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7
Stability

7.1 Exponential stability

The asymptotic properties of the mild solution $z(t) = T(t)z_0$ of the abstract homogeneous Cauchy initial value problem, see also (2.29),

$$\dot{z}(t) = Az(t), \ t \geq 0, \qquad z(0) = z_0, \tag{7.1}$$

depend on the stability properties of the semigroup $T(t)$. The most important concept of stability is exponential stability.

Definition 7.1.1 A C_0-semigroup, $T(t)$, on a Hilbert space Z is *exponentially stable* if there exist positive constants M and α such that

$$\|T(t)\| \leq Me^{-\alpha t} \qquad \text{for } t \geq 0. \tag{7.2}$$

The α is called the *decay rate*, and the supremum over all possible values of α is the *stability margin* of $T(t)$; this is minus its growth bound mentioned in Theorem 2.1.7. We say that $T(t)$ is *β-exponentially stable* if (7.2) holds for $-\alpha < \beta$, i.e., its stability margin is at least $-\beta$. ■

If $T(t)$ is exponentially stable, then the solution to the abstract Cauchy problem (7.1) tends to zero exponentially fast as $t \to \infty$. In Exercise 7.1 we show that this is equivalent to the property that the norm of the semigroup converges to zero as $t \to \infty$.

An important criterion for exponential stability is the following.

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Lemma 7.1.2 *The C_0-semigroup $T(t)$ on Z is exponentially stable if and only if for every $z \in Z$ there exists a positive constant $\gamma_z < \infty$ such that*

$$\int_0^\infty \|T(t)z\|^2 dt \leq \gamma_z. \tag{7.3}$$

Proof The necessity is obvious, so suppose that (7.3) holds. Now Theorem 2.1.7.e implies that there exist numbers $M > 0$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for } t \geq 0. \tag{7.4}$$

Thus for every $n \geq 1$ the operator Q_n defined by

$$Q_n z := \mathbb{1}_{[0,n]}(t)T(t)z$$

is a bounded linear operator from Z to $\boldsymbol{L}_2([0,\infty); Z)$. From equation (7.3) we have that the family $\{Q_n, n \geq 1\}$ is uniformly bounded in n, and thus by the Uniform Boundedness Theorem A.3.19, it follows that

$$\|Q_n\| \leq \gamma \tag{7.5}$$

for some γ independent of n.

For $0 \leq t \leq \delta$, we have that $\|T(t)\| \leq Me^{\omega\delta}$. For $t > \delta$, we calculate

$$\begin{aligned}
\frac{1 - e^{-2\omega t}}{2\omega}\|T(t)z\|^2 &= \int_0^t e^{-2\omega s}\|T(t)z\|^2 ds \\
&\leq \int_0^t e^{-2\omega s}\|T(s)\|^2\|T(t-s)z\|^2 ds \\
&\leq M^2 \int_0^t \|T(s)z\|^2 ds \qquad \text{from (7.4)} \\
&\leq M^2\gamma^2\|z\|^2 \qquad \text{from (7.5).}
\end{aligned}$$

Thus for some $K > 0$ and all $t \geq 0$, we obtain

$$\|T(t)\| \leq K$$

and, moreover,

$$\begin{aligned}
t\|T(t)z\|^2 &= \int_0^t \|T(t)z\|^2 ds \\
&\leq \int_0^t \|T(s)\|^2\|T(t-s)z\|^2 ds \\
&\leq K^2\gamma^2\|z\|^2 \qquad \text{from (7.5).}
\end{aligned}$$

Hence

$$\|T(t)\| \leq \frac{K\gamma}{\sqrt{t}},$$

which implies that

$$\|T(\tau)\| < 1 \qquad \text{for a sufficiently large } \tau.$$

Consequently, $\log(\|T(\tau)\|) < 0$, and so by Theorem 2.1.7 there exist $\tilde{M}$ and $\alpha > 0$ such that

$$\|T(t)\| \leq \tilde{M}e^{-\alpha t} \qquad \text{for all } t \geq 0.$$

■

Lemma 7.1.2 can be used to prove a Lyapunov-type result, which can be of use in establishing stability of the abstract differential equation.

Theorem 7.1.3 *Suppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z. Then $T(t)$ is exponentially stable if and only if there exists a positive operator $P \in \mathcal{L}(Z)$ such that*

$$\langle Az, Pz\rangle + \langle Pz, Az\rangle = -\langle z, z\rangle \qquad \text{for all } z \in \boldsymbol{D}(A). \tag{7.6}$$

Equation (7.6) is called a Lyapunov equation.

If $T(t)$ is exponentially stable, then the unique self-adjoint solution of (7.6) is given by

$$Pz = \int_0^\infty T(s)^* T(s) z ds \qquad \text{for } z \in Z. \tag{7.7}$$

Proof *Necessity*: Since $T(t)$ is exponentially stable, we have that $z(t) = T(t)z_0$ is an element of $\boldsymbol{L}_2([0,\infty); Z)$ for all $z_0 \in Z$. So equation (7.7) defines a nonnegative, self-adjoint operator $P \in \mathcal{L}(Z)$:

For $z \in \boldsymbol{D}(A)$ we have

$$\begin{aligned} &\langle Az, Pz\rangle + \langle Pz, Az\rangle \\ &\quad = \int_0^\infty \langle T(t)z, T(t)Az\rangle dt + \int_0^\infty \langle T(t)Az, T(t)z\rangle dt \\ &\quad = \int_0^\infty \frac{d}{dt}\langle T(t)z, T(t)z\rangle dt. \end{aligned}$$

Now by Theorem 2.1.7.e. there exist positive constants M, ω such that

$$\begin{aligned} \left|\frac{d}{dt}\langle T(t)z, T(t)z\rangle\right| &\leq \|T(t)z\|\|T(t)Az\| + \|T(t)Az\|\|T(t)z\| \\ &\leq 2Me^{-2\omega t}\|Az\|\|z\| \end{aligned}$$

and so $\frac{d}{dt}\langle T(t)z, T(t)z\rangle$ is integrable on $[0, \infty)$.

Hence for all $z \in \boldsymbol{D}(A)$ we have

$$\langle Az, Pz\rangle + \langle Pz, Az\rangle = [\langle T(t)z, T(t)z\rangle]_0^\infty = -\langle z, z\rangle,$$

where we have used the exponential stability of $T(t)$. Thus P is a nonnegative solution of (7.6). Now suppose that there exists an $z_0 \in Z$ such that $Pz_0 = 0$. Equation (7.7) implies that

$$0 = \langle z_0, Pz_0\rangle = \int_0^\infty \|T(s)z_0\|^2 ds,$$

and so $\|T(t)z_0\| = 0$ on $[0, \infty)$ almost everywhere. The strong continuity of $T(t)$ implies that $z_0 = 0$. Thus $P > 0$.

