf{L}_2(0, 1) \text{ and } \frac{dh}{dx}(0) = 0 = \frac{dh}{dx}(1)\}$ . Then we can reformulate



(5.22) in the form (5.6) by defining  $U = \mathbb{C}$ ,

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}$$

with domain

$$\mathbf{D}(\mathfrak{A}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z \mid z_1, \frac{dz_1}{dx} \text{ are absolutely continuous,} \right. \\ \left. \frac{d^2 z_1}{dx^2} \in \mathbf{L}_2(0, 1), \frac{dz_1}{dx}(0) = 0 \text{ and } z_2 \in \mathbf{D}(A_0^{\frac{1}{2}}) \right\}$$

and the boundary operator  $\mathfrak{B} : \mathbf{D}(\mathfrak{B}) \subset Z \mapsto \mathbb{C}$  by

$$\mathfrak{B} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{dz_1}{dx}(1) \quad \text{with} \quad \mathbf{D}(\mathfrak{B}) = \mathbf{D}(\mathfrak{A}).$$

The system (5.22) is a boundary control system, since

$$A = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}$$

with domain  $\mathbf{D}(A) = \mathbf{D}(\mathfrak{A}) \cap \ker(\mathfrak{B}) = \mathbf{D}(A_0) \oplus \mathbf{D}(A_0^{\frac{1}{2}})$  is the infinitesimal generator of a  $C_0$ -semigroup (see Example 3.2.13), and  $Bu$  defined by  $Bu = b(x)u$ , where  $b(x) = \begin{pmatrix} \frac{1}{2}x^2 \\ 0 \end{pmatrix}$ , is contained in the domain of  $\mathfrak{A}$  with  $\mathfrak{B}Bu = u$ , and  $\mathfrak{A}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . ■

## 5.3 Exercises

5.1. In this exercise, we formulate some standard examples with inputs and outputs.

- a. Show that the following systems can be formulated as linear systems  $\Sigma(A, B, C, D)$ . Specify  $A$ ,  $B$ ,  $C$ , and  $D$  and the appropriate input, state, and output spaces.
  - i. The heat equation with Dirichlet boundary conditions:

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + b(x)u(t),$$

$$z(0, t) = 0 = z(1, t),$$

$$y(t) = \int_0^1 c(x)z(x, t)dx,$$

where  $b(x) = \frac{1}{2\varepsilon} \mathbb{1}_{(x_0-\varepsilon, x_0+\varepsilon)}(x)$ ,  $c(x) = \frac{1}{2\nu} \mathbb{1}_{(x_1-\nu, x_1+\nu)}(x)$ .

Hint: See Exercise 3.6.

ii. The perturbed heat equation:

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) - \alpha \frac{\partial z}{\partial x}(x, t) + b(x)u(t),$$

$$\frac{\partial z}{\partial x}(0, t) = 0 = \frac{\partial z}{\partial x}(1, t),$$

$$y(t) = \int_0^1 c(x)z(x, t)dt,$$

where  $b$  and  $c$  are chosen as in i.

Hint: See Exercise 2.13.

iii. The wave equation:

$$\frac{\partial^2 w}{\partial t^2}(x, t) + \frac{\partial^2 w}{\partial x^2}(x, t) = b(x)u(t),$$

$$w(0, t) = 0 = w(1, t),$$

$$y(t) = w(x_1, t), \quad 0 \leq x_1 \leq 1,$$

with  $b$  chosen as in i.

Hint: See Examples 2.3.5 and 3.2.12 and prove that  $C \in \mathcal{L}(Z, \mathbb{C})$  if and only if  $C$  is linear and  $\sum_n |C\phi_n|^2 < \infty$ , where  $\{\phi_n\}$  is a Riesz basis of  $Z$ .

iv. The simply supported beam:

$$\frac{\partial^2 f}{\partial t^2}(x, t) = -\frac{\partial^4 f}{\partial x^4}(x, t) + b(x)u(t),$$

$$f(0, t) = f(1, t) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0, t) = \frac{\partial^2 f}{\partial x^2}(1, t) = 0,$$

$$f(x, 0) = f_1(x), \quad \frac{\partial f}{\partial t}(x, 0) = f_2(x),$$

where  $f(x, t)$  is the displacement of the beam with respect to the position at rest,  $b$  is chosen as in i, and the measurement is taken to be

$$y(t) = f\left(\frac{1}{3}, t\right).$$

Hint: See Exercise 3.9 and prove that  $C \in \mathcal{L}(Z, \mathbb{C})$  if and only if  $C$  is linear and  $\sum_n |C\phi_n|^2 < \infty$ , where  $\{\phi_n\}$  is a Riesz basis of  $Z$ .

## 5.4 Notes and references

The treatment of boundary control systems is very different from that found in other texts such as Bensoussan et al. [19], but it was introduced in Fattorini [59]

as an effective means of studying controllability for such systems. Although it is not completely general and has the disadvantage of introducing an artificial state space, it is a mathematically simple way of formulating many partial differential equations with boundary control, which occur in applications. It has been extended to retarded differential equations with delayed control by Pandolfi in [120].

## References

- [1] W. Arendt and C.J.K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Transactions of the American Mathematical Society*, 306:837–841, 1988.
- [2] W. Arendt, F. Rübiger, and A. Sourour. Spectral properties of the operator equation  $AX + XB = Y$ . *Quart. J. Math. Oxford Ser. (2)*, 45(178):133–149, 1994.
- [3] J.P. Aubin. *Applied Functional Analysis*. John Wiley & Sons, New York, 1979.
- [4] J. Bak and D.J. Newman. *Complex Analysis*. Springer, New York, second edition, 1997.
- [5] A.V. Balakrishnan. *Applied Functional Analysis*. Springer Verlag, Berlin, 1976.
- [6] A.V. Balakrishnan. Strong stabilizability and the steady state Riccati equation. *Applied Mathematics and Optimization*, 7:335–345, 1981.
- [7] M. Balas. Towards a (more) practical control theory for distributed parameter systems, control and dynamic systems. In C.T. Leondes, editor, *Advances in Theory and Applications*, volume 18. Academic Press, New York, 1980.
- [8] J. Ball. Strongly continuous semigroups, weak solutions and the variation of constants formula. *Proc. Amer. Math. Soc.*, 63:370–373, 1977.
- [9] B. Bamieh, F. Paganini, and M.A. Dahleh. Distributed control of spatially invariant systems. *IEEE Trans. Automat. Control*, 47(7):1091–1107, jul 2002.
- [10] C.J.K. Batty. Tauberian theorems for the laplace-stieltjes transform. *Trans. Amer. Math. Soc.*, 322:783–804, 1990.
- [11] C.J.K. Batty and V.Q. Phong. Stability of individual elements under one-parameter semigroups. *Trans. Amer. Math. Soc.*, 322:805–818, 1990.
- [12] F. Bayazit and R. Heymann. Stability of multiplication operators and multiplication semigroups. *arXiv*, math.FA, Feb 2012.

