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6
Input-Output Maps

6.1 Impulse response

In Definition 5.1.1, we denoted the state linear system (5.1), (5.2) in terms of the four operators A, B, C, D which determine particular relationships between the input, the state, and the output. The semigroup specifies the relationship between the states, the controllability map specifies the relationship between the inputs and states, and the observability map specifies the relationship between the initial states and the outputs. The following map specifies the relationships between the inputs and outputs.

Definition 6.1.1 Consider the state linear system $\Sigma(A, B, C, D)$. The *input-output map* of $\Sigma(A, B, C, D)$ on $[0, \tau]$ is the bounded linear map $\mathcal{F}^\tau : \boldsymbol{L}_2([0, \tau]; U) \to \boldsymbol{L}_2([0, \tau]; Y)$ defined by

$$(\mathcal{F}^\tau u)(t) := Du(t) + \int_0^t CT(t-s)Bu(s)ds \qquad \text{for } t \in [0, \tau]. \tag{6.1}$$

■

From equation (6.1) we see that the function $D\delta(t) + CT(t)B$ plays an essential role in the input-output map. Here $\delta(t)$ denotes the Dirac delta function at zero.

Definition 6.1.2 The *impulse response* h of $\Sigma(A, B, C, D)$ is given

$$h(t) = \begin{cases} D\delta(t) + CT(t)B & t \geq 0 \\ 0 & t < 0. \end{cases}$$

■

We remark that the impulse response and the input-output map are related by

$$(\mathcal{F}^\tau u)(t) = \int_0^t h(t-s)u(s)ds \qquad \text{for } t \in [0,\tau]. \tag{6.2}$$

We also need the input-output map on an infinite time interval. Therefore we define the *extended input-output map* $\mathcal{F}^\infty$ on $[0,\infty)$ by

$$\left(\mathcal{F}^\infty u\right)(t) = \int_0^t h(t-s)u(s)ds \text{ for } u \in \boldsymbol{L}_2^{loc}([0,\infty);U), \tag{6.3}$$

where $\boldsymbol{L}_2^{loc}([0,\infty);U)$ denotes the space of functions f on $[0,\infty)$ such that for every $\tau > 0$ the restriction of f to $[0,\tau]$ belongs to $\boldsymbol{L}_2([0,\tau];U)$.

Lemma 6.1.3 *Let $\Sigma(A,B,C,D)$ be a state linear system with impulse response $h(t)$. For zero initial condition and input $u \in \boldsymbol{L}_2^{loc}([0,\infty);U)$ the output y is given by $\mathcal{F}^\infty u$. If, in addition, there exists an $\alpha \in \mathbb{R}$ such that $\int_0^\infty e^{-\alpha t}\|u(t)\|dt < \infty$ and $\int_0^\infty e^{-\alpha t}\|CT(t)B\|dt < \infty$, then the output $y(t)$ satisfies $\int_0^\infty e^{-\alpha t}\|y(t)\|dt < \infty$.*

Proof Since the initial condition is zero, the mild solution of (5.1) is given by, see (5.3),

$$z(t) = \int_0^t T(t-s)Bu(s)ds.$$

Since C and D are bounded, we have that

$$\begin{aligned} y(t) &= Cz(t) + Du(t) \\ &= C\int_0^t T(t-s)Bu(s)ds + Du(s) \\ &= \int_0^t CT(t-s)Bu(s)ds + Du(s) = \int_0^t h(t-s)u(s)ds. \end{aligned}$$

If an input satisfies $\int_0^\infty e^{-\alpha t}\|u(t)\|dt < \infty$, then its is easy to see that it is also an element of $\boldsymbol{L}_2^{loc}([0,\infty);U)$. So we have that the output equals (6.3). Hence

$$\begin{aligned} e^{-\alpha t}y(t) &= \int_0^t e^{-\alpha(t-\tau)}h(t-\tau)e^{-\alpha\tau}u(\tau)d\tau \\ &= \int_0^t e^{-\alpha(t-\tau)}CT(t-\tau)Be^{-\alpha\tau}u(\tau)d\tau + Du(t)e^{-\alpha t}. \end{aligned}$$

Now by Lemma A.6.6 we see that $e^{-\alpha\cdot}y(\cdot) \in \boldsymbol{L}_1([0,\infty);Y)$. ■

For some systems the impulse response can be calculated directly.

Example 6.1.4 Denote by $Z = \ell_2(\mathbb{Z})$, the Hilbert space of doubly infinite sequences $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$, with the usual inner product (see Example A.2.26), and let A be the right shift operator defined by

$$(Az)_k = z_{k-1}.$$

It is easy to show that A is a bounded operator on Z. Consequently, Example 2.1.3 shows that A is the infinitesimal generator of the C_0-semigroup $T(t)$ given by

$$T(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots .$$

Define the following input and output operators

$$Bu := bu, \qquad \text{where } b_n = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Cz := \langle z, c\rangle, \qquad \text{where } c_n = \begin{cases} 1 & \text{for } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

These are bounded linear operators and so $\Sigma(A, B, C, 0)$ is a well defined state linear system on $\ell_2(\mathbb{Z})$. It is readily verified that $CA^kB = 0$ for all $k \geq 0$. Thus the impulse response

$$h(t) = CT(t)B = 0 \qquad \text{for } t \geq 0.$$

Since by definition the impulse response is zero for $t < 0$, we find that h is identically zero. ■

In the previous example we could calculate the impulse response directly. For spatially invariant systems this is also possible.

Lemma 6.1.5 *Consider the spatially invariant system of Example 5.1.2, which by applying the Fourier transform becomes the system $\Sigma(\Lambda_{\check{A}}, \Lambda_{\check{B}}, \Lambda_{\check{C}}, \Lambda_{\check{D}})$ on the state space $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$, the input space $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^m)$, and the output space $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^p)$. The impulse response of this system is given by*

$$h(t) = \Lambda_{h(t,\cdot)},$$

where $h(t,\cdot)$ is zero for $t < 0$ and for $t \geq 0$ it equals the symbol

$$h(t,\phi) = \check{C}(\phi)e^{\check{A}(\phi)t}\check{B}(\phi) + \check{D}(\phi)\delta(t), \quad \phi \in \partial\mathbb{D}.$$

Proof This follows directly from Definition 6.1.2 and the fact that for two bounded multiplication operators Λ_{Q_1} and Λ_{Q_2} we have that $\Lambda_{Q_1}\Lambda_{Q_2} = \Lambda_{Q_1Q_2}$. ■

Riesz-spectral systems with finite-rank inputs and outputs have an explicit expression for h.

Lemma 6.1.6 *Let A be a Riesz-spectral operator. Suppose that $B \in \mathcal{L}(\mathbb{C}^m, Z)$, $C \in \mathcal{L}(Z, \mathbb{C}^k)$. The impulse response of $\Sigma(A, B, C, 0)$ is given by*

$$h(t) = \begin{cases} \sum_{n=1}^{\infty} e^{\lambda_n t} C\phi_n (\overline{B^*\psi_n})^{\mathrm{T}} & t \geq 0 \\ 0 & t < 0. \end{cases} \tag{6.4}$$

Proof The above expressions follow from the representations (3.30) of $T(t)$ and the fact that B and C are bounded operators. For $t \geq 0$ the following holds:

$$\begin{aligned} h(t)u &= CT(t)Bu \\ &= C\left[\lim_{N\to\infty}\sum_{n=1}^{N} e^{\lambda_n t}\langle\cdot, \psi_n\rangle\phi_n\right]Bu \\ &= \lim_{N\to\infty}\sum_{n=1}^{N} e^{\lambda_n t}\langle Bu, \psi_n\rangle C\phi_n \\ &= \sum_{n=1}^{\infty} e^{\lambda_n t}C\phi_n(\overline{B^*\psi_n})^{\mathrm{T}}u, \end{aligned}$$

where we have used the property $\langle v, w\rangle_{\mathbb{C}^m} = \overline{w}^{\mathrm{T}}v$. ■

Using this result we calculate the impulse response for a special case of the heated rod of Example 5.1.3.

Example 6.1.7 Consider the following controlled heat equation:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + 2\cdot \mathbb{1}_{[\frac{1}{2},1]}(x)u(t),$$

$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t),\ z(x,0) = z_0(x),$$

$$y(t) = 2\int_0^{1/2} z(x,t)dx.$$

This is a special case of Example 5.1.3, with $x_0 = \frac{3}{4}$, $\varepsilon = \frac{1}{4}$, and $x_1 = \frac{1}{4}$, $\nu = \frac{1}{4}$. So it can be formulated as a state linear system $\Sigma(A, B, C, 0)$ on $Z = \boldsymbol{L}_2(0,1)$. In Example 3.2.11, we showed that A is a Riesz-spectral operator with $\lambda_n = -n^2\pi^2, n \geq 0$, $\psi_n(x) = \phi_n(x) = \sqrt{2}\cos(n\pi x), n \geq 1$ and $\psi_0(x) = \phi_0(x) \equiv 1$. So applying Lemma 6.1.6 we obtain the following impulse response:

$$h(t) = \begin{cases} \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}\cdot 2\int_0^{\frac{1}{2}}\phi_n(x)dx\cdot 2\int_{\frac{1}{2}}^1\phi_n(x)dx & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Simple calculations give

$$2\int_0^{\frac{1}{2}}\phi_0(x)dx\cdot 2\int_{\frac{1}{2}}^1\phi_0(x)dx = 1$$

and

$$2\int_0^{\frac{1}{2}}\phi_n(x)dx\cdot 2\int_{\frac{1}{2}}^1\phi_n(x)dx = 8\sin^2(n\pi/2), \quad n \geq 1.$$

Thus the impulse response is given by

$$h(t) = \begin{cases} 1 + \sum_{n=1}^{\infty} 8 \sin^2(n\pi/2) e^{-n^2\pi^2 t} & t \geq 0 \\ 0 & t < 0. \end{cases} \tag{6.5}$$

■

Similarly, it is readily verified that the impulse response of Example 5.1.3 is given by the following infinite series

$$h(t) = 1 + \sum_{n=1}^{\infty} \frac{2\cos(n\pi x_0)\sin(n\pi\varepsilon)\cos(n\pi x_1)\sin(n\pi\nu)}{\varepsilon\nu(n\pi)^2} e^{-(n\pi)^2 t}$$

on $t \geq 0$ and zero elsewhere.

6.2 Transfer functions

In this section we introduce a very general definition of a transfer function, which applies not only to systems described by a p.d.e, but also to those described by difference differential equation or integral equations, among others. First we define our general class of systems.

Definition 6.2.1 Let $\mathbb{T} := [0, \infty)$ be the time axis and let U, Y, and X be complex Hilbert spaces. U and Y are the input- and output space, respectively, whereas X contains the remaining variables. A *general system* Σ_{gen} is a subset of $\boldsymbol{L}_1^{loc}([0, \infty); U \times X \times Y)$, i.e., a subset of all locally integrable functions from the time axis $\mathbb{T}$ to $U \times X \times Y$.

Note that two functions f and g are equal in $\boldsymbol{L}_1^{loc}([0, \infty); U \times X \times Y)$ if $f(t) = g(t)$ for almost all $t \geq 0$.

In most of our examples, X will become the state space Z. However, this is not always the case. For instance, in Example 6.2.13 $X = \mathbb{C}$, see (6.22).

Next we define linearity and time-invariance for Σ_{gen}.

Definition 6.2.2 For the general system Σ_{gen} we define the following concepts;

a. Σ_{gen} is *linear* if U, X, and Y are linear spaces, and if $(\alpha u_1 + \beta u_2, \alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \in \Sigma_{\text{gen}}$ whenever (u_1, x_1, y_1) and (u_2, x_2, y_2) are in Σ_{gen} and $\alpha, \beta \in \mathbb{C}$.

b. The system Σ_{gen} is *time-invariant*, if $(u(\cdot + \tau), x(\cdot + \tau), y(\cdot + \tau))$ is in Σ_{gen} for all $\tau > 0$, whenever $(u, x, y) \in \Sigma_{\text{gen}}$.

We are now in a position to define the transfer function of a system Σ_{gen}.

Definition 6.2.3 Let Σ_{gen} be a system, s be an element of $\mathbb{C}$, and $u_0 \in U$. We say that $(u_0 e^{s\cdot}, x(\cdot), y(\cdot))$ is an *exponential solution* in Σ_{gen} if there exist $x_0 \in X$, $y_0 \in Y$, such that $(u_0 e^{st}, x_0 e^{st}, y_0 e^{st}) = (u_0 e^{st}, x(t), y(t))$ for almost all $t \geq 0$.

Let $s \in \mathbb{C}$. If for every $u_0 \in U$ there exists an exponential solution, and the corresponding output trajectory $y_0 e^{st}$, $t \in [0, \infty)$ is unique, then we call the mapping $u_0 \mapsto y_0$ the *transfer function at s*. We denote this mapping by $G(s)$.