Sufficiency: Conversely, suppose that there exists a bounded $P > 0$ such that (7.6) is satisfied. We introduce the following Lyapunov functional:

$$V(t, z) = \langle PT(t)z, T(t)z\rangle.$$

Since P is positive, $V(t, z) \geq 0$ for all $t \geq 0$. For $z \in \boldsymbol{D}(A)$, we may differentiate to obtain

$$\frac{dV}{dt}(t, z) = \langle PAT(t)z, T(t)z\rangle + \langle PT(t)z, AT(t)z\rangle = -\|T(t)z\|^2.$$

On integrating, we obtain

$$0 \leq V(t, z) = V(0, z) - \int_0^t \|T(s)z\|^2 ds$$

and hence

$$\int_0^t \|T(s)z\|^2 ds \leq V(0, z) = \langle Pz, z\rangle \qquad \text{for all } t \geq 0 \text{ and } z \in \boldsymbol{D}(A).$$

This inequality can be extended to all $z \in Z$, since $\boldsymbol{D}(A)$ is dense in Z. In other words, for every $z \in Z$ there exists a $\gamma_z = \langle Pz, z\rangle > 0$ such that

$$\int_0^\infty \|T(s)z\|^2 ds \leq \gamma_z$$

and Lemma 7.1.2 completes the sufficiency proof.

To prove the uniqueness suppose that P_2 is another self-adjoint solution and let $\Delta = P - P_1$. Then we have

$$\langle Az, \Delta z\rangle + \langle \Delta z, Az\rangle = 0.$$

Substituting $z = T(t)z_0$ for $z_0 \in \boldsymbol{D}(A)$ we obtain

$$\langle AT(t)z_0, \Delta T(t)z_0\rangle + \langle \Delta T(t)z_0, AT(t)z_0\rangle = \frac{d}{dt}\langle T(t)z_0, \Delta T(t)z_0\rangle = 0.$$

Hence

$$\langle \Delta T(t)z_0, T(t)z_0\rangle = constant = 0,$$

because $T(t)z_0 \to 0$ as $t \to \infty$. Substituting $t = 0$ gives $\langle \Delta z_0, z_0\rangle = 0$ for all $z_0 \in \boldsymbol{D}(A)$. Since $\boldsymbol{D}(A)$ is dense in Z and Δ is self-adjoint we can conclude that $\Delta = 0$. ■

For the spatially invariant systems of Section 3.1, the solution of (7.6) with the operator $\Lambda_{\check{A}}$ is a multiplicative operator.

Theorem 7.1.4 *For $\check{A} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$ the following assertions are equivalent:*

a. The semigroup generated by $\Lambda_{\check{A}}$ is exponentially stable.

b. *There exists an* $M > 0$ *such that for almost all* $\phi \in \partial\mathbb{D}$ *there exists a positive* $\check{P}(\phi) \in \mathbb{C}^{n\times n}$ *satisfying*

$$\check{A}(\phi)^* \check{P}(\phi) + \check{P}(\phi)\check{A}(\phi) = -I \qquad \text{on } \mathbb{C}^n, \tag{7.8}$$

and $\|\check{P}(\phi)\|_{\mathbb{C}^{n\times n}} \leq M$.

c. *The operator Lyapunov equation (7.6) corresponding to* $A = \Lambda_{\check{A}}$ *has a positive solution which is the multiplicative operator with symbol* $\check{P} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$.

Proof Throughout this proof we denote $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n) = Z$.
a. $\Rightarrow$ *c.* Since the semigroup is exponentially stable the Lyapunov equation (7.6) has the unique solution given by (7.7). Using the fact that the semigroup is the multiplication operator $\Lambda_{e^{\check{A}t}}$ we find that

$$(Pz)(\phi) = \int_0^\infty e^{\check{A}(\phi)^* t} e^{\check{A}(\phi)t} z(\phi) dt, \qquad \phi \in \partial\mathbb{D},$$

where the equality is in Z, i.e, the equality holds pointwise for almost all $\phi \in \partial\mathbb{D}$.
Thus P is a multiplication operator with symbol $\check{P}$ given by

$$\check{P}(\cdot) = \int_0^\infty e^{\check{A}(\cdot)^* t} e^{\check{A}(\cdot)t} dt.$$

Since $\check{A}$ is in $\boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$, it is easy to see that this symbol is measurable. Furthermore, since P is bounded on Z, it follows that $\check{P}$ is bounded on the unit circle, see Property A.6.30. Thus $\check{P} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$.

c. $\Rightarrow$ *b.* This is trivial.

b. $\Rightarrow$ *a.* Taking $\phi \in \partial\mathbb{D}$ such that (7.8) has a solution bounded in norm by M, we find that

$$\int_0^\infty \|e^{\check{A}(\phi)t} v\|^2 dt = v^* \check{P}(\phi) v \leq M\|v\|^2, \qquad v \in \mathbb{C}^n. \tag{7.9}$$

By assumption, the above inequality holds for almost all $\phi \in \partial\mathbb{D}$. Hence for an arbitrary $z \in Z$, we have

$$\int_0^{2\pi} \int_0^\infty \|e^{\check{A}(e^{j\theta})t} z(e^{j\theta})\|^2 dt d\theta \leq \int_0^{2\pi} M\|z(e^{j\theta})\|^2 d\theta = 2\pi M \|z\|_Z^2.$$

Thus by Fubini's theorem A.5.27 we conclude that

$$\begin{aligned} \int_0^\infty \|\Lambda_{e^{\check{A}t}} z\|_Z^2 dt &= \int_0^\infty \|e^{\check{A}t} z\|_Z^2 dt \\ &= \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \|e^{\check{A}(e^{j\theta})t} z(e^{j\theta})\|^2 d\theta dt \leq M\|z\|_Z^2. \end{aligned}$$

From Lemma 7.1.2, we conclude that $\Lambda_{\check{A}}$ generates an exponentially stable semigroup. ■

We remark that the pointwise interpretation in Theorem 7.1.4.b. is useful in simple cases where one can find an explicit solution to the pointwise Lyapunov equation as in Exercise 7.8.d.

A different necessary and sufficient condition for exponential stability is given in the following theorem.

Theorem 7.1.5 *Let A be the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z. Then $T(t)$ is exponentially stable if and only if $(sI - A)^{-1} \in \boldsymbol{H}_\infty(\mathcal{L}(Z))$.*

Proof *Necessity.* By assumption, we know that the C_0-semigroup satisfies $\|T(t)\| \leq Me^{\omega t}$ for some $\omega < 0$. Lemma 2.1.14 implies that $\overline{\mathbb{C}_0^+}$ is contained in the resolvent set of A and, furthermore, for $s \in \overline{\mathbb{C}_0^+}$

$$\|(sI - A)^{-1}\| \leq \frac{M}{\mathrm{Re}(s) - \omega} \leq \frac{M}{-\omega}.$$

So using Lemma A.4.8.c, we conclude that $(sI - A)^{-1} \in \boldsymbol{H}_\infty(\mathcal{L}(Z))$.