- [13] A. Bellini-Morante. *Applied Semigroups and Evolution Equations*. Clarendon Press, Oxford, 1979.
- [14] R. Bellman and K.L. Cooke. Asymptotic behavior of solutions of differential-difference equations. *Mem. Amer. Math. Soc.*, 35, 1959.
- [15] R. Bellman and K.L. Cooke. *Differential-Difference Equations*. Academic Press, 1963.
- [16] C.D. Benchimol. *The Stabilizability of Infinite Dimensional Linear Time Invariant Systems*. PhD thesis, Thesis, UCLA, 1977.
- [17] C.D. Benchimol. Feedback stabilizability in Hilbert spaces. *J. App. Math. and Opt.*, 4:225–248, 1978.
- [18] C.D. Benchimol. A note on weak stabilizability of contraction semigroup. *SIAM J. Control and Optim.*, 16:373–379, 1978.
- [19] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems*, volume 1 of *Systems & Control: Foundations & Applications*. Birkhäuser, Boston, 1992.
- [20] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems*, volume 2 of *Systems & Control: Foundations & Applications*. Birkhäuser, Boston, 1993.
- [21] D.S. Bernstein and D.C. Hyland. The optimal projection equations for finite-dimensional fixed-order dynamic compensation of infinite-dimensional systems. *SIAM J. Contr. and Optim.*, 24:122–151, 1986.
- [22] K.P.M. Bhat. *Regulator Theory for Evolution Systems*. PhD thesis, University of Toronto, 1976.
- [23] H. Blomberg and R. Ylinen. *Algebraic Theory for Multivariable Systems*. Academic Press, 1983.
- [24] S. Bochner and K. Chandrasekharan. *Fourier Transforms*. Number 19 in Annals of Mathematics Studies. Princeton University Press, Princeton, 1949.
- [25] H. Bohr. *Almost Periodic Functions*. Chelsea Publishing Company, New York, 1947.
- [26] A.G. Butkovskiy. *Theory of Optimal Control of Distributed Parameter Systems*. American Elsevier, 1969.
- [27] P.L. Butzer and H. Berens. *Semigroups of Operators and Approximations*. Springer Verlag, 1967.
- [28] F.M. Callier and C.A. Desoer. An algebra of transfer functions for distributed linear time-invariant systems. *IEEE Trans. Circuits and Systems*, CAS-25:651–663, 1978. (Corrections: CAS-26, p. 320, 1979).
- [29] F.M. Callier and C.A. Desoer. Stabilization, tracking and distributed rejection in multivariable convolution systems. *Ann. Soc. Sci. Bruxelles*, 94:7–51, 1980.
- [30] M. Capiński and E. Kopp. *Measure, integral and probability*. Springer Undergraduate Mathematics Series. Springer-Verlag London Ltd., London, second edition, 2004.
- [31] B.M.N. Clarke and D. Williamson. Control canonical forms and eigenvalue assignment by feedback for a class of linear hyperbolic systems. *SIAM J. Control and Optim.*, 19:711–729, 1981.
- [32] B.D. Coleman and V.J. Mizel. Norms and semigroups in the theory of fading memory. *Arch. Rational Mech. Anal.*, 23:87–123, 1966.

- [33] C. Corduneanu. *Almost Periodic Functions*. J. Wiley, New York, 1968.
- [34] R.F. Curtain. Stabilization of boundary control distributed systems via integral dynamic output feedback of a finite-dimensional compensator. In A. Bensoussan and J.L. Lions, editors, *Analysis and Optimization of Systems*, volume 44 of *Lecture Notes in Control and Information Sciences*, pages 761–776. Springer Verlag, 1982.
- [35] R.F. Curtain. Finite-dimensional compensators for parabolic distributed systems with unbounded control and observation. *SIAM J. Control and Optim.*, 22:255–276, 1984.
- [36] R.F. Curtain. A comparison of finite-dimensional controller designs for distributed parameter systems. *Control Theory and Advanced Technology*, 9:609–628, 1993.
- [37] R.F. Curtain, O.V. Iftime, and H. Zwart. System theoretic properties of a class of spatially distributed systems. *Automatica*, 45:1619–1627, 2009.
- [38] R.F. Curtain and K. Morris. Transfer functions of distributed parameter systems; a tutorial. *Automatica*, 45:1101–1116, 2009.
- [39] R.F. Curtain and A.J. Pritchard. The infinite dimensional Riccati equation. *J. Math. Anal. & Appl.*, 47:43–57, 1974.
- [40] R.F. Curtain and A.J. Pritchard. *Infinite-Dimensional Linear Systems Theory*, volume 8 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1978.
- [41] R.F. Curtain and D. Salamon. Finite dimensional compensators for infinite dimensional systems with unbounded input operators. *SIAM J. Control and Optim.*, 24:797–816, 1986.
- [42] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Number 21 in Texts in Applied Mathematics. Springer-Verlag, New York, 1995.
- [43] Ju. L. Dalec'kiĭ and M. G. Kreĭn. *Stability of Solutions of Differential Equations in Banach Space*. American Mathematical Society, Providence, R.I., 1974. Translated from the Russian by S. Smith, Translations of Mathematical Monographs, Vol. 43.
- [44] R. Datko. Extending a theorem of A.M. Lyapunov to Hilbert space. *J. Math. Anal. Appl.*, 32:610–616, 1970.
- [45] E.B. Davies. *One-Parameter Semigroups*. Academic Press, London, 1980.
- [46] M.C. Delfour. State theory for linear hereditary-differential systems. *Anal. Appl.*, 60:8–35, 1977.
- [47] M.C. Delfour. The largest class of hereditary systems defining a  $C_0$ -semigroup on the product space. *Canadian J. Math.*, XXXII(4):969–978, 1980.
- [48] M.C. Delfour and J. Karrakchou. State space theory of linear time invariant systems with delays in state, control and observation variables, part I. *J. Math. Anal. and Appl.*, 125:361–399, 1987.
- [49] M.C. Delfour and J. Karrakchou. State space theory of linear time invariant systems with delays in state, control and observation variables, part II. *J. Math. Anal. and Appl.*, 125:400–450, 1987.
- [50] M.C. Delfour and S.K. Mitter. Hereditary differential systems with constant delays I: General case. *J. Diff. Eqns.*, 12:213–235, 1972.
- [51] M.C. Delfour and S.K. Mitter. Hereditary differential systems with constant delays II: A class of affine systems and the adjoint problem. *J. Diff. Eqns.*, 18:18–28, 1975.