If $G(s)$ is defined for all $s \in \Omega \subset \mathbb{C}$, then the mapping $s \in \Omega \mapsto G(s)$ is called the *transfer function* of the system Σ_{gen} *on* Ω. ■

Under some mild conditions we show that every linear and time-invariant system possesses a transfer function.

Lemma 6.2.4 *Let Σ_{gen} be a linear and time-invariant system. If Σ_{gen} has the property that $(0, 0, y) \in \Sigma_{\text{gen}}$ implies $y = 0$, then $(u_0 e^{s\cdot}, x_0 e^{s\cdot}, y(\cdot)) \in \Sigma_{\text{gen}}$ implies the existence of $y_0 \in Y$ such $y(t) = y_0 e^{st}$ for almost all $t \geq 0$.*

Moreover, if for a given $s \in \mathbb{C}$ there exists an exponential solution for all $u_0 \in U$, and if $(0, x_0 e^{s\cdot}, y(\cdot)) \in \Sigma_{\text{gen}}$ implies $x_0 = 0$, then the transfer function at s exists, and is a linear mapping.

Proof Let $(u_0 e^{s\cdot}, x_0 e^{s\cdot}, y(\cdot))$ be an element of Σ_{gen}, and let $\tau \geq 0$. Combining the linearity and time-invariance of Σ_{gen}, we see that

$$\begin{aligned}(0, 0, y(\cdot + \tau) - e^{s\tau} y(\cdot)) \quad &= \quad (u_0 e^{s(\cdot+\tau)}, x_0 e^{s(\cdot+\tau)}, y(\cdot + \tau)) - \\ &\quad\ (u_0 e^{s\tau} e^{s\cdot}, x_0 e^{s\tau} e^{s\cdot}, e^{s\tau} y(\cdot))\end{aligned}$$

is an element of Σ_{gen}. By assumption, this implies that $y(t + \tau) = e^{s\tau} y(t)$ for all $\tau > 0$ and for almost all $t \geq 0$. This implies that for all $0 \leq h_1 < h_2$ and for all $\tau \geq 0$ we have

$$\int_{\tau+h_1}^{\tau+h_2} y(\alpha)\, d\alpha = \int_{h_1}^{h_2} y(t + \tau)\, dt = \int_{h_1}^{h_2} e^{s\tau} y(t)\, dt = e^{s\tau} \int_{h_1}^{h_2} y(t)\, dt. \quad (6.6)$$

We may differentiate the left and right hand-side with respect to τ to find

$$y(\tau + h_2) - y(\tau + h_1) = s e^{s\tau} \int_{h_1}^{h_2} y(t)\, dt.$$

From this it follows that $y(t)$ is continuous for $t > 0$ and continuous from the right for $t = 0$. In particular, we find that $y(\tau) = \lim_{t\downarrow 0} y(t + \tau) = \lim_{t\downarrow 0} e^{s\tau} y(t) = e^{s\tau} \lim_{t\downarrow 0} y(t)$, which gives the desired result.

Now assume that for a given $s \in \mathbb{C}$ there exists an exponential solution for all $u_0 \in U$. First we show that the exponential solution is unique. When $(u_0 e^{s\cdot}, x_0 e^{s\cdot}, y_0 e^{s\cdot})$ and $(u_0 e^{s\cdot}, \tilde{x}_0 e^{s\cdot}, \tilde{y}_0 e^{s\cdot})$ are both in Σ_{gen}, then by the linearity $(0, (x_0 - \tilde{x}_0) e^{s\cdot}, (y_0 - \tilde{y}_0) e^{s\cdot}) \in \Sigma_{\text{gen}}$. By our assumption this implies that $x_0 = \tilde{x}_0$ and $y_0 = \tilde{y}_0$.

From this we see that we can define a mapping $u_0 \mapsto y_0$. It remains to show that this mapping is linear. Let $(u_{10} e^{s\cdot}, x_{10} e^{s\cdot}, y_{10} e^{s\cdot})$ and $(u_{20} e^{s\cdot}, x_{20} e^{s\cdot}, y_{20} e^{s\cdot})$ be two exponential solutions. By the linearity of Σ_{gen} it is easy to see that

$$((\alpha u_{10} + \beta u_{20}) e^{s\cdot}, (\alpha x_{10} + \beta x_{20}) e^{s\cdot}, (\alpha y_{10} + \beta y_{20}) e^{s\cdot}) \in \Sigma_{\text{gen}}.$$

Hence this implies that $\alpha u_{10} + \beta u_{20}$ is mapped to $\alpha y_{10} + \beta y_{20}$. In other words, the mapping is linear. ■

It turns out that for the class of systems we are considering, the conditions in the above lemma are very weak, and thus the transfer function exists and is a linear operator. We begin by showing this for the state linear system $\Sigma(A, B, C, D)$.

Since the transfer function is defined via the exponential solutions of a general system, we have to associate a general system to our state linear system, i.e., to the abstract equations

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0 \tag{6.7}$$

$$y(t) = Cz(t) + Du(t), \tag{6.8}$$

see (5.1) and (5.2).

Definition 6.2.5 For the state linear system $\Sigma(A, B, C, D)$ the general system is defined as

$$\begin{aligned}\Sigma_{\text{gen}} = \{&(u, z, y) \in \boldsymbol{L}_1^{loc}([0, \infty); U \times Z \times Y) \mid \text{there exists an } z_0 \in Z \\ &\text{such that } z \text{ is the mild solution of (6.7) and} \\ &y \text{ satisfies (6.8)}\}.\end{aligned} \tag{6.9}$$

■

So we see that for state linear systems we chose $X = Z$. Next we show that this Σ_{gen} is linear and time-invariant.

Lemma 6.2.6 *The general system associated to the state linear systems, see (6.9), is linear and time-invariant.*

Furthermore, for $s \in \rho(A)$ the only solution in Σ_{gen} of the form $(0, z_0 e^{s\cdot}, y(\cdot))$ is the zero solution, i.e., $z_0 = 0$ and $y = 0$.

Proof Since the linearity is trivial, we concentrate on the time invariance. We have to show that $(u(\cdot + \tau), z(\cdot + \tau), y(\cdot + \tau))$ is in Σ_{gen} for all $\tau > 0$, whenever $(u, z, y) \in \Sigma_{\text{gen}}$.

Since $z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds$ we have that

$$\begin{aligned} z(t+\tau) &= T(t+\tau)z_0 + \int_0^{t+\tau} T(t+\tau-s)Bu(s)ds \\ &= T(t+\tau)z_0 + \int_0^{\tau} T(t+\tau-s)Bu(s)ds + \\ &\quad \int_\tau^{t+\tau} T(t+\tau-s)Bu(s)ds \\ &= T(t)\left[T(\tau)z_0 + \int_0^{\tau} T(t-s)Bu(s)ds\right] + \\ &\quad \int_0^t T(t-\alpha)Bu(\alpha+\tau)d\alpha. \end{aligned}$$

Hence the $z(\cdot + \tau)$ is the weak solution of (6.7) for initial condition $T(\tau)z_0 + \int_0^\tau T(t-s)Bu(s)ds$ and input $u(\cdot + \tau)$. It is straightforward to see that $y(\cdot + \tau)$ satisfies (6.8) with state $z(\cdot + \tau)$ and input $u(\cdot + \tau)$. Thus (6.9) is time-invariant.

Next we prove the uniqueness property. Assume that $(0, z_0 e^{s\cdot}, y(\cdot)) \in \Sigma_{\text{gen}}$. Since it is a mild solution it satisfies

$$z_0 e^{st} = T(t) z_0, \qquad t \geq 0. \tag{6.10}$$

From this we see that z_0 is an eigenvector of $T(t)$ for all $t \geq 0$. Since the left hand-side is differentiable so is the right hand-side. Thus $z_0 \in D(A)$ and $Az_0 = sz_0$, see also Exercise 2.2. Using the assumption that $s \in \rho(A)$, we see that $z_0 = 0$. That the output is zero follows directly, since it satisfies (6.8). ■

In the following theorem we prove that the state linear system $\Sigma(A, B, C, D)$ possesses a transfer function on $\rho(A)$, and we obtain an explicit expression for the transfer function.

Theorem 6.2.7 *Consider the state linear system $\Sigma(A, B, C, D)$. The general system associated to this system is given by Definition 6.2.5.*

If $(u(t), z(t), y(t))_{t \geq 0}$ is an exponential solution of (6.7)–(6.8) for $s \in \rho(A)$, then z is a classical solution of (6.7). Furthermore, the transfer function exists on $\rho(A)$ and is given by

$$G(s) = C(sI - A)^{-1} B + D. \tag{6.11}$$

Proof The mild solution of (6.7) with initial condition $z(0) = z_0$ is uniquely determined and given by

$$z(t) = T(t) z_0 + \int_0^t T(t - \tau) B u(\tau) d\tau. \tag{6.12}$$

For an exponential solution this equation should equal

$$z_0 e^{st} = T(t) z_0 + \int_0^t T(t - \tau) B u_0 e^{s\tau} d\tau. \tag{6.13}$$

Taking $s \in \rho(A)$ and $z_0 = (sI - A)^{-1} B u_0$ the right hand-side of this equation can be written as

$$T(t) z_0 + \int_0^t T(t - \tau)(sI - A) z_0 e^{s\tau} d\tau =$$
$$T(t) z_0 + e^{st} \int_0^t T(t - \tau) e^{-s(t-\tau)} (sI - A) z_0 d\tau.$$

By Exercise 2.3 the infinitesimal generator of the C_0-semigroup $T(t)e^{-st}$ is given by $A - sI$. Applying Theorem 2.1.13.d. to the above equation we find that

$$T(t) z_0 + e^{st} \int_0^t T(t - \tau) e^{-s(t-\tau)} (sI - A) z_0 d\tau$$
$$= T(t) z_0 - e^{st} \left(T(t) e^{-st} z_0 - z_0 \right) = e^{st} z_0. \tag{6.14}$$

Thus by choosing $z_0 = (sI - A)^{-1}Bu_0$, we see that the state trajectory is given by $z_0 e^{st}$. The output equation of the system yields

$$\begin{aligned} y_0 e^{st} &= y(t) = Cz(t) + Du(t) = Cz_0 e^{st} + Du_0 e^{st} \\ &= C(sI - A)^{-1}Bu_0 e^{st} + Du_0 e^{st}. \end{aligned} \tag{6.15}$$

Thus for every $s \in \rho(A)$ there exists an exponential solution. By the output equation (6.8) and Lemma 6.2.6, we see that the conditions in Lemma 6.2.4 are satisfied and thus there exists a transfer function on $\rho(A)$. That the transfer function is given by (6.11) follows directly from (6.15). ■

We calculate the transfer function for several systems, starting with the system from Example 6.1.4.

Example 6.2.8 Denote by $Z = \ell_2(\mathbb{Z})$, the Hilbert space of doubly infinite sequences $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$, with the usual inner product (see Example A.2.26). Let $\Sigma(A, B, C, 0)$ be the state linear system form Example 6.1.4. Thus A is the right shift operator defined by

$$(Az)_k = z_{k-1}.$$

$$Bu := bu, \qquad \text{where } b_n = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Cz := \langle z, c \rangle, \qquad \text{where } c_n = \begin{cases} 1 & \text{for } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 6.2.7 we know that the transfer function is given by $G(s) = C(sI - A)^{-1}B$, $s \in \rho(A)$. The spectrum of A equals the unit circle. For s with $|s| > 1$, it is not hard to show that

$$\left((sI - A)^{-1}B\right)_n = \begin{cases} -(s^{-n-1}) & n = 0, 1, 2, \ldots \\ 0 & n \leq -1 \end{cases}$$

Whereas for $|s| < 1$, we find that

$$\left((sI - A)^{-1}B\right)_n = \begin{cases} 0 & n = 0, 1, \ldots \\ (s)^{-n-1} & n \leq -1 \end{cases}$$

Combining this with the definition of C we find that

$$G(s) = \begin{cases} 0 & |s| > 1 \\ 1 & |s| < 1 \end{cases}$$

■

Another application of Theorem 6.2.7 yields explicit formulas for the transfer functions of the spatially invariant systems.

Lemma 6.2.9 *Consider the spatially invariant system of Example 5.1.2, which by applying the Fourier transform becomes the system* $\Sigma(\Lambda_{\check{A}}, \Lambda_{\check{B}}, \Lambda_{\check{C}}, \Lambda_{\check{D}})$ *on* *the state space* $\boldsymbol{L}_2(\partial\mathbb{D};\mathbb{C}^n)$*, the input space* $\boldsymbol{L}_2(\partial\mathbb{D};\mathbb{C}^m)$*, and the output space* $\boldsymbol{L}_2(\partial\mathbb{D};\mathbb{C}^p)$*. The transfer function of* $\Sigma(\Lambda_{\check{A}},\Lambda_{\check{B}},\Lambda_{\check{C}},\Lambda_{\check{D}})$ *on* $\rho(\check{A})$ *is given by* $G(s)=\Lambda_{\check{G}(s)}$ *with*

$$\left(\check{G}(s)\right)(\phi)=\check{C}(\phi)(sI-\check{A}(\phi))^{-1}\check{B}(\phi)+\check{D}(\phi),\quad s\in\rho(\check{A}),\phi\in\partial\mathbb{D}.$$

Proof This follows directly from Theorem 6.2.7 and the fact that for two bounded multiplication operators Λ_{Q_1} and Λ_{Q_2} we have that $\Lambda_{Q_1}\Lambda_{Q_2}=\Lambda_{Q_1Q_2}$. ∎

Next we calculate the transfer function for the Riesz spectral systems.