Sufficiency. Suppose that the C_0-semigroup satisfies $\|T(t)\| \leq Me^{(\omega-\varepsilon)t}$ for some positive constants ω and ε. It is easy to see that $e^{-\omega t}T(t)z$ is an element of $\boldsymbol{L}_2([0,\infty); Z)$ for every $z \in Z$. Furthermore, the Laplace transform of $e^{-\omega t}T(t)z$ equals $((s+\omega)I - A)^{-1}z$ (see Property A.6.2.e and Lemma 2.1.14). So, from the Paley-Wiener Theorem A.6.21 we conclude that

$$((s+\omega)I - A)^{-1}z \in \boldsymbol{H}_2(Z).$$

Now, by assumption, $(sI - A)^{-1} \in \boldsymbol{H}_\infty(\mathcal{L}(Z))$, and by Theorem A.6.26.b it follows that

$$(sI - A)^{-1}((s+\omega)I - A)^{-1}z \in \boldsymbol{H}_2(Z).$$

Using the resolvent equation (A.4.5), we conclude that $(sI - A)^{-1}z \in \boldsymbol{H}_2(Z)$, since

$$(sI - A)^{-1}z = ((s+\omega)I - A)^{-1}z + \omega(sI - A)^{-1}((s+\omega)I - A)^{-1}z. \quad (7.10)$$

But the Laplace transform of $T(t)z$ is $(sI - A)^{-1}z$ and so by the Paley-Wiener Theorem A.6.21, we have that $T(t)z \in \boldsymbol{L}_2([0,\infty); Z)$. Finally, Lemma 7.1.2 shows that $T(t)$ is exponentially stable. ■

In finite dimensions, exponential stability can be determined from the spectrum of the operator, since

$$\sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A)) = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t} = \omega_0. \quad (7.11)$$

In Lemma 2.1.10 this was shown to be true for bounded generators on a Hilbert space. However, in general, only the following inequality holds (Lemma 2.1.14):

$$\sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A)) \leq \lim_{t\to\infty} \frac{\log \|T(t)\|}{t} = \omega_0, \quad (7.12)$$

which may be strict as is illustrated in the next example.

Example 7.1.6 Consider the Hilbert space $Z := \{z \in \boldsymbol{L}_2(0,\infty) \mid z$ is absolutely continuous on every finite interval and $\frac{dz}{dx}$ satisfies $\int_0^\infty |\frac{dz}{dx}(x)|^2 e^{-2x}dx < \infty\}$ with the inner product

$$\langle z_1, z_2\rangle := \int_0^\infty z_1(x)\overline{z_2}(x)dx + \int_0^\infty \frac{dz_1}{dx}(x)\overline{\frac{dz_2}{dx}}(x)e^{-2x}dx.$$

On this Hilbert space, we consider the left shift operator given by

$$(T(t)z)(x) := z(t+x) \qquad \text{for } z \in Z \text{ and } x \geq 0.$$

For every $t \geq 0$, $T(t)$ is a bounded linear operator, since

$$\begin{aligned}\|T(t)z\|^2 &= \int_0^\infty |z(t+x)|^2 dx + \int_0^\infty |\frac{dz}{dx}(t+x)|^2 e^{-2x}dx \\ &= \int_t^\infty |z(x)|^2 dx + e^{2t}\int_t^\infty |\frac{dz}{dx}(x)|^2 e^{-2x}dx \leq e^{2t}\|z\|^2.\end{aligned}$$

As in Example 2.1.4, we can show that $T(t)$ is a C_0-semigroup on Z. From the above, we see that the growth bound of $T(t)$ is less than or equal to one. We shall show that it is in fact equal to one. Consider the sequence for a fixed $t \geq 0$

$$z_n(x) = \begin{cases} 0 & \text{for } x \in [0,t) \\ n(x-t) & \text{for } x \in [t, t+\frac{1}{n}) \\ 1 & \text{for } x \in [t+\frac{1}{n}, t+1-\frac{1}{n}) \\ n(t-x+1) & \text{for } x \in [t+1-\frac{1}{n}, t+1) \\ 0 & \text{for } x \in [t+1,\infty). \end{cases}$$

We deduce the following estimates:

$$\begin{aligned}\|z_n\|_Z^2 &= \int_t^{t+\frac{1}{n}} n^2 e^{-2x}dx + \int_{t+1-\frac{1}{n}}^{t+1} n^2 e^{-2x}dx + \int_t^{t+1} |z_n(x)|^2 dx \\ &\leq \int_t^{t+\frac{1}{n}} n^2 e^{-2x}dx + \int_{t+1-\frac{1}{n}}^{t+1} n^2 e^{-2x}dx + 1 \quad \text{since } |z_n(x)| \leq 1 \\ &= \frac{n^2}{2}e^{-2t}\left[1 - e^{-\frac{2}{n}} + e^{-2+\frac{2}{n}} - e^{-2}\right] + 1; \end{aligned} \tag{7.13}$$

$$\begin{aligned}\|T(t)z_n\|_Z^2 &= \int_t^\infty |z_n(x)|^2 dx + e^{2t}\int_t^\infty |\frac{dz_n}{dx}(x)|^2 dx \\ &\geq e^{2t}\int_t^\infty |\frac{dz_n}{dx}(x)|^2 dx \\ &= \frac{n^2}{2}\left[1 - e^{-\frac{2}{n}} + e^{-2+\frac{2}{n}} - e^{-2}\right]. \end{aligned} \tag{7.14}$$

From these inequalities, it follows that

$$\begin{aligned}\frac{\|T(t)z_n\|_Z^2}{\|z_n\|_Z^2} &\geq \frac{\frac{n^2}{2}[1-e^{-\frac{2}{n}}+e^{-2+\frac{2}{n}}-e^{-2}]}{\frac{n^2}{2}e^{-2t}[1-e^{-\frac{2}{n}}+e^{-2+\frac{2}{n}}-e^{-2}]+1} \\ &= \frac{[1-e^{-\frac{2}{n}}+e^{-2+\frac{2}{n}}-e^{-2}]e^{2t}}{[1-e^{-\frac{2}{n}}+e^{-2+\frac{2}{n}}-e^{-2}]+\frac{2e^{2t}}{n^2}}.\end{aligned}$$

For n converging to infinity this last expression converges to e^{2t}, and so the norm of $T(t)$ is larger than or equal to e^t. We had already shown that the norm is less than or equal to e^t. Hence the norm equals e^t, and so the growth bound of $T(t)$ is one.

Next we calculate the infinitesimal generator of $T(t)$. Let $\alpha > 1$ and $z_0 \in Z$. From Lemma 2.1.14, we have that

$$\begin{aligned} h(x) &:= ((\alpha I - A)^{-1}z_0)(x) \\ &= \int_0^\infty z_0(t+x)e^{-\alpha t}dt = e^{\alpha x}\int_x^\infty e^{-\alpha\tau}z_0(\tau)d\tau. \end{aligned} \tag{7.15}$$

Furthermore, the range of $(\alpha I - A)^{-1}$ equals the domain of A. For h in the range of $(\alpha I - A)^{-1}$, it is easily seen that

$$\frac{dh}{dx}(x) = \alpha h(x) - z_0(x), \tag{7.16}$$

which implies that $h \in \{z \in Z \mid \frac{dz}{dx} \in Z\}$. On the other hand, for $h \in \{z \in Z \mid \frac{dz}{dx} \in Z\}$, we define the function z_0 by

$$z_0 := \alpha h - \frac{dh}{dx}. \tag{7.17}$$

Since h and $\frac{dh}{dx}$ are in Z, we have that $z_0 \in Z$. The solution of (7.17) is given by

$$h(x) = e^{\alpha x}h(0) - \int_0^x e^{\alpha(x-\tau)}z_0(\tau)d\tau. \tag{7.18}$$

From this, we see that $h(0)$ is given by

$$h(0) = e^{-\alpha x}h(x) + \int_0^x e^{-\alpha\tau}z_0(\tau)d\tau. \tag{7.19}$$

We shall show that $\lim_{x\to\infty} e^{-\alpha x}h(x) = 0$ by considering the following relationships:

$$\begin{aligned} |h(x)|^2 - |h(0)|^2 &= \int_0^x h(\tau)\overline{\frac{dh}{dx}}(\tau)d\tau + \int_0^x \frac{dh}{dx}(\tau)\overline{h(\tau)}d\tau \\ &\leq 2\left[\int_0^x |h(\tau)|^2 d\tau \int_0^x |\frac{dh}{dx}(\tau)|^2 d\tau\right]^{\frac{1}{2}} \end{aligned} \tag{7.20}$$

by the Cauchy-Schwarz inequality (A.2.1).