- [52] W. Desch and W. Schappacher. Spectral properties of finite-dimensional perturbed linear semigroups. *J. Diff. Eqns.*, 59:80–102, 1985.
- [53] J. Diestel and J. J. Uhl, Jr. *Vector Measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
- [54] G. Doetsch. *Introduction to the Theory and Application of Laplace Transform*. Springer Verlag, Berlin, 1974.
- [55] N. Dunford and J.T. Schwartz. *Linear Operators, part 1*. Interscience, 1959.
- [56] N. Dunford and J.T. Schwartz. *Linear Operators, part 3*. Interscience, 1971.
- [57] H. Dym and H. P. McKean. *Fourier series and integrals*. Academic Press, New York, 1972. Probability and Mathematical Statistics, No. 14.
- [58] K.-J. Engel and R. Nagel. *One-parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafuno, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [59] H.O. Fattorini. Boundary control systems. *SIAM J. Control*, 6:349–388, 1968.
- [60] B.A. Francis. *A Course in  $H_\infty$ -Control Theory*, volume 88 of *LNCIS*. Springer Verlag, Berlin, 1987.
- [61] D. Franke. *Systeme mit Ortlich Verteilten Parametern. Eine Einführung in die Modellbildung, Analyse und Regelung*. Springer-Verlag, Berlin, Heidelberg, 1987.
- [62] J. M. Freeman. The tensor product of semigroups and the operator equation  $SX - XT = A$ . *J. Math. Mech.*, 19:819–828, 1969/1970.
- [63] P. Fuhrman. *Linear Systems and Operators in Hilbert Space*. McGraw-Hill, New York, 1981.
- [64] N. Fuji. Feedback stabilization of distributed parameter systems by a functional observed. *SIAM J. Control and Optim.*, 18:108–121, 1980.
- [65] G. Ghu, P.P. Khargonekar, and E.B. Lee. Approximation of infinite-dimensional systems. *IEEE Trans. Autom. Control*, AC-34:610–618, 1989.
- [66] J.S. Gibson. A note on stabilization of infinite dimensional linear oscillators by compact linear feedback. *SIAM J. Control and Optim.*, 18:311–316, 1980.
- [67] K. Glover, R.F. Curtain, and J.R. Partington. Realisation and approximation of linear infinite dimensional systems with error bounds. *SIAM J. Control and Optim.*, 26:863–898, 1988.
- [68] K. Glover, J. Lam, and J.R. Partington. Rational approximation of a class of infinite dimensional systems I: Singular values of Hankel operators. *MCSS*, 3:325–344, 1990.
- [69] K. Glover, J. Lam, and J.R. Partington. Rational approximation of a class of infinite-dimensional systems II: Optimal convergence rates of  $L_\infty$ -approximants. *MCSS*, 4:233–246, 1991.
- [70] K. Glover, J. Lam, and J.R. Partington. Rational approximation of a class of infinite dimensional systems III: The  $L_2$ -case. In P. Nevai and A. Pinhaus, editors, *Progress in Approximation Theory*, pages 405–440. Academic Press, 1991.
- [71] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of Linear Operators. Vol. II*, volume 63 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1993.

- [72] J.A. Goldstein. *Semigroups of Linear Operators and Applications*. Oxford Mathematical Monographs. Oxford University Press, New York, 1985.
- [73] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, 5:245–256, 1981.
- [74] R. Gressang and G. Lamont. Observers for systems characterized by semigroups. *IEEE Trans. Autom. Control*, AC-20:523–528, 1975.
- [75] G. Gripenberg, S.O. Londen, and O. Staffans. *Volterra Integral and Functional Equations*. Cambridge University Press, Cambridge, 1990.
- [76] B.Z. Guo, H. Zwart, and R.F. Curtain. On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform. Technical Report 1593, University of Twente, the Netherlands, 2001. available at <http://www.math.utwente.nl/publications/>.
- [77] J.K. Hale. *Theory of Functional Differential Equations*. Springer Verlag, New York, 1977.
- [78] H. Helson. *Harmonic Analysis*. Addison-Wesley, London, 1983.
- [79] E. Hille and R.S. Phillips. *Functional Analysis and Semigroups*, volume 31. Amer. Math. Soc. Coll. Publ., Providence, R.I., 1957.
- [80] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York, 1974.
- [81] F. Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. of Diff. Equations*, 1:43–55, 1985.
- [82] F. Huang. Strong asymptotic stability of linear dynamical systems in Banach spaces. *J. Diff. Eqns.*, 104:307–324, 1993.
- [83] B. Jacob and H. Zwart. Equivalent conditions for stabilizability of infinite-dimensional systems with admissible control operators. *SIAM Journal on Control and Optimization*, 37:1419–1455, 1999.
- [84] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, volume 223 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2012.
- [85] C.A. Jacobson and C.N. Nett. Linear state space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans. Autom. Control*, AC-33:541–550, 1988.
- [86] N. Jacobson. *Lectures in Abstract Algebra, Vol. I*. Van Nostrand, New York, 1953.
- [87] E.W. Kamen, P.P. Khargonekar, and A. Tannenbaum. Stabilization of time-delay systems using finite-dimensional compensators. *IEEE Trans. Autom. Control*, AC-30:75–78, 1985.
- [88] T. Kato. *Perturbation Theory of Linear Operators*. Springer Verlag, 1966.
- [89] T. Kawata. *Fourier Analysis in Probability Theory*. Academic Press, New York and London, 1972.
- [90] S. Kitamura, H. Sakairi, and M. Mishimura. Observers for distributed parameter systems. *Electrical Eng. in Japan*, 92:142–149, 1972.
- [91] G. Klambauer. *Real Analysis*. Dover Publication, Inc., Mineola, New York, 2005. Republication of the work published by the American Elsevier Publishing Company, 1973.