Lemma 6.2.10 *Let* $\Sigma(A,B,C,0)$ *be a Riesz-spectral system. The transfer function on* $\rho(A)$ *is given by*

$$G(s)=\sum_{n=1}^{\infty}\frac{1}{s-\lambda_n}C\phi_n(\overline{B^*\psi_n})^{\mathrm{T}}\qquad \text{for } s\in\rho(A). \tag{6.16}$$

Proof The proof is very similar to the proof of Lemma 6.1.6, but now we use the representation (3.28) of $(sI-A)^{-1}$.

For $s\in\rho(A)$ the following holds:

$$\begin{aligned}
G(s)u &= C(sI-A)^{-1}Bu\\
&= C\left[\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{s-\lambda_n}\langle\cdot,\psi_n\rangle\phi_n\right]Bu\\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{s-\lambda_n}\langle Bu,\psi_n\rangle C\phi_n\\
&= \sum_{n=1}^{\infty}\frac{1}{s-\lambda_n}C\phi_n(\overline{B^*\psi_n})^{\mathrm{T}}u,
\end{aligned}$$

where we have used the property $\langle v,w\rangle_{\mathbb{C}^m}=\overline{w}^{\mathrm{T}}v$. ∎

Often it is possible to obtain a closed expression for the transfer function by using Theorem 6.2.7 in the original differential equation as in the following examples.

Example 6.2.11 Consider the following controlled heat equation:

$$\frac{\partial z}{\partial t}(x,t)=\frac{\partial^2 z}{\partial x^2}(x,t)+2\cdot\mathbb{1}_{[\frac{1}{2},1]}(x)u(t),$$

$$\frac{\partial z}{\partial x}(0,t)=0=\frac{\partial z}{\partial x}(1,t),\ z(x,0)=z_0(x),$$

$$y(t)=2\int_0^{1/2}z(x,t)dx.$$

This is a special case of Example 5.1.3, with $x_0 = \frac{3}{4}$, $\varepsilon = \frac{1}{4}$, and $x_1 = \frac{1}{4}$, $\nu = \frac{1}{4}$. So it can be formulated as a state linear system $\Sigma(A, B, C)$ on $Z = \boldsymbol{L}_2(0, 1)$. In Example 3.2.11, we showed that A is a Riesz-spectral operator, and so applying Lemma 6.2.10 we obtain the following transfer function on $\rho(A) = \{s \in \mathbb{C} \mid s \neq -n^2\pi^2, n = 0, 1, 2, \cdots\}$.

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} \frac{2\int_0^{\frac{1}{2}} \phi_n(x)dx \cdot 2\int_{\frac{1}{2}}^{1} \phi_n(x)dx}{s + n^2\pi^2} \\ &= \frac{1}{s} - \sum_{n=1}^{\infty} \frac{8\sin^2(n\pi/2)}{(s + (n\pi)^2)(n\pi)^2} \\ &= \frac{1}{s} - \sum_{r=0}^{\infty} \frac{8}{(s + (2r+1)^2\pi^2)(2r+1)^2\pi^2}, \end{aligned}$$

where we used the calculation done in Example 6.1.7.

It is also possible to obtain a closed-form expression for $G(s)$ by using the exponential solutions, which by Theorem 6.2.7 are classical solutions of the p.d.e.

Hence, given $u(t) = e^{st}u_0$ we are looking for a classical solution of the above p.d.e. of the form $z(t) = e^{st}z_0$, with $z_0 \in D(A)$. Thus $z(t, x) = e^{st}z_0(x)$ and z_0 is twice differentiable with respect to x and it satisfies the boundary condition. Substituting this in the p.d.e. leads to the following ordinary differential equation, where we regard s as a parameter:

$$\begin{aligned} &sz_0(x) = \frac{d^2z_0}{dx^2}(x) + 2 \cdot \mathbb{1}_{[\frac{1}{2},1]}(x)u_0, \\ &\frac{dz_0}{dx}(0) = 0 = \frac{dz_0}{dx}(1). \end{aligned} \tag{6.17}$$

This can be rewritten as the first-order system

$$\frac{d}{dx}\begin{pmatrix} z_0 \\ \frac{dz_0}{dx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}\begin{pmatrix} z_0 \\ \frac{dz_0}{dx} \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\mathbb{1}_{[\frac{1}{2},1]}u_0,$$

which has the usual solution for $s \neq 0$

$$\begin{aligned} \begin{pmatrix} z_0(x) \\ \frac{dz_0}{dx}(x) \end{pmatrix} &= \begin{pmatrix} \cosh(\sqrt{s}x) & \frac{1}{\sqrt{s}}\sinh(\sqrt{s}x) \\ \sqrt{s}\sinh(\sqrt{s}x) & \cosh(\sqrt{s}x) \end{pmatrix}\begin{pmatrix} z_0(0) \\ 0 \end{pmatrix} \\ &\quad -2\int_0^x \begin{pmatrix} \frac{1}{\sqrt{s}}\sinh(\sqrt{s}(x-\xi)) \\ \cosh(\sqrt{s}(x-\xi)) \end{pmatrix}\mathbb{1}_{[\frac{1}{2},1]}(\xi)u_0 d\xi. \end{aligned}$$

In addition, we have

$$\begin{aligned} 0 &= \frac{dz_0}{dx}(1) = \sqrt{s}\sinh(\sqrt{s})z_0(0) - 2\int_{1/2}^{1}\cosh(\sqrt{s}(1-\xi))u_0 d\xi \\ &= \sqrt{s}\sinh(\sqrt{s})z_0(0) + 2\left[\frac{1}{\sqrt{s}}\sinh(\sqrt{s}(1-\xi))\right]_{1/2}^{1} u_0 \\ &= \sqrt{s}\sinh(\sqrt{s})z_0(0) - \frac{2}{\sqrt{s}}\sinh(\frac{\sqrt{s}}{2})u_0. \end{aligned}$$

Thus for all s such that $\sqrt{s}\sinh(\sqrt{s}) \neq 0$ we obtain

$$z_0(0) = \frac{u_0}{s\cosh(\frac{\sqrt{s}}{2})},$$

and we have

$$z_0(x) = \frac{u_0\cosh(\sqrt{s}x)}{s\cosh(\frac{\sqrt{s}}{2})} - 2\int_0^x \frac{\sinh(\sqrt{s}(x-\xi))\mathbb{1}_{[\frac{1}{2},1]}(\xi)}{\sqrt{s}} d\xi u_0. \tag{6.18}$$

Changing the order of integration is justified, and we obtain

$$\begin{aligned} y_0 &= 2\int_0^{1/2} z_0(x)dx = 2\int_0^{1/2}\cosh(\sqrt{s}x)dx\frac{u_0}{s\cosh(\frac{\sqrt{s}}{2})} + 0 \\ &= \frac{2\sinh(\frac{\sqrt{s}}{2})}{s\sqrt{s}\cosh(\frac{\sqrt{s}}{2})}u_0. \end{aligned} \tag{6.19}$$

Thus we may conclude that the transfer function on $\{s \in \mathbb{C} \mid s\sinh(\sqrt{s}) \neq 0\} = \{s \in \mathbb{C} \mid s \neq -r^2\pi^2, r = 0, 1, 2, \cdots\} = \rho(A)$ is given by

$$G(s) = \frac{2\tanh(\frac{\sqrt{s}}{2})}{s\sqrt{s}}.$$

Since for a given domain in $\mathbb{C}$ the transfer function is unique, we have as a corollary that

$$\frac{2\tanh(\frac{\sqrt{s}}{2})}{s\sqrt{s}} = \frac{1}{s} - \sum_{r=0}^{\infty}\frac{8}{(s+(2r+1)^2\pi^2)(2r+1)^2\pi^2}, \qquad s \in \rho(A).$$

■

To show the generality of defining transfer functions via exponential solutions, we consider two simple examples, a partial differential equation and a delay equation, neither of which are state linear systems.

Example 6.2.12 Consider Example 6.2.11 again, but with a different input and a point observation:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad \frac{\partial z}{\partial x}(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = u(t),$$

$$y(t) = z(x_1, t), \qquad 0 \le x_1 \le 1.$$

As in Example 6.2.11 we want to obtain an expression for the transfer function.

The triple $(u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$ is an exponential solution for our system if z_0 satisfies

$$sz_0(x) = \frac{d^2 z_0}{dx^2}(x), \ \frac{dz_0}{dx}(0) = 0, \ \frac{dz_0}{dx}(1) = u_0.$$

For all $s \in \mathbb{C}$ such that $\sqrt{s}\sinh(\sqrt{s}) \neq 0$, this has the solution

$$z_0(x) = \frac{\cosh(\sqrt{s}x)}{\sqrt{s}\sinh(\sqrt{s})} u_0.$$

Hence for all $s \in \Omega := \{s \in \mathbb{C} \mid s \neq -k^2\pi^2, k = 0, 1, 2, \cdots\}$ the transfer function is given by

$$G(s) = \frac{\cosh(\sqrt{s}x_1)}{\sqrt{s}\sinh(\sqrt{s})}.$$

■

Example 6.2.13 Consider the following retarded equation

$$\dot{x}(t) \quad = \quad a_0 x(t) + a_1 x(t-1) + u(t), \quad t \ge 0, \tag{6.20}$$

with the delayed output

$$y(t) = c_0 x(t) + c_1 x(t - \tfrac{1}{2}), \tag{6.21}$$

Any exponential solution to (6.20) and (6.21) satisfies

$$sx_0 \quad = \quad a_0 x_0 + a_1 x_0 e^{-s} + u_0, \tag{6.22}$$

$$y_0 \quad = \quad c_0 x_0 + c_1 e^{-\frac{s}{2}} x_0. \tag{6.23}$$

Thus

$$y_0 = \frac{c_0 + c_1 e^{-\frac{s}{2}}}{s - a_0 - a_1 e^{-s}} u_0.$$

This holds for all $s \in \{s \in \mathbb{C} \mid s - a_0 - a_1 e^{-s} \neq 0\}$. Thus by Definition 6.2.3 for all $s \in \{s \in \mathbb{C} \mid s - a_0 - a_1 e^{-s} \neq 0\} = \rho(A)$ the transfer function is given by

$$G(s) = \frac{c_0 + c_1 e^{-\frac{s}{2}}}{s - a_0 - a_1 e^{-s}}.$$

■

6.3 Transfer functions for boundary control systems

In the previous section we have seen how to obtain the transfer function of boundary control systems (see Example 6.3.2). An alternative approach relies on the following lemma.