This last expression is uniformly bounded in x since $h, \frac{dh}{dx} \in Z$. So $|h(x)|$ is uniformly bounded, and taking the limit in (7.19) as x goes to infinity gives that

$h(0) = \int_0^\infty e^{-\alpha\tau} z_0(\tau) d\tau$, since $\alpha > 0$. Substituting this in (7.18) yields

$$\begin{aligned} h(x) &= e^{\alpha x} \int_0^\infty e^{-\alpha\tau} z_0(\tau) d\tau - \int_0^x e^{\alpha(x-\tau)} z_0(\tau) d\tau \\ &= e^{\alpha x} \int_x^\infty e^{-\alpha\tau} z_0(\tau) d\tau, \end{aligned}$$

and comparing this with (7.15), we see that h is an element of the range of $(\alpha I - A)^{-1}$. From (7.16), it follows that $Ah = \frac{dh}{dx}$. Summarizing, we have shown that

$$Ah = \frac{dh}{dx} \qquad \text{for } h \in \boldsymbol{D}(A) = \{z \in Z \mid \tfrac{dz}{dx} \in Z\}. \tag{7.21}$$

Now we show that every λ with real part larger than zero is in the resolvent set of the infinitesimal generator A. For $\lambda \in \mathbb{C}_0^+$, we introduce the following operator:

$$(Q_\lambda z)(x) := e^{\lambda x} \int_x^\infty e^{-\lambda\tau} z(\tau) d\tau \qquad \text{for } z \in Z.$$

We show that $Q_\lambda \in \mathcal{L}(Z)$. Let $z \in Z$ and $h := Q_\lambda z$. It is easily seen that h is absolutely continuous on every finite interval of $[0, \infty)$ and

$$\frac{dh}{dx}(x) = \lambda h(x) - z(x). \tag{7.22}$$

Combining equations (7.20) and (7.22) shows that

$$\begin{aligned} &|h(x)|^2 - |h(0)|^2 \\ &\quad= \int_0^x h(\tau)\overline{\lambda h(\tau)} d\tau + \int_0^x \lambda h(\tau)\overline{h(\tau)} d\tau - \\ &\qquad \int_0^x h(\tau)\overline{z(\tau)} d\tau - \int_0^x z(\tau)\overline{h(\tau)} d\tau \\ &\quad= 2\mathrm{Re}(\lambda) \int_0^x |h(\tau)|^2 d\tau - \int_0^x h(\tau)\overline{z(\tau)} d\tau - \int_0^x z(\tau)\overline{h(\tau)} d\tau. \end{aligned}$$

From the definition of h, it follows easily that $\lim_{x\to\infty} h(x) = 0$. Hence for sufficiently large x we have that

$$\begin{aligned} 2\mathrm{Re}(\lambda) \int_0^x |h(\tau)|^2 d\tau &\leq \int_0^x h(\tau)\overline{z(\tau)} d\tau + \int_0^x z(\tau)\overline{h(\tau)} d\tau \\ &\leq 2\left[\int_0^x |h(\tau)|^2 d\tau \int_0^x |z(\tau)|^2 d\tau\right]^{\frac{1}{2}}. \end{aligned}$$

Hence

$$[\mathrm{Re}(\lambda)]^2 \int_0^x |h(\tau)|^2 d\tau \leq \int_0^x |z(\tau)|^2 d\tau,$$

and so $h \in \boldsymbol{L}_2(0, \infty)$ and $\|h\|_{\boldsymbol{L}_2(0,\infty)} \leq \frac{1}{\mathrm{Re}(\lambda)} \|z\|_{\boldsymbol{L}_2(0,\infty)}$. Using the fact that $z \in Z$ and equation (7.22) shows that $h \in Z$ and $\|Q_\lambda z\|_Z = \|h\|_Z \leq \gamma \|z\|_Z$ for some $\gamma > 0$, which proves the assertion that $Q_\lambda \in \mathcal{L}(Z)$ for every $\lambda \in \mathbb{C}_0^+$.

Finally, we show that Q_λ is the inverse of $(\lambda I - A)$. Since $h := Q_\lambda z$ and z are elements of Z, we have from (7.22) that $h \in \boldsymbol{D}(A)$, and with (7.21) we conclude that $(\lambda I - A)Q_\lambda = I$. It remains to calculate $Q_\lambda(\lambda I - A)$. For $z \in Z$, consider

$$\begin{aligned}(Q_\lambda(\lambda I - A)z)(x) &= e^{\lambda x}\int_x^\infty e^{-\lambda\tau}[\lambda z(\tau) - \frac{dz}{d\tau}(\tau)]d\tau \\ &= \lambda e^{\lambda x}\int_x^\infty e^{-\lambda\tau} z(\tau)d\tau - \lambda e^{\lambda x}\int_x^\infty e^{-\lambda\tau}z(\tau)d\tau + \\ &\quad z(x) \qquad \text{using integration by parts} \\ &= z(x).\end{aligned}$$

So every $\lambda \in \mathbb{C}_0^+$ is in the resolvent set of A, but the growth bound of $T(t)$ is one. Hence A does not satisfy (7.11). ■

In Exercise 7.9, an example is given for which the difference between the growth bound of the C_0-semigroup and $\sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A))$ is infinite. Fortunately, most examples encountered in applications do achieve equality in (7.12). We introduce the terminology *spectrum determined growth assumption* for this case, i.e., when

$$\sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A)) = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t} = \omega_0. \tag{7.23}$$

We remark that the spectrum determined growth assumption does not imply that $\|T(t)\| \le Me^{\omega_0 t}$, but only that for every $\omega > \sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A))$, we can find an $M_\omega > 0$ such that $\|T(t)\| \le M_\omega e^{\omega t}$, see Exercise 2.1.

In Lemma 2.1.10. we prove that when A is a bounded operator the spectrum determined growth assumption is satisfied. For another proof we refer to Exercise 7.3. Hence our class of spatial invariant systems satisfies the spectrum determined growth assumption. In Theorem 3.2.8.d we proved that if A is a Riesz-spectral operator, then it satisfies the spectrum determined growth assumption. In Theorem **??** we will show that our other class of examples, the delay differential equations, also satisfies it.

Now we derive necessary and sufficient conditions for a C_0-semigroup to satisfy the spectrum determined growth assumption.

Theorem 7.1.7 *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on the Hilbert space Z and define*

$$\omega_\sigma := \sup(\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)). \tag{7.24}$$

Then $T(t)$ satisfies the spectrum determined growth assumption if and only if for every $\omega > \omega_\sigma$ $((s+\omega)I - A)^{-1} \in \boldsymbol{H}_\infty(\mathcal{L}(Z))$.

Proof *Necessity.* Suppose that $T(t)$ satisfies the spectrum determined growth assumption. Then for every $\omega > \omega_\sigma$, $e^{-\omega t}T(t)$ is an exponentially stable semigroup with infinitesimal generator $(-\omega I + A)$ (see Exercise 2.3). From Theorem 7.1.5, we conclude that $(sI - (-\omega I + A))^{-1} = ((s+\omega)I - A)^{-1} \in \boldsymbol{H}_\infty(\mathcal{L}(Z))$.