- [92] E. Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley & Sons, New York, 1978.
- [93] B. Ya. Levin. *Lectures on Entire Functions*, volume 150 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.
- [94] N. Levison and R.M. Redheffer. *Complex Variables*. Holden-Day, Inc., San Francisco, 1970.
- [95] J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problem, I, II, III*. Springer Verlag, Berlin, 1972.
- [96] H. Logemann. *Funktionentheoretische Methoden in der Regelungstheorie Unendlichdimensionaler Systeme*. PhD thesis, Institut für Dynamische Systeme, Universität Bremen, Germany, 1986. Report nr. 156.
- [97] H. Logemann. Finitely generated ideals in certain algebras of transfer functions of infinite-dimensional systems. *Int. J. Control*, 45:247–250, 1987.
- [98] H. Logemann. Stability and stabilizability of linear infinite-dimensional discrete-time systems. *IMA Journal of Mathematical Control & Information*, 9:255–263, 1992.
- [99] L.H. Loomis. *An introduction to abstract harmonic analysis*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1953.
- [100] G. Lumer and R.S. Phillips. Dissipative operators in a Banach space. *Pacific J. Math.*, 11:679–698, 1961.
- [101] Gunter Lumer and Marvin Rosenblum. Linear operator equations. *Proc. Amer. Math. Soc.*, 10:32–41, 1959.
- [102] S. MacLane and G. Birkhoff. *Algebra*. MacMillan, New York, 1965.
- [103] P.M. Mäkilä. Laguerre series approximation of infinite-dimensional systems. *Automatica*, 26:985–996, 1990.
- [104] A. Manitius and R. Triggiani. Function space controllability of linear retarded systems: A derivation from abstract operator conditions. *SIAM J. Control and Optim.*, 16:599–645, 1978.
- [105] A. Manitius and R. Triggiani. Sufficient conditions for function space controllability and feedback stabilizability of linear retarded systems. *IEEE Trans. Autom. Control*, AC-23:659–665, 1978.
- [106] M. Marcus. *Introduction to Modern Algebra*. Marcel Dekker Inc., New York, 1978.
- [107] S.M. Melzer and B.C. Kuo. Optimal regulation of systems described by a countably infinite number of objects. *Automatica*, 7:359–366, 1971.
- [108] A.S. Morse. System invariants under feedback and cascade control. In *Proc. Int. Conf. on Math. Syst. Theory*, Udine, Italy, 1976.
- [109] S. Mossaheb. On the existence of right coprime factorizations for functions meromorphic in a half-plane. *IEEE Trans. Autom. Control*, AC-25:550–551, 1980.
- [110] A.W. Naylor and G.R. Sell. *Linear Operator Theory in Engineering and Science*, volume 40 of *Applied Mathematical Sciences*. Springer, New York, 1982.
- [111] S.A. Nefedov and F.A. Sholokhovich. A criterion for the stabilizability of dynamical systems with finite-dimensional input. *Differentsial'nye Uravneniya*, 22:163–166, 1986.

- [112] C.N. Nett. The fractional representation approach to robust linear feedback design: A self-contained exposition. Master's thesis, Dept. of ECSE, Rensselaer Polytechnic Institute, Troy, NY, USA, 1984.
- [113] N. K. Nikol'skiĭ. *Treatise on the Shift Operator*, volume 273 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1986. Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.
- [114] R. Ober and S. Montgomery-Smith. Bilinear transformation of infinite-dimensional state-space systems and balanced realisation of nonrational transfer function. *SIAM J. Control and Optim.*, 28:438–465, 1990.
- [115] A.W. Olbrot. Stabilizability, detectability and spectrum assignment for linear autonomous systems with general time delays. *IEEE Trans. Autom. Control*, AC-23:887–890, 1978.
- [116] J.C. Oostveen. *Strongly Stabilizable Distributed Parameter Systems*. SIAM, Philadelphia, 2000.
- [117] M. Opmeer and R.F. Curtain. The suboptimal Nehari problem for well-posed infinite-dimensional systems. *SIAM J. Control and Optimization*, 44, 2005.
- [118] P.A. Orner and A.M. Foster. A design procedure for a class of distributed parameter control systems. *Trans. A.S.M.E. Series G. Journal of Dynamical Systems, Measurement & Control*, 93:86–93, 1971.
- [119] L. Pandolfi. On feedback stabilization of functional differential equations. *Boll. UHI 4, Il Supplimento al Fascicolo 3*, XI(IV):626–635, 1975.
- [120] L. Pandolfi. Generalized control systems, boundary control systems, and delayed control systems. *MCSS*, 3:165–181, 1990.
- [121] J. R. Partington. *Linear Operators and Linear Systems*, volume 60 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004. An analytical approach to control theory.
- [122] J.R. Partington, K. Glover, H.J. Zwart, and R.F. Curtain.  $L_\infty$ -approximation and nuclearity of delay systems. *Systems and Control Letters*, 10:59–65, 1988.
- [123] A. Pazy. Asymptotic behavior of the solution of an abstract evolution equation and some applications. *J. Diff. Eqns.*, 4:493–509, 1968.
- [124] A. Pazy. On the applicability of Lyapunov's theorem in Hilbert spaces. *SIAM J. Math. Anal.*, 3:291–295, 1972.
- [125] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer Verlag, New York, 1983.
- [126] Vũ Quốc Phong. The operator equation  $AX - XB = C$  with unbounded operators  $A$  and  $B$  and related abstract Cauchy problems. *Math. Z.*, 208(4):567–588, 1991.
- [127] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory, A Behavioral Approach*, volume 26 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1998.
- [128] A.J. Pritchard and J. Zabczyk. Stability and stabilizability of infinite dimensional systems. *SIAM Review*, 23, 1981.
- [129] J. Prüss. On the spectrum of  $C_0$ -semigroups. *Trans. Am. Math. Soc.*, 284:847–856, 1984.

- [130] K.M. Przyluski. The Lyapunov equations and the problem of stability for linear bounded discrete-time systems in Hilberts space. *Appl. Math. Optim.*, 6:97–112, 1980.
- [131] C. R. Putnam. *Commutation Properties of Hilbert Space Operators and Related Topics*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36. Springer-Verlag New York, Inc., New York, 1967.
- [132] W.H. Ray. *Advanced Process Control*. McGraw-Hill, New York, London, 1981.
- [133] R. Rebarber. Spectral assignability for distributed parameter systems with unbounded scalar control. *SIAM J. Control and Optim.*, 27:148–169, 1989.
- [134] R. Rebarber. Necessary conditions for exponential stabilizability of distributed parameter systems with infinite-dimensional unbounded control. *Systems and Control Letters*, 14:241–248, 1990.
- [135] M. Rosenblum. On the operator equation  $BX - XA = Q$ . *Duke Math. J.*, 23:263–269, 1956.
- [136] M. Rosenblum and J. Rovnyak. *Hardy Classes and Operator Theory*. Oxford University Press, New York, 1985.
- [137] H. L. Royden. *Real Analysis*. The Macmillan Co., New York, second edition, 1963.
- [138] W. Rudin. *Functional Analysis*. McGraw-Hill Book Company, New York, 1973.
- [139] W. Rudin. *Principals of Mathematical Analysis*. McGraw-Hill Book Company, New York, third edition, 1976.
- [140] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, New York, third edition, 1987.
- [141] D.L. Russell. Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems. *J. Math. Anal.*, 62:182–255, 1968.
- [142] D.L. Russell. Linear stabilization of the linear oscillator in Hilbert space. *J. Math. Anal. Appl.*, 25:663–675, 1969.
- [143] D.L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open problems. *SIAM Review*, 20:639–739, 1978.
- [144] Y. Sakawa. Feedback control of second order evolution equations with damping. *SIAM J. Control and Optim.*, 22:343–361, 1984.
- [145] Y. Sakawa and T. Matsushita. Feedback stabilization for a class of distributed systems and construction of a state estimator. *IEEE Trans. Autom. Control*, AC-20:748–753, 1975.
- [146] D. Salamon. *Control and Observation of Neutral Systems*. Number 91 in Research Notes in Mathematics. Pitman Advanced Publishing Program, Boston, 1984.
- [147] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21:147–164, 1989.
- [148] J.M. Schumacher. *Dynamic Feedback in Finite and Infinite-Dimensional Linear Systems*, volume 143 of *Mathematical Centre Tracts*. Mathematical Centrum, Amsterdam, 1981.
- [149] J.M. Schumacher. A direct approach to compensator design for distributed parameter systems. *SIAM J. Control and Optim.*, 21:823–836, 1983.
- [150] Sen-Yen Shaw and S. C. Lin. On the equations  $Ax = q$  and  $SX - XT = Q$ . *J. Funct. Anal.*, 77(2):352–363, 1988.