Lemma 6.3.1 *Consider the boundary control system of Defnition 5.2.2*

$$\dot{z}(t) = \mathfrak{A}z(t), \quad z(0) = z_0, \quad \mathfrak{P}z(t) = u(t)$$

with the onbservation

$$y(t) = \mathfrak{Q}z(t), \quad \boldsymbol{D}(\mathfrak{Q}) \subset Z \to U, \quad \boldsymbol{D}(\mathfrak{A}) \subset \boldsymbol{D}(\mathfrak{Q}).$$

Let A be the operator defined as $Az = \mathfrak{A}z$ for $z \in D(A) = D(\mathfrak{A}) \cap \ker \mathfrak{P}$.

a. For every $s \in \rho(A)$ and for every input function of the form $u(t) = e^{st}u_0$, $t \geq 0$, there exists a unique mild solution of the form $z(t) = e^{st}z_0$, where

$$z_0 = (sI - A)^{-1}\left[-Bu_0 s + \mathfrak{A}Bu_0\right] + Bu_0. \tag{6.24}$$

b. z_0 is characterized as the solution of $(sI - \mathfrak{A})z_0 = 0$ where $\mathfrak{P}z_0 = u_0$.

c. The trajectories $z(t), u(t)$ form a classical solution of the boundary control system.

d. The transfer function on $\rho(A)$ is given by

$$z_0 = \mathfrak{Q}(sI - A)^{-1}\left[-Bu_0 s + \mathfrak{A}Bu_0\right] + \mathfrak{Q}Bu_0. \tag{6.25}$$

Proof *a*. Since $u(t) = u_0 e^{st}$ is differentiable, we know that the mild solution is given by, see (5.12)

$$\begin{aligned} z(t) \quad = \quad & Bu(t) - T(t)Bu(0) + T(t)z_0 - \\ & \int_0^t T(t-\tau)B\dot{u}(\tau)d\tau + \int_0^t T(t-\tau)\mathfrak{A}Bu(\tau)d\tau. \end{aligned}$$

Substituting the expression for $u(t)$ gives

$$z(t) = Bu_0 e^{st} - T(t)Bu_0 + T(t)z_0 + \int_0^t T(t-\tau)\left[-Bu_0 s + \mathfrak{A}Bu_0\right] e^{ss\tau} d\tau.$$

In the integral we substitute $\tilde{\tau}$ for $t - \tau$, and we write $z_0 e^{st}$ for $z(t)$, this gives

$$\left[z_0 - Bu_0\right] e^{st} - T(t)\left[z_0 - Bu_0\right] = e^{st} \int_0^t T(\tilde{\tau})\left[-Bu_0 s + \mathfrak{A}Bu_0\right] e^{-s\tilde{\tau}} d\tilde{\tau}.$$

Since the left-hand side is differentiable at zero, so must the right-hand side. This gives that $z_0 - Bu_0$ is in the domain of A. Furthermore, the derivative at zero equals

$$s\left[z_0 - Bu_0\right] - A\left[z_0 - Bu_0\right] = \left[-Bu_0 s + \mathfrak{A}Bu_0\right].$$

Hence if $s \in \rho(A)$, then we see that z_0 is unique and is given by

$$z_0 = (sI - A)^{-1}\left[-Bu_0 s + \mathfrak{A}Bu_0\right] + Bu_0,$$

which proves (6.24).

b. Since the the original problem the operators are $\mathfrak{A}$ and $\mathfrak{P}$, we want to characterize this initial condition using these operators. Using the fact that $z_0 - Bu_0 \in \boldsymbol{D}(A) \subset \ker \mathfrak{P}$, we find that

$$\mathfrak{P}z_0 = u_0.$$

Furthermore,

$$(sI - \mathfrak{A})z_0 = [-Bu_0 s + \mathfrak{A}Bu_0] + (sI - \mathfrak{A})Bu_0 = 0.$$

It remains to show that the above equation uniquely determines z_0. Assume the contrary, and let z_1 and z_2 be two solutions in the domain of $\mathfrak{A}$, then $\mathfrak{P}(z_1 - z_2) = 0$. Hence $z_1 - z_2 \in \boldsymbol{D}(A)$. Furthermore,

$$(sI - A)(z_1 - z_2) = (sI - \mathfrak{A})(z_1 - z_2) = 0.$$

Since $s \in \rho(A)$, we conclude that $z_1 - z_2 = 0$.

c. That $z(t)$ is a classical solution follows from Theorems 5.2.3 and 5.2.4.

d. From Definition 6.2.3 the map $u_0 \to y_0$ determines the transfer function when $y(t) = y_0 e^{st}$ and $z(t) = z_0 e^{st}$. So using part *a*. we obtain (6.25).

■

We apply this result to the heat equation from Example 6.2.12 with a point observation.

Example 6.3.2 Consider

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad \frac{\partial z}{\partial x}(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = u(t), \quad z(x,0) = 0,$$

$$y(t) = z(x_1, t), \qquad 0 \le x_1 \le 1.$$

In Example 5.2.5 we determined A, B and found that $\mathfrak{P}B = I = \mathfrak{A}B$. From (6.24) for $u(t) = e^{st}u_0$ and $z(t) = e^{st}z_0$, with $s \in \rho(A)$ we have

$$\begin{aligned} z_0 &= (sI - A)^{-1}[-Bu_0 s + \mathfrak{A}Bu_0] + Bu_0 \\ &= (sI - A)^{-1}[-Bu_0 s + \mathfrak{A}Bu_0] + (sI - A)(sI - A)^{-1}Bu_0 \\ &= -A(sI - A)^{-1}Bu_0 + (\lambda I - A)^{-1}u_0 \\ &= \left[\sum_{n=1}^{\infty} \frac{2(-1)^n \cos(n\pi\cdot)}{s + (n\pi)^2} + \frac{1}{s}\right] u_0, \end{aligned} \tag{6.26}$$

where we have used Theorem 3.2.8 a and Example 3.2.11 to calculate the resolvent $(sI - A)^{-1}$.

From Lemma 6.3.1 b. we know that $z_0 \in D(\mathfrak{A})$. In particular this implies that z_0 can be evaluated in any point $x_1 \in [0,1]$. Equivalently, we see that that the infinite sum in (6.26) is absolutely convergent for $s \in \rho(A)$ and $x_1 \in [0,1]$, since

$$\begin{aligned} &\left|\sum_{n=1}^{\infty} \frac{2(-1)^n \cos(n\pi x_1)}{s + (n\pi)^2} + \frac{1}{s}\right| \\ &\quad \le \sum_{n=0}^{\infty} \left|\frac{2}{s + (n\pi)^2}\right| \le \sum_{n=0}^{\infty} \frac{2}{|\mathrm{Re}(s) + (n\pi)^2|} < \infty. \end{aligned}$$

Thus applying Lemma 6.3.1 d. we obtain the transfer function

$$\mathfrak{G}(s) = \sum_{n=1}^{\infty} \frac{2(-1)^n \cos(n\pi x_1)}{s + (n\pi)^2} + \frac{1}{s}, \quad s \neq -(n\pi)^2, n \in \mathbb{Z}.$$

In Example 6.2.12 we obtained the following closed-form expression for the transfer function

$$G(s) = \frac{\cosh(\sqrt{s}x_1)}{\sqrt{s}\sinh(\sqrt{s})}, \quad \sinh s \neq 0.$$

Since the transfer function is unique, we obtain the equality:

$$G(s) = \frac{\cosh(\sqrt{s}x_1)}{\sqrt{s}\sinh(\sqrt{s})} = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2(-1)^n \cos n\pi x_1}{s + (n\pi)^2}. \tag{6.27}$$

■

6.4 Transfer functions and the Laplace transform of the impulse response

From finite-dimensional systems theory it is well-known that the Laplace transform of the impulse response equals the transfer function and it is given by

$$\hat{h}(s) := \int_0^\infty e^{-st} CT(t)Bdt. \tag{6.28}$$

For infinite-dimensional systems with finite-dimensional inputs this is well-defined as a Lebesgue integral, but if U is infinite-dimensional, (6.28) is only defined as a Pettis integral (see Definition A.5.16). This is because $T(t)$ is only strongly measurable (see Example A.5.18). However, we can define $\hat{h}$ as a function with values in $\mathcal{L}(U, Y)$ as follows.

Definition 6.4.1 Consider the state linear system $\Sigma(A, B, C, D)$ with impulse response $h(t)$. For $s \in \mathbb{C}$ such that for every $u \in U$ the Laplace transform of the function $h(t)u$ exists at s we define $\hat{h}(s)u$ by

$$\hat{h}(s)u = \int_0^\infty e^{-st} CT(t)Budt + Du. \tag{6.29}$$

■

In fact, $\widehat{hu}(s)$, rather than $\hat{h}(s)u$, is the correct notation for the Laplace transform of $h(t)u$. However, we find it more convenient to use the notation $\hat{h}(s)u$, and with this understanding we call $\hat{h}(s)$ the *Laplace transform of the impulse response*. In the following theorem we show that $\hat{h}(s)$ defines a bounded linear operator from U to Y for s in the right half-plane bounded by the growth bound of the semigroup. Moreover, $\hat{h}(s) = G(s)$.

Theorem 6.4.2 *Let $\Sigma(A, B, C, D)$ be a state linear system with impulse response $h(t)$ and transfer function $G(s)$. Let ω_0 be the growth bound of the semigroup $T(t)$, then on $\{s \in \mathbb{C} \mid Re(s) > \omega_0\}$ the Laplace transform of the impulse response exists, $\hat{h}(s) \in \mathcal{L}(U, Y)$ and there holds*

$$\hat{h}(s) = G(s). \tag{6.30}$$

Proof First we show that $\hat{h}(s) \in \mathcal{L}(U, Y)$. The linearity follows directly from (6.29). For $\omega > \omega_0$ we can find an M such that $\|T(t)\| \leq Me^{\omega t}$, and thus $\|CT(t)Bu\| \leq \|C\|\|B\|\|u\|Me^{\omega t}$. Using this inequality, we find for $s \in \mathbb{C}$ with $\text{Re}(s) > \omega$ the Laplace transform of $h(t)u$ exists (see Definition A.6.1), and

$$\begin{aligned} \|\hat{h}(s)u\| &\leq \int_0^\infty \|e^{-st}CT(t)Budt\|dt + \|Du\| \\ &\leq \int_0^\infty e^{-Re(s)t}e^{\omega t}\|C\|M\|B\|\|u\|dt + \|D\|\|u\| \\ &= \left(\frac{\|C\|M\|B\|}{Re(s) - \omega} + \|D\|\right)\|u\|. \end{aligned}$$

From this we see that $\hat{h}(s) \in \mathcal{L}(U, Y)$.

It remains to show (6.30). By Lemma 2.1.14.a we have for $\text{Re}(s) > \omega_0$, the following:

$$\begin{aligned} C(sI - A)^{-1}Bu &= C\int_0^\infty e^{-st}T(t)Budt \\ &= \int_0^\infty e^{-st}CT(t)Budt \qquad \text{since } C \text{ is bounded.} \end{aligned}$$

Thus for all $u \in U$ and $\text{Re}(s) > \omega_0$

$$(D + C(sI - A)^{-1}B)u = \int_0^\infty e^{-st}h(t)udt = \hat{h}(s)u,$$

where we used (6.29). Since this holds for all $u \in U$, we have proved the assertion. ∎

For finite-dimensional systems it is well-known that the Laplace transform of the input and the output are related via the transfer function. For infinite-dimensional systems something similar holds, but the relation goes via the Laplace transform of the impulse response.

Theorem 6.4.3 *Let $\Sigma(A, B, C, D)$ be a state linear system with impulse response $h(t)$. If there exists an $\alpha \in \mathbb{R}$ such that $\int_0^\infty e^{-\alpha t}\|u(t)\|dt < \infty$ and $\int_0^\infty e^{-\alpha t}\|CT(t)B\|dt < \infty$, then the output $y(t)$ corresponding to the initial condition zero satisfies $\int_0^\infty e^{-\alpha t}\|y(t)\|dt < \infty$, and*

$$\hat{y}(s) = \hat{h}(s)\hat{u}(s) \qquad \textit{for } s \in \mathbb{C} \textit{ with } Re(s) > \alpha, \tag{6.31}$$

where $\hat{h}$ denotes the Laplace transform of the impulse response h.

Proof The first part of the theorem follows from Lemma 6.1.3. Furthermore, there holds

$$\begin{aligned} e^{-\alpha t}y(t) &= \int_0^t e^{-\alpha(t-\tau)}h(t-\tau)e^{-\alpha\tau}u(\tau)d\tau \\ &= \int_0^t e^{-\alpha(t-\tau)}CT(t-\tau)Be^{-\alpha\tau}u(\tau)d\tau + Du(t)e^{-\alpha t}. \qquad (6.32) \end{aligned}$$

Thus by Definition A.6.1 y and u are Laplace transformable in $\mathrm{Re}(s) > \alpha$ and by (6.32)

$$\begin{aligned} \hat{y}(s) &= D\hat{u}(s) + \int_0^\infty e^{-st}\int_0^t CT(t-\tau)Bu(\tau)d\tau dt \\ &= D\hat{u}(s) + \int_0^\infty \int_\tau^\infty e^{-s(t-\tau)}CT(t-\tau)Be^{-s\tau}u(\tau)dtd\tau, \end{aligned}$$

where interchanging of the order of integration is valid by Theorem A.5.27.

Hence for $\mathrm{Re}(s) > \alpha$ and $y_0 \in Y$ we have

$$\begin{aligned} &\langle y_0, \int_0^\infty \int_0^\infty e^{-s\sigma}CT(\sigma)Bd\sigma e^{-s\tau}u(\tau)d\tau\rangle \\ &\quad= \int_0^\infty \langle \int_0^\infty B^*T(t)^*C^*e^{-\bar{s}\sigma}y_0 d\sigma, e^{-s\tau}u(\tau)\rangle d\tau \\ &\quad= \langle \int_0^\infty B^*T(t)^*C^*e^{-\bar{s}\sigma}y_0 d\sigma, \hat{u}(s)\rangle \\ &\quad= \langle \hat{h}_0(s)^* y_0, \hat{u}(s)\rangle = \langle y_0, \hat{h}_0(s)\hat{u}(s)\rangle, \end{aligned}$$

where $\hat{h}_0(s) = \hat{h}(s) - D$. Thus

$$\hat{y}(s) = \hat{h}(s)\hat{u}(s).$$

■

Theorem 6.4.2 gives an expression for the transfer function for s in the right half-plane bounded to the left by the growth bound of the semigroup. The region of convergence of h could be larger than this right-half plane, and one would expect that the equality (6.30) would hold on the region of convergence of h. However, a combination of Examples 6.1.4 and 6.2.8 shows that this is not the case.