Sufficiency. Let ω be larger than ω_σ. From Theorem 7.1.5, it follows that the operator $-\omega I + A$ generates an exponentially stable semigroup. This semigroup is given by $e^{-\omega t}T(t)$ (see Exercise 2.3). Hence we conclude that the growth bound of $T(t)$ is less than ω. This holds for all $\omega > \omega_\sigma$, and so the growth bound of $T(t)$ is less than or equal to ω_σ. Together with equation (7.12), this implies that the growth bound equals ω_σ. ∎

Summarizing the results in this section, for almost all the examples considered in this book that satisfy $\sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A)) = -\beta_0$ (spatially invariant systems, retarded delay differential equations and Riesz-spectral operators), we may deduce the β-exponential stability of the semigroup for any $\beta > -\beta_0$.

7.2 Weak and strong stability

In the previous section we studied the exponential stability of the semigroup $T(t)$. For the abstract differential equation (7.1) this means that for all $z_0 \in Z$ the solution $z(t) = T(t)z_0$ converges to zero exponentially as $t \to \infty$. A weaker property is when the solution to the abstract homogeneous Cauchy initial value problem (2.29) is *asymptotically stable*, i.e., $z(t) \to 0$ as $t \to \infty$ for all $z_0 \in Z$. The convergence rate depends on the initial condition, but this would be acceptable for many applications. Consequently, in this section, we examine weaker concepts of stability for semigroups.

Definition 7.2.1 The C_0-semigroup $T(t)$ on the Hilbert space Z is *strongly stable* if for every $z \in Z$, $T(t)z$ converges to zero as t tends to ∞.

$T(t)$ is *weakly stable* if for every $z_1, z_2 \in Z$, $\langle z_1, T(t)z_2\rangle$ converges to zero as t tends to ∞. ∎

One example of a semigroup that is strongly, but not exponentially stable is the left shift semigroup from Example 2.1.4. Its adjoint is only weakly stable.

Example 7.2.2 Let $Z = \boldsymbol{L}_2(0,\infty)$ and recall the shift semigroup

$$(T(t)f)(x) = f(t+x) \qquad \text{for } f \in Z, x \geq 0.$$

We show that it is strongly stable:

$$\|T(t)f\|_2^2 = \int_0^\infty |f(t+x)|^2 dx = \int_t^\infty |f(\alpha)|^2 d\alpha \to 0 \quad \text{as } t \to \infty.$$

However, for $f(x) = 1/(1+x)$ we see that $\|T(t)f\|_2^2 = 1/(1+t)$, which can never be bounded by $Me^{-\omega t}$ for some positive M and ω. Hence the semigroup is not exponentially stable.

The dual semigroup is given by

$$\left(T^*(t)g\right)(x) = \begin{cases} g(x-t) & x > t \\ 0 & x \in [0,t] \end{cases}$$

and

$$\|T^*(t)g\|_2^2 = \int_t^\infty |g(x-t)|^2 dx = \int_0^\infty |g(\alpha)|^2 d\alpha = \|g\|_2^2.$$

So $T^*(t)$ is not strongly stable, but is is weakly stable, since

$$\langle T^*(t)g, f\rangle = \langle g, T(t)f\rangle \to 0 \qquad \text{as } t \to \infty$$

for all $f, g \in Z$. ■

In the previous example, we saw that the weakly stable semigroup was bounded. This a general result.

Lemma 7.2.3 *Let $T(t)$ be a C_0-semigroup on the Hilbert space Z. If $T(t)$ is strongly stable, then it is weakly stable. Furthermore, if it is weakly stable, then it is bounded in norm for $t \geq 0$.*

Proof The first assertion follows directly from the definition, and so we concentrate on the second one. Let z be an arbitrary element of Z, then since $T(t)$ is weakly stable, the sequence $\{T(n)z, n \in \mathbb{Z}\}$ is weakly convergent to zero. By Lemma A.3.36 we have that $\|T(n)z\|$ is uniformly bounded. This holds for each $z \in Z$. The Uniform Boundedness Theorem A.3.19 shows that, in fact, $\|T(n)\| \leq M_1$ for some constant $M_1 > 0$.

Now given $t > 0$ there exits an integer n such that $0 \leq t - n < 1$, and

$$\sup_{t\geq 0} \|T(t)\| = \sup_{t\geq 0} \|T(t-n)T(n)\| \leq \sup_{0\leq s<1} \|T(s)\|M_1 \leq M_2 M_1 = M < \infty,$$

where we have used Theorem 2.1.7. ■

If the resolvent of the infinitesimal generator is compact, weak stability of the semigroup implies strong stability.

Lemma 7.2.4 *If $T(t)$ is a weakly stable semigroup and A has compact resolvent, then $T(t)$ is strongly stable.*

Proof *a*. By Lemma 7.2.3 there exists an $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. Since $T(n)z$ is weakly convergent and $(\beta I - A)^{-1}$ is compact, $(\beta I - A)^{-1}T(n)z$ has a convergent subsequence $(\beta I - A)^{-1}T(n_j)z \to 0$ as $j \to \infty$ (Definition A.3.21). Choose $\varepsilon > 0$, and let j be such that $\|(\beta I - A)^{-1}T(n_j)z\| \leq \varepsilon/M$, then for $t \geq n_j$ we have that

$$\begin{aligned}
\|T(t)(\beta I - A)^{-1}z\| &= \|(\beta I - A)^{-1}T(t)z\| \\
&= \|(\beta I - A)^{-1}T(t-n_j)T(n_j)z\| \\
&= \|T(t-n_j)(\beta I - A)^{-1}T(n_j)z\| \\
&\leq \|T(t-n_j)\|\|(\beta I - A)^{-1}T(n_j)z\| \leq \varepsilon.
\end{aligned}$$

So $T(t)(\beta I - A)^{-1}z \to 0$ as $t \to \infty$.

b. Since $\boldsymbol{D}(A)$ is dense in Z for every $z_0 \in Z$ and every $\varepsilon > 0$ there there exists $z_1 \in \boldsymbol{D}(A)$ with $\|z_0 - z_1\| \leq \varepsilon/M$. Define $z = (\beta I - A)z_1$ and choose $t_1 > 0$ such that $\|T(t)(\beta I - A)^{-1}z\| \leq \varepsilon$ for $t \geq t_1$ which by part a. is possible. Then for $t \geq t_1$

$$\begin{aligned}
\|T(t)z_0\| &= \|T(t)(z_0 - z_1 + z_1)\| \\
&\leq \|T(t)\|\|z_0 - z_1\| + \|T(t)(\beta I - A)^{-1}z\| \\
&\leq M\frac{\varepsilon}{M} + \varepsilon = 2\varepsilon.
\end{aligned}$$

Since this holds for any $\varepsilon > 0$, this shows that $\|T(t)z_0\| \to 0$ as $t \to \infty$ for all $z_0 \in Z$. ■

The following result provides necessary and sufficient conditions for the strong stability of spatially invariant systems.