- [151] M. Slemrod. A note on complete controllability and stabilizability of linear control systems in Hilbert space. *SIAM J. Control*, 12:500–508, 1974.
- [152] O. J. Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.
- [153] O.J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM Journal on Control and Optimization*, 36:1268–1292, 1998.
- [154] E.M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
- [155] S.H. Sun. On spectrum distribution of complete controllable systems. *SIAM J. Control and Optim.*, 19:730–743, 1981.
- [156] H. Tanabe. *Equations of Evolution*. Pitman, 1979.
- [157] A.E. Taylor. *Introduction to Functional Analysis*. John Wiley, 1958.
- [158] A.E. Taylor and D.C. Lay. *Introduction to Functional Analysis*. John Wiley & Sons, New York-Chichester-Brisbane, second edition, 1980.
- [159] E.G.F. Thomas. Totally summable functions with values in locally convex spaces. In A. Bellow and D. Kölzow, editors, *Measure Theory*, volume 541 of *Lecture Notes in Mathematics*, pages 117–131, Berlin, 1976. Springer Verlag. Proceedings of the Conference held in Oberwolfach, 15–21 June, 1975.
- [160] E.G.F. Thomas. Vector-valued integration with applications to the operator valued  $H_\infty$  space. *Journal of Mathematical Control and Information*, 14:109–136, 1997.
- [161] E. C. Titchmarsh. *The Theory of Functions*. Oxford University Press, 1939. Second edition.
- [162] Y. Tomilov. A resolvent approach to stability of operator semigroups. *J. Operator Theory*, 46:63–98, 2001.
- [163] R. Triggiani. On the stabilization problem in Banach space. *J. Math. Anal. Appl.*, 52:383–403, 1975.
- [164] R. Triggiani. Lack of uniform stabilization for noncontractive semigroups under compact perturbations. *Proc. Amer. Math. Soc.*, 105:375–383, 1989.
- [165] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.
- [166] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press, Cambridge, Mass., USA, 1985.
- [167] M. Vidyasagar, H. Schneider, and B.A. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Trans. Autom. Control*, AC-27:880–894, 1982.
- [168] J.A. Walker. On the application of Lyapunov’s direct method to linear dynamical systems. *J. Math. Analysis and Appl.*, 53:187–220, 1976.
- [169] P.K.C. Wang. Modal feedback stabilization of a linear distributed system. *IEEE Trans. Autom.*, AC-17:552–553, 1972.
- [170] G. Weiss and R.F. Curtain. Dynamic stabilization of regular linear systems. *IEEE Transactions on Automatic Control*, 42:4–21, 1997.
- [171] G. Weiss and R. Rebarber. Optimizability and estimatability for infinite-dimensional systems. *SIAM Journal of Control and Optimization*, 39:1204–1232, 2000.

- [172] W.M. Wonham. *Linear Multivariable Control; A Geometric Approach*. Springer Verlag, New York, 1974.
- [173] K. Yosida. *Functional Analysis*. Springer Verlag, 1966.
- [174] J. Zabczyk. A note on  $C_0$ -semigroups. *Bull. l'Acad. Pol. de Sc. Serie Math.*, 23:895–898, 1975.
- [175] J. Zabczyk. Stabilization of boundary control systems. *J. Diff. Eqns*, 32, 1979.
- [176] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms and approximate inverses. *IEEE Trans. Autom. Control*, AC-26:301–320, 1981.
- [177] O. Zariski and P. Samuel. *Commutative Algebra, Vol. 1*. Van Nostrand, New York, 1958.
- [178] E. Zauderer. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1989.
- [179] H. Zwart. Transfer functions for infinite-dimensional systems. *Systems Control Lett.*, 52(3-4):247–255, 2004.
- [180] H.J. Zwart. *Geometric Theory for Infinite-Dimensional Systems*, volume 115 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1989.
- [181] H.J. Zwart, R.F. Curtain, J.R. Partington, and K. Glover. Partial fraction expansions for delay systems. *Systems & Control Letters*, 10:235–244, 1988.

## Index

- $>$ , for operators, 272  
 $\mathcal{Q}'$ , 259  
 $T^*$ , 265  
 $V^\perp$ , 243  
 $X$ -homotopic, 234  
 $X'$ , 254  
 $X''$ , 257  
 $Z_1$ , 45  
 $Z_{-1}$ , 46  
 $Z_\alpha$ , 245  
 $\Lambda_F$ , 315  
 $\Lambda_{\tilde{\lambda}}$ , 53  
 $\Sigma(A, -, C, -)$ , 114  
 $\Sigma(A, B, -, -)$ , 114  
 $\Sigma(A, B, C, -)$ , 114  
 $\Sigma(A, B, C, D)$ , 113  
 $*$ , 304  
 $C[0, 1]$ , 239  
 $C([a, b]; X)$ , 251  
 $C^1([0, \tau]; Z)$ , 90  
 $D(T)$ , 247  
 $H_2(Z)$ , 309  
 $H_\infty[H_\infty]^{-1}$ , 325  
 $H_\infty$ , 309  
 $H_\infty(X)$ , 309  
 $L(\Omega; Z)$ , 293  
 $L_2(\partial\mathbb{D}; \mathbb{C}^m)$ , 314  
 $L_2((-\infty, \infty); Z)$ , 305  
 $L_2(\partial\mathbb{D}^2; \mathbb{C}^n)$ , 78  
 $L_2^{loc}([0, \infty); U)$ , 127  
 $L_\infty(\partial\mathbb{D}; \mathbb{C}^{k \times m})$ , 314  
 $L_\infty(\partial\mathbb{D}^2; \mathbb{C}^{m \times n})$ , 78  
 $L_p(a, b)$ , 238  
 $L_p(\Omega; Z)$ , 293  
 $L_\infty(\Omega; Z)$ , 293  
 $L_\infty(a, b)$ , 238  
 $P(\Omega; \mathcal{L}(Z_1, Z_2))$ , 293  
 $P_p(\Omega; \mathcal{L}(Z_1, Z_2))$ , 293  
 $P_\infty((-\infty, \infty); \mathcal{L}(U, Y))$ , 305  
 $P_\infty(\Omega; \mathcal{L}(Z_1, Z_2))$ , 293  
 $\hat{h}$ , 303  
 $\ell_2(\mathbb{Z}; \mathbb{C}^m)$ , 319  
 $\ell_2(\mathbb{Z}; \mathbb{C}^n)$ , 51  
 $\ell_p(\mathbb{Z}; \mathbb{C}^{k \times m})$ , 240  
 $\ell_p(\mathbb{Z}; \mathbb{C}^n)$ , 240  
 $\ell_p(\mathbb{N})$ , 236  
 $\ell_p(\mathbb{Z}^+)$ , 237  
 $\ell_p(\mathbb{Z})$ , 237  
 $\ell_\infty(\mathbb{N})$ , 237  
 $\ell_\infty(\mathbb{Z})$ , 237  
 $\geq$ , for operators, 272  
 $\tilde{\mathcal{A}}(\beta)$ , 336  
 $\hat{h}$ , 302  
 $\hat{h}(s)$ , 141