Example 6.4.4 Let $\Sigma(A, B, C)$ be the state linear system from Example 6.1.4 and 6.2.8. Thus A is the right shift operator on $Z = \ell_2(\mathbb{Z})$ defined by

$$(Az)_k = z_{k-1}.$$

$$Bu := bu, \qquad \text{where } b_n = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Cz := \langle z, c\rangle, \qquad \text{where } c_n = \begin{cases} 1 & \text{for } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

From Example 6.1.4 we know that $h(t) = 0$, $t \geq 0$, whereas from Example 6.2.8 the transfer function is given by

$$G(s) = \begin{cases} 0 & |s| > 1 \\ 1 & |s| < 1 \end{cases}$$

Hence we see that $G(s)$ equals $\hat{h}(s)$ on the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$, but not on a larger half-plane. Since the spectrum of A equals to unit disc and its norm is one, the growth bound of e^{At} is one. ■

This example shows that the result of Theorem 6.4.2 is sharp. However, in many examples the equality (6.30) does hold on a larger set. For instance, it is easy to see that the transfer function of Example 6.2.11 is the meromorphic extension of the Laplace transfer of its impulse response, see Example 6.1.7. This is a general result.

Theorem 6.4.5 *Consider the state linear system $\Sigma(A, B, C, D)$ with transfer function $G(s)$. If the spectrum of A consists out of isolated points and the transfer function is meromorphic on $\mathbb{C}$, then the transfer function and the Laplace transform of the impulse response are equal on the intersection of their domains. Furthermore, this Laplace transform has a unique meromorphic extension to $\mathbb{C}$, which equals the transfer function.*

Proof Since the transfer function is defined on $\rho(A)$, and since this function is meromorphic, we have that the spectrum of A consists of isolated points.

By Theorem 6.4.2 we know that the transfer function and the Laplace transform of the impulse response, $\hat{h}$, are equal on $\mathbb{C}^+_{\omega_0} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \omega_0\}$, where ω_0 is the growth bound of the semigroup generated by A. Both functions are holomorphic on their domains, and thus they are both holomorphic on the intersection of their domains. By the assumption on the spectrum of A the intersection of both domains is again a domain. Thus the transfer function can be seen as a holomorphic continuation of $\hat{h}$ to the intersection of both domains. Since the holomorphic continuation is unique, we have that the transfer function must be equal to $\hat{h}$ on the intersection their domains.

By assumption the transfer function G is meromorphic on $\mathbb{C}$. Let $u \in U$ and $y \in Y$ and assume that there exists a meromorphic function f on $\mathbb{C}$ such that $f(s) = \langle y, G(s)u\rangle$ for all $s \in \mathbb{C}^+_{\omega_0}$. By Theorem A.1.16 there exist entire functions $n, d, n_{u,y}$, and $d_{u,y}$ such that

$$f(s) = \frac{n(s)}{d(s)} \qquad \text{and} \qquad \langle y, G(s)u\rangle = \frac{n_{u,y}(s)}{d_{u,y}(s)}, \quad s \in \mathbb{C}.$$

Since f equals the transfer function on $\mathbb{C}^+_{\omega_0}$, we have that

$$n(s)d_{u,y}(s) = n_{u,y}(s)d(s) \qquad \text{on } \mathbb{C}^+_{\omega_0} \tag{6.33}$$

Since the functions on both sides of the equality sign are entire functions, we can see the left-hand side as a holomorphic continuation of the right-hand side to $\mathbb{C}$. Since this continuation is unique, we have that (6.33) holds on $\mathbb{C}$. Thus

$f(s) = \langle y, G(s)u \rangle$. In other words, for this u and y the meromorphic continuation is unique. Since u and y are arbitrary, we conclude that $\hat{h}(s)$ has a unique meromorphic continuation to $\mathbb{C}$, which equals G. ■

Using this result we can show that for many Riesz spectral and delay systems the equality (6.30) holds everywhere.

Corollary 6.4.6 *Let $\Sigma(A, B, C, D)$ be a Riesz spectral system with $B \in \mathcal{L}(\mathbb{C}^m, Z)$, and $C \in \mathcal{L}(Z, \mathbb{C}^k)$. Assume further that the eigenvalues are* isolated, *i.e., for every $\lambda_n \in \sigma(A)$ there exists a contour Γ_n such that λ_n is the only element of the spectrum lying inside Γ_n. Then the Laplace transform of the impulse response (6.4) has a unique meromorphic continuation to $\mathbb{C}$, which equals the transfer function G as given in equation (6.16).*

Proof From the expression of G, see (6.16), and the assumption on the spectrum it follows that G is meromorphic on $\mathbb{C}$. Now the assertion follows from Theorem 6.4.5. ■

6.5 Exercises

6.1. For the state linear system $\Sigma(A, B, C, D)$, define the following operator for $t \in [t_0, t_1]$:

$$(\mathcal{B}_0 u)(t) := \int_{t_0}^{t} T(t-\sigma)Bu(\sigma)d\sigma,$$

and

$$(\mathcal{F}_0 u)(t) := \int_{t_0}^{t} CT(t-\sigma)Bu(\sigma)d\sigma + Du(t).$$

a. Prove that $\mathcal{B}_0$ is bounded from $\boldsymbol{L}_2([t_0, t_1]; U)$ to $\boldsymbol{L}_2([t_0, t_1]; Z)$;
b. Prove that $\mathcal{F}_0$ is bounded from $\boldsymbol{L}_2([t_0, t_1]; U)$ to $\boldsymbol{L}_2([t_0, t_1]; Y)$.

6.2. In Theorem 6.2.7 we showed that for $s \in \rho(A)$ any exponential solution is also a classical solution. By using equation (6.13) show that the assumption $s \in \rho(A)$ is not needed.

6.3. In this exercise we prove some well-known property of transfer functions. Let $\Sigma_{\text{gen},1}$ and $\Sigma_{\text{gen},2}$ be two systems, i.e, $\Sigma_{\text{gen},1} \subset \boldsymbol{L}_1^{loc}([0,\infty); U_1 \times X_1 \times Y_1)$ and $\Sigma_{\text{gen},2} \subset \boldsymbol{L}_1^{loc}([0,\infty); U_2 \times X_2 \times Y_2)$. Assume that for a given $s \in \mathbb{C}$ both systems have a transfer function. Furthermore, we assume that for both systems the output is determined by the input and the state, that is, if $(u_0 e^{st}, r_0 e^{st}, y(t))_{t \geq 0} \in \Sigma_{\text{gen}}$, then $y(t) = y_0 e^{st}$ for some y_0 and for almost every $t \geq 0$.

a. Assume that $Y_1 = U_2$. The *series connection* $\Sigma_{\text{gen,series}} \subset \boldsymbol{L}_1^{loc}([0,\infty); U_1 \times (X_1 \times X_2) \times Y_2)$ of $\Sigma_{\text{gen},1}$ and $\Sigma_{\text{gen},2}$ is defined as follows, see

Figure 6.1,

$$(u_1, (x_1, x_2), y_2) \in \Sigma_{\text{gen,series}} \text{ if there exists a } y_1 \text{ such that } (u_1, x_1, y_1) \in \Sigma_{\text{gen},1} \text{ and } (y_1, x_2, y_2) \in \Sigma_{\text{gen},2}.$$

Show that the series connection $\Sigma_{\text{gen,series}}$ has the transfer function $G(s) = G_2(s)G_1(s)$ at s.

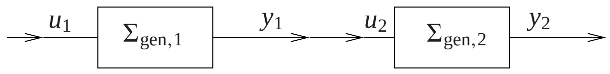

Figure 6.1. Series connection

b. Assume that $U_1 = U_2$ and $Y_1 = Y_2$. The *parallel connection* $\Sigma_{\text{gen,parallel}} \subset \boldsymbol{L}_1^{loc}([0,\infty); U_1 \times (X_1 \times X_2) \times Y_2)$ of $\Sigma_{\text{gen},1}$ and $\Sigma_{\text{gen},2}$ is defined as follows

$$(u_1, (x_1, x_2), y) \in \Sigma_{\text{gen,parallel}} \text{ if there exists a } y_1 \in Y \text{ and } y_2 \in Y_2 \text{ such that } (u_1, x_1, y_1) \in \Sigma_{\text{gen},1}, (u_1, x_2, y_2) \in \Sigma_{\text{gen},2}, \text{ and } y = y_1 + y_2.$$

Show that the parallel connection $\mathfrak{S}_{\text{parallel}}$ has the transfer function $G_1(s) + G_2(s)$ at s.

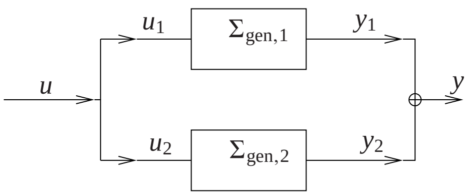

Figure 6.2. Parallel connection

c. Assume that $U_1 = Y_2$ and $Y_1 = U_2$. The *feedback connection* $\Sigma_{\text{gen,feedback}} \subset \boldsymbol{L}_1^{loc}([0,\infty); U_1 \times (X_1 \times X_2) \times Y_1)$ of $\Sigma_{\text{gen},1}$ and $\Sigma_{\text{gen},2}$ is defined as follows

$$(u, (r_1, r_2), y_1) \in \Sigma_{\text{gen,feedback}} \text{ if there exists a } u_1 \text{ and } y_2 \text{ such that } (u_1, x_1, y_1) \in \Sigma_{\text{gen},1}, (y_1, x_2, y_2) \in \Sigma_{\text{gen},2}, \text{ and } u_1 = u - y_2.$$

Show that the feedback connection $\mathfrak{S}_{\text{feedback}}$ has the transfer function $G_1(s) \cdot (I + G_2(s)G_1(s))^{-1}$ at s, provided $I + G_2(s)G_1(s)$ is invertible.

6.4. Determine the transfer functions of the systems given in Exercise 5.1.

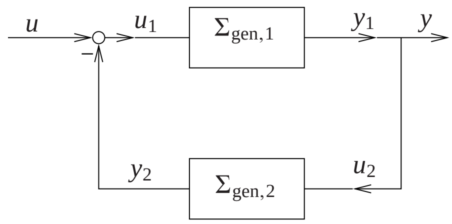

Figure 6.3. Feedback connection

6.5. For mathematical convenience, we have assumed that the state, input, and output spaces are Hilbert spaces over the field of complex numbers. However, in applications we usually deal with systems with real values and the transfer function is also *real*, i.e.,

$$G(s) = \overline{G(\overline{s})} \qquad \text{for all } s \in \mathbb{C}^+_{\omega_0},$$

where ω_0 is the growth bound of the semigroup corresponding to the system.
Prove that for a state linear system $\Sigma(A, B, C, D)$ with finite-rank inputs and outputs the transfer function is real if and only if its impulse response $h(t)$ takes its values in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^k)$ for all $t \geq 0$.

6.6. Consider the heat equation given by

$$\frac{\partial z}{\partial t}(x,t) = \alpha \frac{\partial^2 z}{\partial x^2}(x,t) + b(x)u(t),$$

$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t),$$

$$y(t) = \int_0^1 c(x)z(x,t)dx,$$

for a positive α and non-overlapping shaping functions b and c given by

$$\begin{aligned} b(x) &= \frac{1}{2\varepsilon}\mathbb{1}_{[x_0-\varepsilon, x_0+\varepsilon]}(x), \\ c(x) &= \frac{1}{2\nu}\mathbb{1}_{[x_1-\nu, x_1+\nu]}(x), \end{aligned}$$

with $[x_0 - \varepsilon, x_0 + \varepsilon] \cap [x_1 - \nu, x_1 + \nu] = \emptyset$.

a. Suppose that $\alpha = 1$ and that $x_1 < x_0$. Obtain the following closed form for the transfer function on $\{s \in \mathbb{C} \mid s \neq -k^2\pi^2, k = 0, 1, 2, \cdots\}$

$$g(s) = \frac{\sinh(\sqrt{s}\varepsilon)\sinh(\sqrt{s}\nu)\cosh(\sqrt{s}x_1)\cosh(\sqrt{s}(1-x_0))}{\nu\varepsilon s\sqrt{s}\sinh(\sqrt{s})}.$$

b. Prove that for the case $\alpha = 1$ and $x_1 > x_0$ the transfer function on $\{s \in \mathbb{C} \mid s \neq -k^2\pi^2, k = 0, 1, 2, \cdots\}$ is given by

$$g(s) = \frac{\sinh(\sqrt{s}\varepsilon)\sinh(\sqrt{s}\nu)\cosh(\sqrt{s}x_0)\cosh(\sqrt{s}(1-x_1))}{\nu\varepsilon s\sqrt{s}\sinh(\sqrt{s})}.$$

c. Prove that the transfer function for an arbitrary positive α is given by $\frac{1}{\alpha}g(\frac{s}{\alpha})$.