Theorem 7.2.5 *Suppose that $\check{A} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$. The C_0- semigroup generated by $\Lambda_{\check{A}}$ is strongly stable if and only if the following two conditions hold:*

a. *The semigroup generated by $\Lambda_{\check{A}}$ is uniformly bounded in norm for $t \geq 0$, i.e.,*

$$\sup_{t\geq 0} \operatorname*{ess\,sup}_{\phi\in\partial\mathbb{D}} \|e^{\check{A}(\phi)t}\|_{\mathbb{C}^{n\times n}} < \infty, \tag{7.25}$$

b. *The finite-dimensional semigroups $\{e^{\check{A}(\phi)t} \mid \phi \in \partial\mathbb{D}\}$ are exponentially stable except for a set of measure zero.*

Proof *Sufficiency:* Let z be an element of $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$. By the uniform boundedness condition (7.25) we know that there exists a set $\boldsymbol{\Omega} \subset \partial\mathbb{D}$ and an $M \geq 0$, independent of k, such that the complement of $\boldsymbol{\Omega}$ has measure zero and

$$\|e^{\check{A}(\phi)k}z(\phi)\|^2_{\mathbb{C}^n} \leq M^2\|z(\phi)\|^2_{\mathbb{C}^n} \tag{7.26}$$

for all $\phi \in \boldsymbol{\Omega}$ and $k \in \mathbb{N}$.

Furthermore, we know that for almost every $\phi \in \partial\mathbb{D}$

$$\lim_{k\to\infty} \|e^{\check{A}(\phi)k}z(\phi)\|^2_{\mathbb{C}^n} = 0. \tag{7.27}$$

From (7.26) and (7.27) we see that we may apply the Lebesgue Dominated Convergence Theorem A.5.26 to conclude that

$$\lim_{k\to\infty} \|e^{\check{A}k}z\|^2_{\boldsymbol{L}^2(\partial\mathbb{D},\mathbb{C}^n)} = \lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \|e^{\check{A}(e^{\jmath\theta})k}z(e^{\jmath\theta})\|^2_{\mathbb{C}^n} d\theta = 0.$$

Using the uniform boundedness condition (7.25) once more, we see that for all $t \geq k$ there holds

$$\|e^{\check{A}t}z\|^2_{\boldsymbol{L}^2(\partial\mathbb{D},\mathbb{C}^n)} \leq M\|e^{\check{A}k}z\|^2_{\boldsymbol{L}^2(\partial\mathbb{D},\mathbb{C}^n)}$$

Thus

$$\lim_{t\to\infty} \|e^{\check{A}t}z\|^2_{\boldsymbol{L}^2(\partial\mathbb{D},\mathbb{C}^n)} = 0,$$

which proves the sufficiency part.

Necessity: If the semigroup generated by $\Lambda_{\check{A}}$ is strongly stable, then by Lemma 7.2.3 it is bounded in norm, i.e. $\sup_{t\geq 0}\|e^{\check{A}(\cdot)t}\| \leq M$ for some $M > 0$. From the definition of the norm, Definition A.6.29, we conclude that (7.25) holds. Assume now that there exists a subset Ψ of $\partial\mathbb{D}$ of positive measure such that $e^{\check{A}(\phi)t}$ is not exponentially stable for $\phi \in \Psi$.

Let $\{z_1, \cdots, z_n\}$ be a basis for $\mathbb{C}^n$, and define Ψ_k as

$$\Psi_k = \{\phi \in \partial\mathbb{D} \mid \inf_{t\geq 0} \|e^{\breve{A}(\phi)t} z_k\| > 0\}, \quad k = 1, \cdots, n.$$

This is measurable, since by the uniform boundedness in norm of the semigroup we have that $\Psi_k = \{\phi \in \partial\mathbb{D} \mid \inf_{m\in\mathbb{N}} \|e^{\breve{A}(\phi)m} z_k\| > 0\}$. This set is measurable by Property A.5.5.

We claim that at least one of Ψ_k has a positive measure. For if the measure of Ψ_k were zero for all $k \in \{1, \cdots, n\}$, then, using the fact that $\{z_k, k = 1, \cdots, n\}$ is a basis of $\mathbb{C}^n$, we would conclude that $e^{\breve{A}(\phi)t}$ is exponentially stable for almost all $\phi \in \partial\mathbb{D}$. However, our assumption above excludes this case, and so there does exist a Ψ_k with positive measure. Without loss of generality, we assume that this is Ψ_1.

Now we define $z \in \boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$ by

$$z(\phi) = \begin{cases} 0 & \phi \notin \Psi_1 \\ z_1 & \phi \in \Psi_1. \end{cases}$$

The definition of Ψ_1 implies that z is not equal to zero in $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$, and that $\|e^{\breve{A}t} z\|$ is bounded away from zero. This contradicts the assumption of strong stability of the semigroup. Thus the finite-dimensional semigroups $\{e^{\breve{A}(\phi)t} \mid \phi \in \partial\mathbb{D}\}$ are exponentially stable except for a set of measure zero. ■

If $\breve{A}(\phi)$ is continuous in ϕ on $\partial\mathbb{D}$, then the ess sup in (7.25) can be replaced by max. However, even when $\breve{A}(\phi)$ is continuous the exponential stability of $e^{\breve{A}(\phi)t}$ for almost all $\phi \in \partial\mathbb{D}$ is not sufficient to conclude strong stability. For a counterexample see Exercise 7.8. Hence (7.25) is necessary.

For a given $\breve{A}$ condition b. of Theorem 7.2.5 can be checked by calculating the eigenvalues of $\breve{A}(\phi)$ for every $\phi \in \partial\mathbb{D}$. However, to check condition (7.25) can be much harder. The following lemma gives a sufficient condition formulated in terms of the eigenvalues.

Lemma 7.2.6 *Suppose that $\breve{A} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$. For $\phi \in \partial\mathbb{D}$ we denote the n eigenvalues of $\breve{A}(\phi)$ by $\{\lambda_k(\phi), k = 1, \cdots, n\}$.*

The C_0-semigroup generated by $\Lambda_{\breve{A}}$ is strongly stable if the following two conditions hold:

a. For almost all $\phi \in \partial\mathbb{D}$ the real part of $\lambda_k(\phi)$, $k = 1, \cdots, n$ is negative;

b. There exists a $\delta > 0$ such that for all $k, l \in \{1, \cdots, n\}$, $k \neq l$ and almost all $\phi \in \partial\mathbb{D}$ there holds

$$|\lambda_k(\phi) - \lambda_l(\phi)| \geq \delta.$$

Proof The first condition implies that the finite dimensional semigroups $e^{\breve{A}(\phi)t}$ are exponentially stable for almost every $\phi \in \mathbb{D}$. Hence condition b of Theorem 7.2.5 is satisfied. It remains to show that (7.25) holds.

Suppose that conditions a and b hold and let Ω denote the subset of $\partial\mathbb{D}$ for which both conditions hold. Note that this implies that the complement of Ω has measure zero. Let ϕ_0 be an arbitrary element of Ω, and let δ be the positve number of item b. We show that there exits a constant M, depending only on $\|\check{A}\|_\infty$, n, and δ such that for all $\phi_0 \in \Omega$ and $t \geq 0$ there holds $\|e^{\check{A}(\phi_0)t}\| \leq M$.

By assumption, the finite dimensional semigroup $e^{\check{A}(\phi_0)t}$ is stable. This finite-dimensional semigroup $e^{\check{A}(\phi_0)t}$ is the inverse Laplace transform of $(sI_n - \check{A}(\phi_0))^{-1}$, where I_n denotes the identity matrix on $\mathbb{C}^n$.