- ker  $T$ , 248
- $\langle \cdot, \cdot \rangle$ , 241
- $\mathbb{C}(s)$ , 325
- $\mathbb{C}_p(s)$ , 325
- $\mathbb{C}_\beta^+$ , 302
- $\mathbb{C}_\delta^+$ , 191
- $\mathbb{C}_\delta^-$ , 191
- $\mathbb{D}$ , 314
- $\mathbb{R}(s)$ , 325
- $\mathbb{R}_p(s)$ , 325
- $\mathbb{Z}^+$ , 237
- $\underline{H}_2$ , 309
- $\underline{V}$ , 238
- $\mathbb{C}_\beta^+$ , 302
- $\| \cdot \|$ , 236
- $\partial\mathbb{D}$ , 314
- $\perp$ , 243
- ran  $T$ , 247
- $\rho(A)$ , 274
- $\sigma_c(A)$ , 276
- $\sigma_r(A)$ , 276
- $\sigma(A)$ , 276
- $\sigma_p(A)$ , 276
- $\sigma_\delta^+(A)$ , 191
- $\sigma_\delta^-(A)$ , 191
- $\hookrightarrow$ , 250
- $r_\sigma(T)$ , 280
- $\mathcal{A}(\beta)$ , 333
- $\mathcal{L}(X)$ , 249
- $\mathcal{L}(X, Y)$ , 249
- $\mathcal{MA}$ , 328
- $\mathcal{MB}$ , 328
- $\mathcal{MR}$ , 328
- $\mathcal{R}(\beta)$ , 325
- $\mathcal{R}^r(\beta)$ , 325
- $\mathcal{R}_\infty^r(\beta)$ , 325
- $\mathcal{R}_\infty(\beta)$ , 325
- $S^+$ , 315
- $\mathcal{F}^\infty$ , 127
- $\mathcal{F}^r$ , 126
- $\mathfrak{F}_c(h)$ , 303
- m, 287
- $m^*$ , 287
- abstract boundary control problems, 117
- abstract differential equation, 90
- abstract evolution equation, 90
- addition, 235
- adjoint operator, 265, 269
- algebra, 323
- algebraic inverse, 248
- almost periodic, 337
- analytic, 228
- antistable, 6
- approximation
  - in  $H_\infty$ , 338
  - in  $L_\infty$ , 306
  - of holomorphic functions, 233
- approximation error, 210
- associative, 321
- associative property, 235
- asymptotically stable, 176
- balanced realization, 8
- Banach algebra, 323
- Banach space, 240
- Banach Steinhaus theorem, 251
- basis, 244
- beam equation, 82
  - $C_0$ -semigroup, 84
  - boundary control, 156, 159
  - Riesz-spectral generator, 84
- $\beta$ -exponentially detectable, 189
- $\beta$ -exponentially stabilizable, 189
- $\beta$ -exponentially stable, 163
- Bezout identity, 327, 332
- biorthogonal, 61
- biorthogonal sequence, 61
- biproper, 325
- Bochner integrals, 289
- boundary control system, 117, 161
- boundary operator, 117
- bounded
  - operator, 248
  - set, 238
- bounded algebraic inverse, 274
- bounded inverse, 274
- bounded linear functional, 253
- bounded linear operator, 248
- Cauchy sequence, 239
- Cauchy's residue theorem, 232
- Cauchy's theorem, 231, 302
- Cauchy-Schwarz inequality, 241
- characteristic function
  - transfer function
  - equality, 144
- circle criterion, 148

- classical solution, 34, 106
  - on  $[0, \infty)$ , 91
  - on  $[0, \tau]$ , 91
- closed
  - operator, 259
  - set, 238
- closed curve, 231
- closed graph theorem, 262
- closed-loop system, 209
- closure, 238
- coercive, 272
- cogenerator, 47
- commutative, 321, 323
- commutative property, 235
- commutative ring, 321
- compact
  - operator, 251
  - set, 238
- compact, normal resolvent, 285
- compensator, 209
- complete, 239
- completion, 240
- complex vector space, 236
- continuous, 251
  - strongly, 251
  - uniformly, 251
- continuous at  $x_0$ , 248
- continuous embedding, 250
- continuous on  $\mathbf{D}(F)$ , 248
- continuous spectrum, 276
- contour, 231
  - closed, 231
  - positively oriented, 231
  - simple, 231
- contraction mapping theorem, 246
- contraction semigroup, 37
- controllability gramian, 6
- controllability map, 5
- controllable, 5
- convergence
  - strong, 251
  - uniform, 250
  - weak, 257
- converges, 238
- convolution algebra, 333
- convolution operator, 52
- convolution product, 304, 333
- coprime, 327, 332
  - left, 332
  - over  $\hat{\mathcal{A}}(\beta)$ , 337
  - right, 332
- curve
  - closed, 231
  - rectifiable, 230
  - simple, 231
- $C_0$ -semigroup, 15
  - bounded generator, 15
  - contraction, 37
  - dual, 41
  - growth bound, 21
  - holomorphic, 48
  - infinitesimal generator, 24
  - measurable, 290
  - normal, 48
  - perturbed, 97, 105
  - self-adjoint, 47
- $C_0$ -semigroup
  - strongly stable, 176
- decay rate, 163
- delay equation/system
  - $\beta$ -exponentially detectable, 206
  - $\beta$ -exponentially stabilizable, 206
  - $\beta$ -exponentially stable, 173
  - spectrum determined growth
    - assumption, 173
  - transfer function, 145
- dense, 238
- dense injection, 250
- derivative, 297
  - Fréchet, 296
- detectable, *see* exponentially detectable
  - $\beta$ -exponentially, 189
  - exponential, 9
    - spatial invariant system, 199
  - exponentially, 189
- difference equation
  - power stable, 186
  - strongly stable, 186
  - weakly stable, 186
- differentiable, 297
  - strongly, 300
  - uniformly, 300
  - weakly, 300
- differential, *see* Fréchet differential
- differential equation
  - abstract, 90
- dimension, 236



- discrete Fourier transform, 319
- discrete-time system
  - stability, 186
  - stabilizability, 225
- dissipative operators, 50
- distributive, 321
- divisors of zero, 321
- domain
  - complex, 228
  - of an operator, 247
- doubly coprime factorization, 332
- dual operator, 259, 262
- dual semigroup, 41
- dual space, 254
- duality pairing, 264
- Dunford integral, 281
  