6.7. *Circle Criterion*: Let g be a meromorphic function on $\mathbb{C}$. Define for $r, R \in \mathbb{R}$ the contour $\Gamma := \{s \in \mathbb{C} \mid \mathrm{Re}(s) = r, |\mathrm{Im}(s)| \leq R\} \cup \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq r, |s - r| = R\}$. Assume that there are no poles or zeros of g on Γ, and let $P(\Gamma)$ and $N(\Gamma)$ denote the number of poles and zeros inside Γ, respectively. From Theorem A.1.20, we have that $N(\Gamma) - P(\Gamma)$ equals the number of times that $\{g(\gamma); \gamma \in \Gamma\}$ winds around the origin, if γ traverses Γ once counterclockwise.
We use this result to calculate the number of poles and zeros of g in $\mathbb{C}_r^+$.

a. Let r be a real number. Suppose that the function g has a nonzero limit at infinity in $\overline{\mathbb{C}_r^+}$. Thus for some nonzero g_∞, $\lim_{s\in\overline{\mathbb{C}_r^+}, |s|\to\infty} g(s) = g_\infty$, i.e.,

$$\lim_{\alpha\to\infty}\left[\sup_{\{\mathrm{Re}(s)\geq r\}\cap\{|s|>\alpha\}} |g(s) - g_\infty|\right] = 0.$$

Define $\Gamma_r := \{s \in \mathbb{C} \mid \mathrm{Re}(s) = r\}$ and assume that there are no poles or zeros of g on Γ_r, and let $P(\Gamma_r)$ and $N(\Gamma_r)$ denote the number of poles inside $\mathbb{C}_r^+$ and the number of zeros inside $\mathbb{C}_r^+$, respectively.
Prove that $N(\Gamma_r) - P(\Gamma_r)$ is the number of times that $\{g(\gamma); \gamma \in \Gamma_r\}$ winds around the origin, if γ traverses Γ_r from $r + \jmath\infty$ to $r - \jmath\infty$.

b. Consider the state linear system $\Sigma(A, B, C, 0)$ on the Hilbert space Z, where A is a Riesz-spectral operator, $B \in \mathcal{L}(\mathbb{C}, Z)$, and $C \in \mathcal{L}(Z, \mathbb{C})$. Assume that the eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$ have no finite limit points and that they lie within a cone. More specifically, there exist real numbers a and m such that $a \geq 0$, $\sup_{n\in\mathbb{N}}\{\mathrm{Re}(\lambda_n)\} \leq m$ and $|\mathrm{Im}(\lambda_n)| \leq a[m - \mathrm{Re}(\lambda_n)]$ for all $n \in \mathbb{N}$.

i. Prove that $g(s) := C(sI - A)^{-1}B$ is meromorphic on $\mathbb{C}$.

ii. Prove that for every real r the following holds:

$$\lim_{s\in\overline{\mathbb{C}_r^+}, |s|\to\infty} sC(sI - A)^{-1}B = CB.$$

Thus we can use the result in part a to calculate the number of zeros of $g(s) := C(sI - A)^{-1}B$ inside $\mathbb{C}_r^+$.

6.8. Consider the following system on $\boldsymbol{L}_2(0,1)$:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + b(x)u(t);$$
$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t);$$
$$y(t) = \int_0^1 z(x,t)c(x)dx,$$

where $b(x) = c(x) = \mathbb{1}_{[1/2,1]}(x)$.

a. Prove that the transfer function of this system is given by

$$g(s) = \frac{-\sinh(\frac{\sqrt{s}}{2}) + \sqrt{s}\cosh(\frac{\sqrt{s}}{2})}{2s\sqrt{s}\cosh(\frac{\sqrt{s}}{2})}.$$

This example is similar to Example 6.2.11.

b. Use Exercise 6.7 to calculate the maximum real part of the zeros of the transfer function, correct to 5 percent. Deduce that there is one real zero with this property.

6.9. In this exercise, we define zeros of linear state systems $\Sigma(A, B, C, 0)$ with scalar input and output operators, $C \in \mathcal{L}(Z, \mathbb{C})$, $B \in \mathcal{L}(\mathbb{C}, Z)$. As usual, A is the infinitesimal generator of a C_0-semigroup on the Hilbert space Z. The *zeros* of $\Sigma(A, B, C, 0)$ are defined to be those points $s_0 \in \mathbb{C}$ for which $g(s) = C(sI - A)^{-1}B$ is zero.

a. Prove that if s_0 is a zero of the scalar linear system $\Sigma(A, B, C, 0)$, then the kernel of the operator

$$K(s_0) = \begin{pmatrix} s_0 I - A & B \\ C & 0 \end{pmatrix}$$

considered from the domain $\boldsymbol{D}(A) \oplus \mathbb{C}$ to $Z \oplus \mathbb{C}$ is nonzero.

b. Suppose that A is a Riesz-spectral operator whose spectrum comprises of isolated eigenvalues. Prove that if $\Sigma(A, B, C, 0)$ is approximately controllable and approximately observable and the kernel of the operator $K(s_0)$ defined in a is nonzero, then s_0 is a zero of $\Sigma(A, B, C, 0)$.

c. Consider now the closed-loop system for some $k \in \mathbb{R}$

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \qquad t \geq 0 \\ y(t) &= Cz(t) \\ u(t) &= ky(t) + v(t). \end{aligned}$$

i. Find the closed-loop transfer function $g_c(s)$ from v to y.

ii. Prove that $\lambda \in \rho(A)$ is an eigenvalue of $(A + kBC)$ if and only if $g(\lambda) = \frac{1}{k}$.

iii. Prove that if A is self-adjoint and $B^* = C$, then $\sigma(A) \cap \sigma(A + kBC) = \{\lambda_n \mid C\phi_n = 0\}$, where ϕ_n is the eigenfunction corresponding to the eigenvalue λ_n.

iv. Show that if $|k| \to \infty$, then the poles of the closed-loop transfer function that are convergent converge to zeros of $\Sigma(A, B, C, 0)$.

v. Prove that if s_0 is a zero of $\Sigma(A, B, C, 0)$, then there is a sequence of eigenvalues of $A + kBC$ that converges to s_0 as $|k| \to \infty$.

6.10. In this exercise we define the transfer function of the discrete-time system $\Sigma_d(A_d, B_d, C_d, D_d)$, i.e., the difference equation

$$z(n+1) = A_d z(n) + B_d u(n), \quad z(0) = z_0, \quad n \in \mathbb{N} \tag{6.34}$$

$$y(n) = C_d z(n) + D_d u(n). \tag{6.35}$$

Definition 6.5.1 An *exponential solution* of (6.34)–(6.35) is a solution of the form $(u(n), z(n), y(n)) = (u_0\zeta^n, z_0\zeta^n, y_0\zeta^n)$, $n \in \mathbb{N}$.
If for $\zeta \in \mathbb{C}$ there exists a unique exponential solution for every $u_0 \in U$, then we call the mapping $u_0 \to y_0$ the *transfer function* of $\Sigma_d(A_d, B_d, C_d, D_d)$ at ζ. ■

Show that for every $\zeta \in \rho(A_d)$ the transfer function exists, and it is given by

$$G_d(\zeta) = C_d(\zeta I - A_d)^{-1} B_d + D_d, \qquad \zeta \in \rho(A_d). \tag{6.36}$$

6.11. In this exercise, we examine a relationship between the continuous-time and discrete-time transfer function.
Consider the linear system $\Sigma(A, B, C, D)$ with the transfer function $G(s) = D + C(sI - A)^{-1}B$ on $\rho(A)$, and $1 \in \rho(A)$. We introduce the matrix function $G_d(\zeta) := G(\frac{\zeta-1}{\zeta+1})$.

a. Show that the matrix function $G_d(\zeta)$ is holomorphic and bounded outside the closed unit disc if and only if $G(s)$ is holomorphic and bounded in the open right half plane.

b. Show that $G_d(\zeta)$ is the transfer function of the discrete-time state linear system $\Sigma_d(A_d, B_d, C_d, D_d)$ by verifying that

$$G_d(\zeta) = D_d + C_d(\zeta I - A_d)^{-1} B_d,$$

where $B_d = \sqrt{2}\,(I - A)^{-1}B$, $C_d = \sqrt{2}\,C(I - A)^{-1}$, $D_d = D + C(I - A)^{-1}B$ and $A_d = (I - A)^{-1}(I + A)$ is bounded.

6.12. Consider the heat equation with Dirichlet boundary control action

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t); \; z(0, t) = u(t), \; z(1, t) = 0 \\ z(x, 0) &= z_0(x). \end{aligned} \tag{6.37}$$

If $u \in \boldsymbol{C}^2([0, \tau])$, then, as in Section 5.2, we can reformulate this as a bounded control system. From Exercise 3.6, we know that the operator

$$A = \frac{d^2}{dx^2} \qquad \text{with}$$

$$\begin{aligned} \boldsymbol{D}(A) &= \{h \in \boldsymbol{L}_2(0,1) \mid h \text{ and } \tfrac{dh}{dx} \text{ are absolutely con-} \\ & \quad \text{tinuous, } \tfrac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1) \text{ and } h(0) = h(1) = 0\} \end{aligned}$$

is the infinitesimal generator of the C_0-semigroup $T(t)$ on $\boldsymbol{L}_2(0,1)$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \langle z, \sqrt{2}\sin(n\pi x)\rangle_{\boldsymbol{L}_2} \sqrt{2}\sin(n\pi \cdot). \tag{6.38}$$

We define the related, but different, operator $\mathfrak{A}$ on $\boldsymbol{L}_2(0,1)$:

$$\mathfrak{A} = \frac{d^2}{dx^2} \qquad \text{with}$$

$$\begin{aligned} \boldsymbol{D}(\mathfrak{A}) &= \{h \in \boldsymbol{L}_2(0,1) \mid h \text{ and } \tfrac{dh}{dx} \text{ are absolutely} \\ & \quad \text{continuous, } \tfrac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1) \text{ and } h(1) = 0\}, \end{aligned}$$

and the boundary operator $\mathfrak{P} : \boldsymbol{L}_2(0,1) \to \mathbb{C}$ given by

$$\mathfrak{P}h = h(0), \qquad \boldsymbol{D}(\mathfrak{P}) = \boldsymbol{C}(0,1). \tag{6.39}$$

a. Show that every λ in $\mathbb{C}$ is an eigenvalue of $\mathfrak{A}$ and hence conclude that $\mathfrak{A}$ cannot be the infinitesimal generator of a C_0-semigroup.
b. Show that (6.37) defines a boundary control system.
   Hint: Try $Bu = b(x)u$ with $b(x) = 1 - x$.
c. Show that for absolutely continuous inputs the mild solution of (6.37) is given by

$$\begin{aligned} z(t) &= T(t)z_0 + bu(t) - T(t)Bu(0) - \\ & \quad \int_0^t T(t-s)B\dot{u}(s)ds. \end{aligned} \tag{6.40}$$

d. Find the transfer function of the system with the observation $y(t) = z(x_1, t)$ where $0 \le x_1 \le 1$.

6.13. In this exercise, we prove that the following model for the evolution of a population introduced in Example 1.1.4 is well posed:

$$\begin{aligned} \frac{\partial p}{\partial t}(r,t) &= -\frac{\partial p}{\partial r}(r,t) - \mu(r,t)p(r,t), \\ p(r,0) &= p_0(r), \\ p(0,t) &= u(t), \end{aligned} \tag{6.41}$$

where $p(r,t)$ represents the population of age r at time t, $\mu(r,t) \ge 0$ is the mortality function, $p_0(r)$ is the given initial age distribution, and $u(t)$ is the number of individuals born at time t. First we shall assume that there is no control term, $u(t) \equiv 0$.

a. Assume that $\mu \equiv 0$. Prove that for $p_0 \in \boldsymbol{C}^1(0, \infty)$ with $p_0(0) = 0$ the solution of (6.41) is given by

$$p(r, t) = \begin{cases} p_0(r - t) & \text{for } r \geq t \\ 0 & \text{for } r < t. \end{cases}$$

Show that the semigroup associated with (6.41) is given by

$$T(t)p_0 = 1_{[0,\infty)}(\cdot - t)p_0(\cdot - t)$$

on the Hilbert space $Z = \boldsymbol{L}_2(0, \infty)$.

b. Reformulate (6.41) in the abstract form (4.25), and give the expression for A, $\boldsymbol{D}(A)$, and $D(t)$.

c. Assume that $\mu(r, t) \geq 0$ is independent of t and hence $D(\cdot) \equiv D$. Prove that $A + D$ is the infinitesimal generator of a contraction semigroup and show that the mild solution of (6.41) satisfies

$$\begin{aligned} p(r, t) &= 1_{[0,\infty)}(r - t)p_0(r - t) - \\ &\quad \int_{\max(0,t-r)}^{t} \mu(r - t + s)p(r - t + s, s)ds. \end{aligned}$$

Hint: See Example 2.3.4 and Exercise 2.7.

d. Now we shall assume that $u(t) \not\equiv 0$. Let $\mu(r, t)$ be independent of t and reformulate (6.41) as a boundary control system.