We study an arbitrary element of $(sI_n - \check{A}(\phi_0))^{-1}$. By Cramer's rule we know that such an element has the form

$$\frac{p(s,\phi_0)}{\det(sI_n - \check{A}(\phi_0))} = \frac{p(s,\phi_0)}{\Pi_{k=1}^n (s - \lambda_k(\phi_0))},$$

where p is the determinant of some sub-matrix of $sI_n - \check{A}(\phi_0)$. Note that condition b implies that $\check{A}(\phi_0)$ has no multiple eigenvalues. Thus this rational function has a partial fraction expansion of the form

$$\frac{p(s,\phi_0)}{\Pi_{k=1}^n (s - \lambda_k(\phi_0))} = \sum_{k=1}^n \frac{a_k(\phi_0)}{(s - \lambda_k(\phi_0))}, \tag{7.28}$$

with

$$a_k(\phi_0) = \frac{p(\lambda_k(\phi_0),\phi_0)}{\Pi_{m=1,m\neq k}^n (\lambda_m(\phi_0)) - \lambda_k(\phi_0))}.$$

By Exercise **??** we know that the $\lambda_k(\phi_0)$ are bounded independently of $\phi_0 \in \Omega$ and $k \in \{1, \cdots, n\}$. From the same exercise it follows that $p(\lambda_k(\phi_0), \phi_0)$ is bounded by a constant depending on n and $\|\check{A}\|_\infty$. Hence the numerator of $a_k(\phi_0)$ is bounded by a constant depending only on n and $\|\check{A}\|_\infty$.

By condition b and the definition of δ, the denominator is larger than or equal to δ^{n-1}. Combining this with the other estimate, we see that $a_k(\phi_0)$ is bounded by a constant M_0 depending only on n, $\|\check{A}\|_\infty$ and δ.

The expression in (7.28) represents the Laplace transform of an arbitrary element of $(sI_n - \check{A}(\phi_0))^{-1}$. Hence the corresponding element of the matrix $e^{\check{A}(\phi_0)t}$ is

$$\sum_{k=1}^n a_k(\phi_0) e^{\lambda_k(\phi_0))t}.$$

Combining this with condition a, we find that

$$|\sum_{k=1}^n a_k(\phi_0) e^{\lambda_k(\phi_0))t}| \leq \sum_{k=1}^n |a_k(\phi_0)||e^{\lambda_k(\phi_0)t}| \leq \sum_{k=1}^n M_0 e^{\mathrm{Re}(\lambda_k(\phi_0))t} \leq nM_0.$$

Hence for all $t \geq 0$, and $\phi_0 \in \Omega$ every element of $e^{\check{A}(\phi_0)t}$ is bounded by nM_0 and $\|e^{\check{A}(\phi_0)t}\| \leq M$ as claimed. Since the complement of Ω is a set of measure zero, Theorem 7.2.5 completes the proof. ■

For Riesz-spectral operators weak and strong stability are equivalent and can be determined from the spectrum.

Lemma 7.2.7 *For the Riesz-spectral operator A with eigenvalues $\{\lambda_n, n \geq 1\}$ the following assertions are equivalent:*

a. A generates a strongly stable semigroup;

b. A generates a weakly stable semigroup;

c. $Re(\lambda_n) < 0$ for all $n \in \mathbb{N}$.

Proof The implication $a. \Rightarrow b.$ always holds, and so we concentrate on the other implications.

$b. \Rightarrow c.$ Suppose this does not hold. Then there exists an n_0 such the $\mathrm{Re}(\lambda_{n_0}) \geq 0$. From Theorem 3.2.8 we see that $\langle \psi_{n_0}, T(t)\phi_{n_0}\rangle = e^{\lambda_{n_0} t}$. Since this does not converge to zero, $T(t)$ is not weakly stable. This provides a contraction.

$c. \Rightarrow a.$ If all eigenvalues of A lie in the left half-plane, then by Theorem 3.2.8 we know that A generates a C_0-semigroup. For a given $z_0 \in Z$ and $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |\langle z_0, \psi_n\rangle|^2 \leq \varepsilon$. We have

$$\begin{aligned}
\|T(t)z_0\|^2 &= \|\sum_{n=1}^{\infty} e^{\lambda_n t}\langle z_0, \psi_n\rangle \phi_n\|^2 \\
&\leq M \sum_{n=1}^{\infty} |e^{\lambda_n t}\langle z_0, \psi_n\rangle|^2, \qquad \text{by (3.18)} \\
&\leq M \sum_{n=1}^{N} |e^{\lambda_n t}\langle z_0, \psi_n\rangle|^2 + M \sum_{n=N+1}^{\infty} |\langle z_0, \psi_n\rangle|^2 \\
&\leq M \sum_{n=1}^{N} |e^{\lambda_n t}\langle z_0, \psi_n\rangle|^2 + M\varepsilon, \qquad (7.29)
\end{aligned}$$

where we have used that $|e^{\lambda_n t}| \leq 1$ for all $n \geq 1$ and all $t \geq 0$. By choosing t sufficiently large, we can make the first term in (7.29) as small as we like. Hence we conclude that $T(t)z_0$ converges to zero as $t \to \infty$. ■

So for Riesz spectral operators weak and strong stability are equivalent. For our class of delay differential equations, strong and exponential stability are equivalent. To show this we need the following lemma.

Lemma 7.2.8 *Let A be the infinitesimal generator of the strongly continuous semigroup $T(t)$. If the spectrum determined growth assumption is satisfied, and there exists an eigenvalue λ_0 such that*

$$\sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\} = \mathrm{Re}(\lambda_0), \tag{7.30}$$

then the semigroup is strongly stable if and only if it is exponentially stable.

Proof We already know that exponential stability implies strong stability. So assume that $T(t)$ is strongly stable. Let ϕ_0 be an eigenvector corresponding to the eigenvalue λ_0. From Exercise 2.2 we have that $T(t)\phi_0 = e^{\lambda_0 t}\phi_0$. Since the semigroup is strongly stable, we must have $\mathrm{Re}(\lambda_0) < 0$. By equation (7.30) this implies that

$$\sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\} = \mathrm{Re}(\lambda_0) < 0.$$

Since the spectrum determined growth assumption is satisfied, we can conclude exponential stability. ∎

7.3 Exercises

7.1. Show that for a C_0-semigroup $T(t)$ on the Hilbert space Z the following assertions are equivalent:

a. $T(t)$ is exponentially stable;
b. $\lim_{t\to\infty} \|T(t)\| = 0$;
c. There exists an $\varepsilon > 0$ such that for all $z \in Z$ there holds:

$$\lim_{t\to\infty} e^{\varepsilon t}\|T(t)z\| = 0.$$

7.2. Let $T(t)$ be a C_0-semigroup with growth bound one and the supremum of $\mathrm{Re}(\lambda)$, $\lambda \in \sigma(A)$, equal to zero. Let $\omega_0, \omega_\sigma \in \mathbb{R}$ with $\omega_0 > \omega_\sigma$. Show that the following operator is a C_0-semigroup with growth bound ω_0 and $\sup(\mathrm{Re}(\lambda), \lambda \in \sigma(A))$ equal to ω_σ:

$$S(t) := e^{\omega_\sigma t} T((\omega_0 - \omega_\sigma)t).$$

7.3. Let A be a bounded operator on the Hilbert space Z. Use Theorem 7.1.7 to show that e^{At} satisfies the spectrum determined growth assumption.
Hint: Use Corollary A.4.10.