- eigenfunction, *see* eigenvector
- eigenvalue, 276
  - isolated, 276
  - multiplicity, 276
  - order, 276
- eigenvector, 276
  - generalized, 276
- entire, 228
- entire function
  - order, 229
  - type, 229
- equivalent norms, 238
- exponential solution, 130
  - discrete-time, 150
- exponentially detectable, 9, 189
- exponentially stabilizable, 9, 189
- exponentially stable, 6, 163
- extended input-output map, 127
  
- feedback, 189
- feedback connection, 147
- feedback operator, 189
- field, 322
- finite rank, 247
- fixed point, 246
- Fourier coefficients, 244
- Fourier expansion, 244
- Fourier series, 319
- Fourier transform, 303
  - discrete, 319
  - inverse, 308
- Fréchet derivative, 296
- Fréchet differentiable at  $x$ , 296
- Fréchet differential at  $x$ , 296
- fractional representation theory, 5
- Fubini's theorem, 295
- functional, 246
  
- general system, 130
- generalized eigenvectors, 276
- generator of mild evolution operator, 104
- graph, 259
- graph norm, 45
- greatest common divisor, 326
- greatest common left divisor, 328
- greatest common right divisor, 328
- Gronwall's lemma, 305
- group, 321
- growth bound, 21
  
- Hadamard's theorem, 229
- Hahn-Banach theorem, 254
- Hankel operator, 7
- Hankel singular values, 7
- Hardy space, 309
- heat equation, 13, 110
  - $C_0$ -semigroup, 69
  - $\beta$ -exponentially detectable, 218
  - $\beta$ -exponentially stabilizable, 218
  - as state linear system, 114
  - boundary control, 116, 120, 151, 157, 160
  - control, 154
  - Dirichlet boundary conditions, 81
  - impulse response, 129
  - inhomogeneous, 95
  - stabilizing compensator, 213
  - transfer function, 135, 137, 140, 148
  - zero, 149
- high gain feedback, 221
- Hilbert space, 241
- Hilbert's inequality, 320
- Hille-Yosida theorem, 29
- Hölder inequality, 255
- holomorphic, 228, 301
- holomorphic continuation, 230
- holomorphic semigroup, 48
- homotopic, 234
- homotopic invariant, 234
  
- ideal, 321

- identity, 321
- impulse response, 126
  - Riesz-spectral system, 128
- impulse response function, 7
- $\text{ind}(g)$ , 234
- indented imaginary axis, 234
- index, *see* Nyquist index
- induced norm, 241
- infinite-dimensional, 236
- infinitesimal generator, 24
- injective, 248
- inner product, 241
- inner product space, 241
- input space, 113
- input stabilizable, 216
- input-output map, 7, 126
  - extended, 127
- input-output stable, 6
- inputs, 113
- instability due to delay, 220
- integral
  - Bochner, 289
  - complex, 231
  - Dunford, 281
  - Lebesgue, 289
  - Pettis, 291
- integral domain, 321
- integral operators, 253
- invariant
  - shift, 52, 314, 315
  - spatially, 52
- inverse, 236, 321
  - algebraic, 248
  - bounded, 274
  - in Banach algebra, 323
  - right, 267
- inverse element, 321
- inverse Fourier transform, 308
- invertible, 247
  - over  $\hat{\mathcal{A}}(\beta)$ , 337
- isolated, 145
- isolated eigenvalue, 276
- isometrically isomorphic, 238
- isomorphic
  - isometrically, 238
  - topologically, 238
- ker, 248
- kernel, 248
- Laplace transform, 302
  - impulse response, 142
  - two-sided, 303
- Laplace transform of the impulse response, 142
- Laplace-transformable functions, 302
- Laurent operator, 315
- Laurent series, 232
- Lebesgue integrable, 289
- Lebesgue integral, 289
- Lebesgue measure, 287
- Lebesgue-dominated convergence theorem, 295
- left coprimeness, 328
- left divisors, 328
- left multiple, 328
- left shift, 315
- left shift operator, 16, 52
- left-coprime, 332
- left-coprime factorization over  $\mathcal{MR}$ , 329
- Legendre polynomials, 245
- length of an interval, 287
- linear
  - system, 130
- linear combination, 236
- linear functional, 253
  - bounded, 253
- linear operator, 247
- linear quadratic gaussian or LQG, 212
- linear space, *see* linear vector space
  - normed, 236
- linear subspace, 236
- linear vector space, 235
- linearly dependent, 236
- linearly independent, 236
- Liouville's theorem, 229
- Luenberger observer, 210
- Lyapunov equation, 165, 184
  - discrete-time, 187
- Lyapunov equations, 6
- Lyapunov inequality, 185
- maximal, 244
- McMillan degree, 330
- measurable
  - function, 287
  - of semigroups, 290
  - set, 287

- strong, 288
- uniform, 288
- weak, 288
- measure
  - Lebesgue, 287
- meromorphic, 232
- mild evolution operator, 102
- mild evolution operator generated by
  - $A + D(\cdot)$ , 104
- mild solution, 35, 93, 106, 111, 120
- minimal realization, 7
- Minkowski inequality, 237, 238
- minor, 330
- model reduction, 8
- multiplication operator, 53, 315
- multiplicative subset, 324
- multiplicity
  - algebraic, 276
- natural embedding, 257
- nonnegative, 272
- nonzero limit at  $\infty$  in  $\overline{\mathbb{C}_0^+}$ , 233
- norm, 236
  - equivalent, 238
  - induced by inner product, 241
  - operator, 248
- normal  $C_0$ -semigroup, 48
- normal operator, 268
  - unbounded, 273
- normed linear space, 236
- Nyquist index, 234
- Nyquist theorem, 233
- observability gramian, 6
- observability map, 6
- observable, 6
- observer, *see* Luenberger observer
- open, 238
- open mapping theorem, 251
- operator, 247
  - adjoint
    - bounded, 265
    - unbounded, 269
  - algebraic inverse, 248
  - bounded, 248
  - closed, 259
  - coercive, 272
  - compact, 251
  - dual
    - bounded, 259
    - unbounded, 262
  - finite rank, 247
  - inverse, 248, 274
  - linear, 247
  - multiplication, 53
  - nonnegative, 272
  - norm, 248
  - normal, 268
  - positive, 272
  - self-adjoint, 271
  - square root, 272
  - symmetric, 271
  - unbounded, 259
  - unitary, 268
- order, 276
  - entire function, 229
  - of a pole, 231
  - of a zero, 230
- orientation, 231
- orthogonal, 243
- orthogonal complement, 243
- orthogonal projection, 273
- orthogonal projection lemma, 273
- orthogonal projection on  $V$ , 273
- orthonormal basis, 244
- orthonormal set, 243
- outer measure, 287
- output, 113
- output injection, 189
- output injection operator, 189
- output space, 113
- output stabilizable, 216
- Paley-Wiener theorem, 311, 312
- parallel connection, 146
- parallelogram law, 241
- Parseval equality, 244
- Parseval's equality, 308
- Pettis integrable, 291
- Pettis integral, 291
- piecewise constant functions, 294
- pivot space, 264
- Plancherel theorem, 320
- point spectrum, 276
- poles, 231, 330
- positive, 272
- positively oriented, 231
- power stabilizable, 225