6.14. Let us consider the one-dimensional rod shown in Figure 6.4, Ray [132, example 4.2.2]. Heat is added from a steam chest at $x = 0$, and the end at

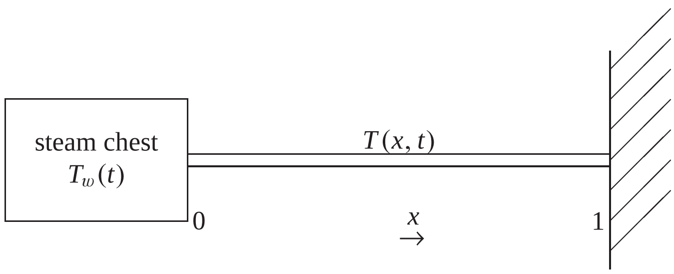

Figure 6.4. A one-dimensional heated rod

$x = 1$ is perfectly insulated. If we define the variables $z(x, t) = T(x, t) - T_d$, $u(t) = T_w - T_{wd}$, where $T_w(t)$ is the temperature of the steam chest and

T_d, T_{wd} are set-point values, then we obtain the partial differential equation

$$\begin{aligned} &\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \\ &\frac{\partial z}{\partial x}(0,t) - \beta z(0,t) = -\beta u(t), \\ &\frac{\partial z}{\partial x}(1,t) = 0, \\ &z(x,0) = z_0(x), \end{aligned} \tag{6.42}$$

with $\beta > 0$. We want to formulate this as a boundary control system.

Let $A_0 = \dfrac{d^2}{dx^2}$ with $\boldsymbol{D}(A_0) = \{h \in \boldsymbol{L}_2(0,1) \mid h, \frac{dh}{dx}$ are absolutely continuous, $\frac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1)$, $\frac{dh}{dx}(0) - \beta h(0) = 0$, and $\frac{dh}{dx}(1) = 0\}$.

a. Use Exercise 2.5 to show that A_0 is self-adjoint.
b. For the uncontrolled situation, $u \equiv 0$, formulate (6.42) as an abstract differential equation $\dot{z}(t) = Az(t)$ on the state space $Z = \boldsymbol{L}_2(0,1)$ with A the infinitesimal generator of a C_0-semigroup.
c. Formulate equation (6.42) in the abstract form (5.6).
d. Prove that the system obtained in b is a boundary control system.
e. Find the transfer function of the system with the observation $y(t) = z(x_1, t)$, where $0 \le x_1 \le 1$.

6.15. In Section 5.2, we considered distributed parameter systems that were controlled on the boundary, as distinct from systems controlled in the interior. In this exercise, we consider both types of control simultaneously, as in the following example:

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + b_d(x)u_d(t); \\ z(0,t) &= u(t), \ z(1,t) = 0 \\ z(x,0) &= z_0(x), \end{aligned} \tag{6.43}$$

where $b_d \in \boldsymbol{L}_2(0,1)$. This is a special case of the following mixed abstract boundary control system

$$\begin{aligned} \dot{z}(t) &= \mathfrak{A}z(t) + B_d u_d(t), \quad z(0) = z_0 \\ \mathfrak{P}z(t) &= u(t), \end{aligned} \tag{6.44}$$

where $\mathfrak{A} : \boldsymbol{D}(\mathfrak{A}) \subset Z \mapsto Z$, $B_d \in \mathcal{L}(U_d, Z)$, $\mathfrak{P} : \boldsymbol{D}(\mathfrak{P}) \subset Z \mapsto U$ satisfies $\boldsymbol{D}(\mathfrak{A}) \subset \boldsymbol{D}(\mathfrak{P})$, and Z, U_d, U are separable Hilbert spaces. Furthermore, we suppose that $\mathfrak{A}, \mathfrak{P}$ form a boundary control system as defined in Definition 5.2.2. Under the above conditions, the following abstract differential equation is well defined for $\dot{u}, u_d$ in $\boldsymbol{L}_1([0,\tau]; U)$ and $\boldsymbol{L}_1([0,\tau]; U_d)$, respectively:

$$\begin{aligned} \dot{v}(t) &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B_d u_d(t), \\ v(0) &= v_0. \end{aligned} \tag{6.45}$$

a. Prove that if $u \in \boldsymbol{C}^2([0,\tau];U)$, $u_d \in \boldsymbol{C}^1([0,\tau];U)$, and $\upsilon_0 = z_0 - Bu(0) \in \boldsymbol{D}(A)$, the classical solutions of (6.44) and (6.45) are related by $\upsilon(t) = z(t) - Bu(t)$.

b. Prove that if $u \in \boldsymbol{C}^2([0,\tau];U)$, $u_d \in \boldsymbol{C}^1([0,\tau];U)$, and $\upsilon_0 \in \boldsymbol{D}(A)$, then the abstract differential equation

$$\begin{aligned} \dot{z}^e(t) &= \begin{pmatrix} 0 & 0 \\ \mathfrak{A}B & A \end{pmatrix} z^e(t) + \\ &\quad \begin{pmatrix} I & 0 \\ -B & B_d \end{pmatrix} \begin{pmatrix} \tilde{u}(t) \\ u_d(t) \end{pmatrix}, \qquad (6.46) \\ z^e(0) &= \begin{pmatrix} (z_0^e)_1 \\ (z_0^e)_2 \end{pmatrix} \end{aligned}$$

with $(z_0^e)_1 = u(0)$, $(z_0^e)_2 = \upsilon_0$ and $\tilde{u} = \dot{u}$ has the unique classical solution $z^e(t) = \begin{pmatrix} u(t) \\ \upsilon(t) \end{pmatrix}$, where $\upsilon(t)$ is the unique classical solution of (6.44).

Prove also that in the case $z_0 = \upsilon_0 + Bu(0)$, the classical solution is given by

$$\begin{aligned} z(t) &= \begin{pmatrix} B & I \end{pmatrix} z^e(t) \\ &= Bu(t) - T(t)Bu(0) + T(t)z_0 - \\ &\quad \int_0^t T(t-s)B\dot{u}(s)ds + \int_0^t T(t-s)\mathfrak{A}Bu(s)ds + \\ &\quad \int_0^t T(t-s)B_d u_d(s)ds. \qquad (6.47) \end{aligned}$$

c. Formulate (6.43) as a mixed boundary control system.
Hint: See Exercise 6.12.

6.16. Mathematical models for chemical reactions are often highly nonlinear. An example of such a model from Ray [132, example 4.4.1] in dimensionless coordinates is

$$\begin{aligned} \frac{\partial h}{\partial t}(x,t) &= \alpha \frac{\partial^2 h}{\partial x^2}(x,t) - \frac{\partial h}{\partial x}(x,t) + \\ &\quad B \exp\left[\frac{\gamma h(x,t)}{1+h(x,t)}\right] + \beta[u(t) - h(x,t)], \\ -\upsilon(t) &= \alpha \frac{\partial h}{\partial x}(0,t) - h(0,t), \qquad (6.48) \\ \frac{\partial h}{\partial x}(1,t) &= 0, \end{aligned}$$

where $0 \le x \le 1,\ t \ge 0$; α, β, B, and γ are positive constants and u and υ are control inputs.

a. Obtain a linearization of this nonlinear partial differential equation about the steady state h_s, u_s, υ_s satisfying

$$\begin{aligned} 0 &= \alpha \frac{d^2 h_s}{dx^2}(x) - \frac{dh_s}{dx}(x) + B \exp\left[\frac{\gamma h_s(x)}{1+h_s(x)}\right] + \\ &\quad \beta[u_s - h_s(x)],\ 0 \le x \le 1,\ t \ge 0, \\ -\upsilon_s &= \alpha \frac{dh_s}{dx}(0) - h_s(0), \\ \frac{dh_s}{dx}(1) &= 0, \text{ and} \\ h_s(x) &\neq -1\ \forall x \in [0,1]. \end{aligned}$$

b. Suppose that the two inputs have constant values: $\upsilon(t) = \upsilon_s$ and $u(t) = u_s$. Obtain an abstract formulation $\dot{z}(t) = Az(t)$ of your linearization on the state space $Z = \boldsymbol{L}_2(0,1)$.

c. Use Exercise 2.5 to show that the operator A_0 defined by

$$A_0 h = -\alpha \frac{d^2 h}{dx^2} + \frac{dh}{dx}$$

with domain $\boldsymbol{D}(A_0) = \{h \in \boldsymbol{L}_2(0,1) \mid h, \frac{dh}{dx}$ are absolutely continuous with $\frac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1)$ and $\alpha \frac{dh}{dx}(0) = h(0),\ \frac{dh}{dx}(1) = 0\ \}$ is self-adjoint and positive on $\boldsymbol{L}_2(0,1)$ with inner product $\langle \cdot, \cdot \rangle_\alpha$ given by

$$\langle h_1, h_2 \rangle_\alpha := \int_0^1 h_1(x)\overline{h_2(x)} e^{-\frac{x}{\alpha}} dx,$$

provided that $\alpha > 0$.

d. Show that A is the infinitesimal generator of a C_0-semigroup.

e. Suppose that the input $u(t)$ is kept at the constant value u_s, but that the boundary input $\upsilon(t)$ may vary with time. How would you formulate the corresponding linearized system as a boundary control system?

6.17. Consider the undamped beam equation of Exercise 3.9 with the following boundary control action:

$$\begin{aligned} &\frac{\partial^2 f}{\partial t^2}(x,t) = -\frac{\partial^4 f}{\partial x^4}(x,t), \\ &f(0,t) = f(1,t) = 0, \\ &\frac{\partial^2 f}{\partial x^2}(0,t) = 0,\ \frac{\partial^2 f}{\partial x^2}(1,t) = u(t), \\ &f(x,0) = f_1(x),\ \frac{\partial f}{\partial t}(x,0) = f_2(x). \end{aligned} \tag{6.49}$$

Recall from Exercise 3.9 that if $u = 0$, this can be formulated as an abstract differential equation on the Hilbert space $Z = \boldsymbol{D}(A_0^{\frac{1}{2}}) \oplus \boldsymbol{L}_2(0,1)$, where $A_0 = \dfrac{d^4}{dx^4}$ on the domain $\boldsymbol{D}(A_0) = \{h \in \boldsymbol{L}_2(0,1) \mid h, \frac{dh}{dx}, \frac{d^2h}{dx^2}, \frac{d^3h}{dx^3}$ are absolutely continuous, $\frac{d^4h}{dx^4} \in \boldsymbol{L}_2(0,1)$ and $h(0) = h(1) = \frac{d^2h}{dx^2}(0) = \frac{d^2h}{dx^2}(1) = 0\}$.

a. Reformulate equation (6.49) as a boundary control system with $\mathfrak{A}B = 0$.
b. Find the transfer function of the system with the observation $y(t) = \frac{\partial z}{\partial t}(x_1, t)$, where $0 \le x_1 \le 1$.

6.18. (Ray [132, example 4.2.1, p. 140]) Consider the feedback control of the steam-jacket tubular heat exchanger. Thermocouples measure the temperature of the fluid in the tube at three points $T(\frac{1}{4}, t)$, $T(\frac{1}{2}, t)$, and $T(\frac{3}{4}, t)$. These are used to determine the adjustment of the temperature of the steam at the inlet valve at $x = 0$ in order to control the exchanger.
The mathematical model for the process takes the form

$$\frac{\partial T}{\partial t}(x,t) + \upsilon \frac{\partial T}{\partial x}(x,t) = -\frac{hA}{\rho C_p} T(x,t)$$
$$T(0,t) = T_f(t), \quad T(x,0) = T_0(x),$$

where $T(x,t)$ is the temperature at time t at point x along the tube measured from $x = 0$ to $x = 1$. υ, h, A, ρ, and C_p are physical parameters of the exchanger and $T_f(t)$ is the temperature of steam at the inlet valve at $x = 0$. We wish to follow the temperature profile $T_d(x)$ as closely as possible, and T_{f_d} is the nominal inlet temperature to achieve this. If we define the deviation variables

$$z(x,t) = T(x,t) - T_d(x), \quad z_0(x) = T(x,0) - T_d(x),$$
$$u(t) = T_f(t) - T_{f_d},$$

then we obtain the following model

$$\frac{\partial z}{\partial t}(x,t) + \nu \frac{\partial z}{\partial x}(x,t) = -az(x,t)$$
$$z(0,t) = u(t), \quad z(x,0) = z_0(x),$$

where $a = \frac{hA}{\rho C_p}$.