- power stable, 186
- principal ideal, 321
- principal ideal domain, 321
- principal-axis balanced, 8
- principle of the argument, 233
- projection, 273
- proper, 325
- Pythagoras' theorem, 243
  
- quasi-isolated, 64
- quotient algebra, 324
- quotient field, 324
- quotient field of  $H_\infty$ , 325
- quotient ring, 324
  
- ran, 247
- range, 247
- rational controller
  - for Riesz-spectral system, 223
- real transfer function, 147
- real vector space, 236
- realization, 7
  - balanced, 8
  - minimal, 7
  - principal-axis balanced, 8
  - truncated balanced, 8
- rectifiable curve, 230
- reflexive, 257
- region, 228
- relatively compact, 238
- removable singularity, 231
- Res-spectral operator
  - weakly stable, 181
- residual spectrum, 276
- residue, 232
- resolvent equation, 277
- resolvent operator, 27, 274
- resolvent set, 274
- Riemann-Lebesgue lemma, 303
- Riesz basis, 59
- Riesz representation theorem, 262
- Riesz-spectral operator, 64, 80
  - $C_0$ -semigroup, 65, 88
  - resolvent, 64, 88
  - strongly stable, 181
- Riesz-spectral system, 113
  - $\beta$ -exponentially detectable, 201, 203
  - $\beta$ -exponentially stabilizable, 201, 203
  - impulse response, 128
  - transfer function, 135, 145
- right divisor, 328
- right inverse, 267
- right multiple, 328
- right shift operator, 52
- right-coprime, 328, 332
- right-coprime factorization
  - over  $\mathcal{MR}$ , 329
- ring, 321
- Rouché's theorem, 231
  
- saturated, 324
- scalar multiplication, 235
- Schmidt decomposition, 284
- Schmidt pairs, 284
- self-adjoint, 271
  - spectrum, 280
- self-adjoint  $C_0$ -semigroup, 47
- semigroup, *see*  $C_0$ -semigroup
  - spatially invariant, 57
- semilinear, 241
- separable, 239
- series connection, 146
- set
  - bounded, 238
  - closed, 238
  - compact, 238
  - dense, 238
  - maximal, 244
  - open, 238
  - orthogonal, 243
  - relatively compact, 238
- shift
  - left, 52, 315
  - right, 52
- shift invariant, 52, 315
- shift operator, 44
  - stabilizability, 219
- shift semigroup, *see* shift operator
  - left, 16
- shift-invariant, 314
- simple, 231, 288
- singular values, 249, 284
- skew-adjoint, 49
- Sobolev spaces, 243
- solution
  - classical, 34, 91, 106
  - exponential, 130
  - mild, 35, 93, 106, 111

- weak, 46, 94
- span, 236
- spatial invariant system
  - exponential detectable, 199
  - exponential stabilizable, 199
- spatially invariant operator
  - strongly stable, 178
- spatially invariant operators, 52
- spatially invariant semigroup, 57
- spatially invariant system, 113
  - as state linear system, 114
  - stabilizing compensator, 212
  - transfer function, 134
- Spectral Mapping Theorem, 281
- spectral radius, 280
- spectrum, 276
  - continuous, 276
  - point, 276
  - residual, 276
- spectrum decomposition assumption at  $\delta$ , 194
- spectrum determined growth assumption, 172, 188, 226
- square root, 272
- stability
  - discrete-time, 186
  - not determined by  $\rho(A)$ , 169, 183
- stability margin, 163
- stabilizability
  - discrete-time, 225
- stabilizability by high gain feedback, 221
- stabilizable, *see* exponentially stabilizable
  - $\beta$ -exponentially, 189
  - exponential
    - spatial invariant system, 199
  - exponentially, 9, 189
- stable, *see* exponentially stable
  - $\beta$ -exponentially, 163
  - asymptotically, 176
  - exponentially, 6, 163
  - power, 186
  - strongly, 176
    - discrete-time, 186
  - weakly, 176
    - discrete-time, 186
- state, 113
- state linear system, 113
- state space, 113
- steam chest, 153
- strictly proper, 306, 325
- strip, *see* vertical strip
- strong convergence, 251
- strong stability, 188, 227
- strongly (Lebesgue) measurable, 288
- strongly continuous at  $t_0$ , 251
- strongly continuous group, 49
- strongly continuous semigroup, *see*  $C_0$ -semigroup
- strongly differentiable at  $t_0$ , 300
- strongly input stabilizable, 216
- strongly measurable, 288
- strongly output stabilizable, 216
- strongly stable
  - discrete-time, 186
  - spatially invariant operator, 178
- Sturm-Liouville operators, 43
- symbol, 315
- symmetric, 271
- system
  - general, 130
  - linear, 130
  - time-invariant, 130
- time-invariant
  - system, 130
- topological dual space, 254
- topologically isomorphic, 238
- tracking, 224
- transfer function, 5, 7
  - characteristic function
    - equality, 144
  - delay system, 145
  - discrete-time, 150
  - not unique on  $\rho(A)$ , 127, 144
  - Riesz-spectral system, 135, 145
  - spatially invariant system, 134
- transfer function at  $s$ , 131
- transfer function on  $\Omega$ , 131
- triangular inequality, 236
- truncated balanced realization, 8
- two-sided Laplace transform, 303
- type
  - entire function, 229
- unbounded, 259
- unbounded  $B$  operator, 115
- uniform boundedness theorem, 251
- uniform convergence, 250

- uniformly (Lebesgue) measurable, 288
- uniformly bounded in norm, 48
- uniformly continuous at  $t_0$ , 251
- uniformly differentiable at  $t_0$ , 300
- uniformly measurable, 288
- unimodular, 328
- uniqueness of the Laplace transform, 302
- unit, 321
- unitary group, 49
- unitary operator, 268
  
- variation of constants formula, 14
- vector space
  - complex, 236
  - linear, 235
  - real, 236
- vertical strip, 337
  
- wave equation, 40, 86
  - $C_0$ -semigroup, 87
  - as Riesz-spectral system, 70
  - boundary control, 122
  - strongly system stable, 216
- weak convergence, 257
- weak solution, 46, 94
- weak solution of the partial differential equation, 298
- weakly (Lebesgue) measurable, 288
- weakly differentiable at  $t_0$ , 300
- weakly measurable, 288
- weakly stable
  - discrete-time, 186
- weakly stable, 176
  
- zero, 150, 230
- zero element, 235, 321