a. Formulate the above as a boundary control system using the results of Exercise 6.13. Show that $A = -\upsilon\frac{d}{dx} - aI$, $\boldsymbol{D}(A) = \{h \in \boldsymbol{L}_2(0,\infty) \mid h$ is absolutely continuous on finite intervals, $\frac{dh}{dx} \in \boldsymbol{L}_2(0,\infty)$ and $h(0) = 0\}$ generates a C_0-semigroup on $Z = \boldsymbol{L}_2(0,\infty)$. Find $B \in \mathcal{L}(\mathbb{C}, Z)$.
b. Obtain the extended state linear system $\Sigma(A^e, B^e, C^e, -)$ on $Z^e = \mathbb{C} \oplus Z$ for the observation

$$y(t) = \int_0^1 z(x,t)c(x)dx, \qquad c(x) = \frac{1}{2\nu} 1_{[x_1-\nu, x_1+\nu]}(x).$$

c. Show that

$$\left((sI-A)^{-1}h\right)(x) = \frac{1}{\upsilon}\int_0^x e^{-(\frac{s+a}{\upsilon})(x-t)}h(t)dt.$$

Hence derive an expression for the transfer function from $u(t)$ to $y(t)$ in b.

d. Obtain an expression for the transfer function from $u(t)$ to $y(t)$, where

$$y(t) = \left(z(\frac{1}{4},t), z(\frac{1}{2},t), z(\frac{3}{4},t)\right)^{\mathrm{T}}.$$

6.19. Consider the following boundary control problem of Exercise 6.12:

$$\begin{aligned}&\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad z(0,t)=u(t), \quad z(1,t)=0,\\ &z(x,0)=z_0(x)\end{aligned} \tag{6.50}$$

with the observation

$$y(t) = \int_0^1 z(x,t)c(x)dx,$$

where $c(x) = \frac{1}{2\nu}1_{[x_1-\nu,x_1+\nu]}(x)$.

a. Using the results of Exercise 6.12, formulate this as a linear system $\Sigma(A^e, B^e, C^e, -)$ on the state space $Z^e = \mathbb{C}\oplus \boldsymbol{L}_2(0,1)$.

b. Using Lemma 6.3.1, show that the transfer function from u to y has the form

$$g(s) = -\langle A(sI-A)^{-1}b, c\rangle_{\boldsymbol{L}_2}, \quad s\in\rho(A),$$

where A and b are as in Exercise 6.12. Hence show that

$$g(s) = \sum_{n=1}^{\infty}\frac{2\sin(n\pi x_1)\sin(n\pi\nu)}{\nu(s+n^2\pi^2)}, \quad s\neq -(n\pi)^2, n\in\mathbb{Z}.$$

c. Find the following closed-form expression for the transfer function

$$g_2(s) = \frac{\sinh(\sqrt{s}\nu)\sinh((1-x_1)\sqrt{s})}{\nu\sqrt{s}\sinh(\sqrt{s})}, \quad \sinh s\neq 0.$$

d. Establish the equality of $g(s)$ and $g_2(s)$ as in Example 6.3.2.

6.20. In Example 6.3.2, we have indirectly proved the equality of two expressions for the transfer function, namely,

$$\frac{\cosh(\sqrt{s}x_1)}{\sqrt{s}\sinh(\sqrt{s})} = \frac{1}{s} + \sum_{n=1}^{\infty}\frac{2(-1)^n\cos n\pi x_1}{s+(n\pi)^2}. \tag{6.51}$$

In this exercise, we consider an alternative approach.

a. Obtain the Fourier series for the following smooth function of x on $0 \le x \le 1$ in terms of $(1, \sqrt{2}\cos(n\pi x), n \ge 1)$

$$f(s,x) = \frac{\cosh(sx)}{s\sinh(s)}.$$

Hence prove that

$$\frac{\cosh(sx)}{s\sinh(s)} = \frac{1}{s^2} + \sum_{n=1}^{\infty} \frac{2(-1)^n \cos n\pi x}{s^2 + (n\pi)^2}$$

pointwise for $0 < x < 1$ and all values of the parameter s, except $s = 0$, $s^2 = -(n\pi)^2$.

b. Prove (6.51).

c. Establish the equality of $g(s)$ and $g_2(s)$ as follows:

i. Show that

$$\frac{\sinh((1-x)s)}{\sinh(s)} = \sum_{n=1}^{\infty} \frac{2n\pi \sin(n\pi x)}{s^2 + (n\pi)^2}.$$

ii. Integrate over $[x_1 - \nu, x_1 + \nu]$ with respect to x.

6.21. Consider the undamped beam equation with boundary control from Exercise 6.17 (see also Exercise 3.9):

$$\begin{aligned} &\frac{\partial^2 f}{\partial t^2}(x,t) + \frac{\partial^4 f}{\partial x^4}(x,t) = 0, \quad f(0,t) = 0 = f(1,t) \\ &\frac{\partial^2 f}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 f}{\partial x^2}(1,t) = u(t), \end{aligned} \tag{6.52}$$

with the observation

$$y(t) = f(x_1, t).$$

a. Using the results of Exercise 6.17, formulate this as a linear system $\Sigma(A^e, B^e, C^e, -)$ on the extended state space $Z^e = \mathbb{C} \oplus \boldsymbol{D}(A_0^{\frac{1}{2}}) \oplus \boldsymbol{L}_2(0,1)$, where A_0, $A_0^{\frac{1}{2}}$, etc., are specified as in Exercises 3.9 and 6.17. Prove that C^e is bounded.

b. Using the formulation $\Sigma(A^e, B^e, C^e, -)$ from a, derive the following expression for the transfer function from u to y:

$$g(s) = -\chi(x_1) \begin{pmatrix} I & 0 \end{pmatrix} A(sI - A)^{-1} \begin{pmatrix} b \\ 0 \end{pmatrix},$$

where $\chi(x_1)$ denotes the evaluation map in the point x_1, $b(x) = \frac{1}{6}(x^3 - x)$, and $A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}$. Hence, using the spectral expansion of A from Exercise 3.9, show that the transfer function is given by

$$g(s) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n\pi \sin n\pi x_1}{s^2 + (n\pi)^4}.$$

c. Find the following closed-form expression for the transfer function

$$\begin{aligned} g_2(s) &= [\cos(\lambda)\sinh(\lambda)\sin(\lambda x_1)\cosh(\lambda x_1) - \\ &\quad \sin(\lambda)\cosh(\lambda)\cos(\lambda x_1)\sinh(\lambda x_1)] \cdot \\ &\quad \left[s(\sin^2(\lambda) + \sinh^2(\lambda)\right]^{-1}, \end{aligned}$$

where $\lambda^2 = \frac{s}{2}$.

d. Verify that $g_2(s)$ is indeed a well defined function of s by showing that $\lambda = \sqrt{s}$ and $\lambda = -\sqrt{s}$ both yield the same expression.

e. Show that $g_2(s)$ equals $g(s)$.

6.22. Consider the heated rod example of Exercise 6.14:

$$\begin{aligned} &\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \\ &\frac{\partial z}{\partial x}(0,t) - \beta z(0,t) = -\beta u(t), \\ &\frac{\partial z}{\partial x}(1,t) = 0, \end{aligned} \tag{6.53}$$

with the observation

$$y(t) = z(x_1, t).$$

a. In Exercise 6.14, it was shown how to formulate the above as an extended system $\Sigma(A^e, B^e, -, -)$ on the state space $\mathbb{C} \oplus \boldsymbol{L}_2(0,1)$, where $A^e = \begin{pmatrix} 0 & 0 \\ -1 & A_0 \end{pmatrix}$, $B^e = \begin{pmatrix} 1 \\ -b \end{pmatrix}$, $b(x) = 2 + \beta x - \frac{1}{2}\beta x^2$, and $A_0 = \frac{d^2}{dx^2}$, $\boldsymbol{D}(A_0) = \{h \in \boldsymbol{L}_2(0,1) \mid h$ and $\frac{dh}{dx}$ are absolutely continuous, $\frac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1)$ and $\frac{dh}{dx}(0) - \beta h(0) = 0, \frac{dh}{dx}(1) = 0\}$.
Show that A_0 has the eigenvalues $\lambda_n = -\gamma_n^2$, where γ_n are the solutions of $\beta\cos(\gamma) = \gamma\sin(\gamma)$ and that the corresponding orthonormal eigenfunctions of A_0 are $\phi_n(x) = \alpha_n\beta\sin(\gamma_n x) + \alpha_n\gamma_n\cos(\gamma_n x)$, where $\alpha_n = \left[\frac{\lambda_n^2 + \beta + \beta\cos^2(\lambda_n)}{2}\right]^{-\frac{1}{2}}$.

b. Find the following expression for the transfer function from u to y:

$$g(s) = \chi(x_1)\sum_{n=1}^{\infty}\frac{1}{s-\lambda_n}[-\beta\langle 1, \phi_n\rangle - \lambda_n\langle b, \phi_n\rangle]\phi_n.$$

c. Find the following closed-form expression of the transfer function

$$g(s) = \frac{\beta\cosh(\sqrt{s}(1-x_1))}{\beta\cosh(\sqrt{s}) + \sqrt{s}\sinh(\sqrt{s})}.$$

d. Verify that the expression given in part c is a well defined function of s by showing that $\lambda = \sqrt{s}$ and $\lambda = -\sqrt{s}$ both yield the same expression.

6.23. (Ray [132, exercise 4.4, p. 242]) Consider the boundary control of a thin metal rod that has one end in a water bath at 25°C and the other end inserted in a steam chest. Air at 25°C is blowing transversely across the rod. The temperature of the right-hand end is assumed fixed at 25°C, while the temperature of the left-hand end may be controlled by adjusting the steam pressure. Thus, for time $t > 0$ the system may be modelled by

$$\rho C_p \frac{\partial T}{\partial t}(x,t) = k\frac{\partial^2 T}{\partial x^2}(x,t) - k(T(x,t) - 25),\ 0 < x < L,$$
$$T(0,t) = f(t), \quad T(L,t) = 25,$$

where $T(x,t)$ is the temperature at position x at time t and h, ρ, k, C_p are known parameters of the rod. f denotes the steam pressure applied at one end. The temperature is measured at one point αL, $0 < \alpha < 1$. Defining

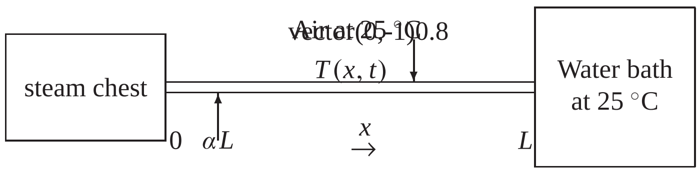

Figure 6.5. A one-dimensional heated rod

$z = T - 25$, $u = f - 25$, $\beta = \frac{k}{\rho C_p L^2}$, and $\xi = \frac{x}{L}$, we obtain the model

$$\frac{\partial z}{\partial t}(\xi,t) = \beta\frac{\partial^2 z}{\partial \xi^2}(\xi,t) - \beta L^2 z(\xi,t),$$
$$z(0,t) = u(t), \quad z(1,t) = 0,$$
$$y(t) = z(\alpha,t).$$

a. Show that the above system defines a boundary control system and hence obtain a series expansion for the transfer function of the system.
Hint: $\frac{d^2}{dx^2} - L^2 I$ is a bounded perturbation of the operator in Exercise 3.6 and the resolvent of $\beta[\frac{d^2}{dx^2} - L^2 I]$ is easily obtained from the resolvent of $\frac{d^2}{dx^2}$.

b. Find a closed-form expression for the transfer function.

c. Suppose that instead of measuring $z(\alpha,t)$ we have a delay in the observations, and so we actually measure

$$y(t) = z(\alpha, t - \varepsilon).$$

Furthermore, suppose that the adjustment of the steam pressure is implemented through a device with linear dynamics and with the delayed measurement as input according to:

$$\dot{w}(t) = aw(t) + ky(t),$$

$$u(t) = \gamma w(t),$$

where a, γ, and k are certain constants.
What is the transfer function of the closed-loop system?
Hint: Introduce the new input $v = u + u_0$ in the partial differential equation and find the transfer function from u_0 to y.

6.6 Notes and references

The idea of defining the transfer function via exponentially solutions is an old one, but it has hardly been investigated for distributed parameter systems. The paper by Zwart [179] was the first where this approach was used for infinite-dimensional systems. There the term characteristic function was used for what we here define as the transfer function.

One may find the exponential solution in Polderman and Willems [127], where all solutions of this type are called the exponential behaviour.

The formula for the transfer function, $G(s) = C(sI - A)^{-1}B + D$ can also easily be derived using the Laplace transform, see e.g.Curtain and Zwart [42]. However, using this approach the function is only defined in some right half-plane and not on the whole resolvent set of A. In finite-dimensional spaces the transfer function is rational, and there are no mathematical difficulties in extending the transfer function to $\rho(A)$. The situation is different for infinite-dimensional spaces, since transfer functions can contain terms like $\sqrt{s}$, and for these functions it is less clear how to extend them. For further examples of transfer functions of distributed parameter systems, see Curtain and Morris [38].

Our treatment of the frequency domain interpretation of input and output stability follows Opmeer and Curtain [117].

For more extensions of these ideas to more general infinite-dimensional systems see Staffans [152].

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