7.4. Let $T(t)$ be a C_0-semigroup on the Hilbert space Z. Prove that $T(t)$ is exponentially stable if and only if $\sigma(A) \subset \mathbb{C}_0^-$ and for every $z_0 \in Z$

$$\lim_{\rho\to\infty} \left[\sup_{\{s\in\overline{\mathbb{C}_0^+} \mid |s|>\rho\}} \|(sI - A)^{-1} z_0\| \right] = 0.$$

7.5. Let A be a self-adjoint operator on the Hilbert space Z satisfying $\langle Az, z\rangle \leq \alpha\|z\|^2$, $z \in \boldsymbol{D}(A)$.

a. For which values of $\alpha \in \mathbb{R}$ will A generate an exponentially stable C_0-semigroup?
b. For the case that $0 \in \rho(A)$ find the positive solution to the Lyapunov equation (7.6). Show that it is coercive if and only if A is a bounded operator.

7.6. Examine the stability properties of the following spatially invariant systems of the form (3.1) with the component matrices

i.

$$A_0 = \begin{bmatrix} 0 & 1 \\ -\beta & -\mu \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

and all other $A_r = 0$, where β and μ are positive numbers.

ii.

$$A_0 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and all other $A_r = 0$.

7.7. Consider the following operator on the Hilbert space H which has the orthonormal basis $\{\phi_n, n \geq 1\}$;

$$Az = -\sum_{n=1}^{\infty} \frac{1}{n} \langle z, \phi_n \rangle \phi_n, \quad z \in Z.$$

a. Show that it is a bounded operator and find an expression for the semigroup e^{At}.
b. Show that e^{At} is strongly stable, but not exponentially stable.

7.8. In this exercise it is shown that in Theorem 7.2.5 it is not sufficient to show that $e^{\check{A}(\phi)t}$ is exponentially stable for almost all $\phi \in \partial\mathbb{D}$.
Consider the spatially invariant system (3.1) with the following non zero component 2×2 matrices

$$A_0 = 2I, \quad A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = -A_1, \quad A_2 = A_{-2} = -I.$$

a. Calculate $\check{A}$ and show that all its eigenvalues and hence those of A are in the closed left half-plane.
b. Show that $e^{\check{A}(\phi)t}$ is exponentially stable for almost all $\phi \in \partial\mathbb{D}$.
c. Show that $\Lambda_{e^{\check{A}t}}$ and e^{At} are unbounded in norm and hence not strongly stable.
d. Solve the pointwise Lyapunov equation (7.8) and show that it is not bounded in norm on $\partial\mathbb{D}$.

7.9. In this exercise, we shall prove that there exists an infinitesimal generator A of the C_0-semigroup $T(t)$, with an empty spectrum but a growth bound equal to zero. Consider the linear space $Z := \{z \in \boldsymbol{L}_2(0,\infty) \mid \int_0^\infty |z(x)|^2 e^{2x^2} dx < \infty$, z is absolutely continuous on every finite interval and $\frac{dz}{dx} \in \boldsymbol{L}_2(0,\infty)\}$.

a. Show that Z is a Hilbert space with inner product given by

$$\langle z_1, z_2 \rangle := \int_0^\infty z_1(x)\overline{z_2}(x) e^{2x^2} dx + \int_0^\infty \frac{dz_1}{dx}(x) \overline{\frac{dz_2}{dx}}(x) dx.$$

b. On this space, we consider the left shift operator

$$(T(t)z)(x) = z(t+x) \qquad \text{for } z \in Z, x \geq 0.$$

Show that $T(t)$ is a bounded operator on Z with norm bound less than or equal to one.

c. Prove that $T(t)$ is a C_0-semigroup on Z.

d. Use the sequence

$$z_n(x) = \begin{cases} 0 & \text{for } x \in [0, t) \\ n(x-t) & \text{for } x \in [t, t+\frac{1}{n}) \\ 1 & \text{for } x \in [t+\frac{1}{n}, t+1-\frac{1}{n}) \\ n(t-x+1) & \text{for } x \in [t+1-\frac{1}{n}, t+1) \\ 0 & \text{for } x \in [t+1, \infty) \end{cases}$$

to prove that the norm of $T(t)$ equals one. Conclude that the growth bound of $T(t)$ is zero.

e. Prove that for $s \in \mathbb{C}_0^+$ the resolvent operator of the infinitesimal generator A is given by

$$((sI - A)^{-1}h)(x) = e^{sx} \int_x^\infty e^{-s\tau} h(\tau) d\tau. \tag{7.31}$$

f. Prove that the infinitesimal generator is given by $Ah = \frac{dh}{dx}$ with domain $\boldsymbol{D}(A) = \{h \in Z \mid \frac{dh}{dx} \in Z\}$.

g. Prove that the point spectrum of A is empty.

h. Consider the left shift semigroup on the larger state space $Z_e := \{z \in \boldsymbol{L}_2(0, \infty) \mid \int_0^\infty |z(x)|^2 e^{2x^2} dx < \infty\}$. Show that the operator defined in (7.31) is in $\mathcal{L}(Z_e)$ for all $s \in \mathbb{C}$.
Hint: Show that the shift semigroup has growth bound $-\infty$ on Z_e and use Lemma 2.1.14.

i. Prove that the spectrum of A is empty.
Hint: Show that the operator given in (7.31) is the inverse operator for all $s \in \mathbb{C}$.

7.10. Let A be the infinitesimal generator of a C_0- semigroup and let E be a self-adjoint operator in $\mathcal{L}(Z)$. Consider the Lyapunov equation

$$\langle Az_1, Xz_2 \rangle + \langle z_1, XAz_2 \rangle = \langle z_1, Ez_2 \rangle \qquad \text{for } z_1, z_2 \in \boldsymbol{D}(A). \tag{7.32}$$

Show that if $X \in \mathcal{L}(Z)$ is the unique solution of (7.32), then X is self-adjoint.

7.11. Let A be the infinitesimal generator of an exponentially stable semigroup and let Q be a self-adjoint operator in $\mathcal{L}(Z)$. Consider the Lyapunov equation for all $z_1, z_2 \in \boldsymbol{D}(A)$

$$\langle Az_1, Pz_2 \rangle + \langle Pz_1, Az_2 \rangle = -\langle z_1, Qz_2 \rangle. \tag{7.33}$$

a. Show that equation (7.33) has a solution given by

$$Pz = \int_0^\infty T^*(t) Q T(t) z dt.$$

b. Show that the solution of (7.33) is unique.
Hint: Substitute $z_1 = T(t)z_{0,1}$, $z_2 = T(t)z_{0,2}$ and integrate (7.33).

c. Show that P is nonnegative if Q is nonnegative.

d. Show that P is positive if Q is positive.

7.12. Let A be the infinitesimal generator of a C_0-semigroup and let Q be a self-adjoint operator in $\mathcal{L}(Z)$. Consider the *Lyapunov inequality*

$$\langle Az_1, Pz_1\rangle + \langle Pz_1, Az_1\rangle \le -\langle z_1, Qz_1\rangle, \qquad z_1 \in \boldsymbol{D}(A). \tag{7.34}$$

a. Let Q be a coercive operator (see Definition A.3.76). Show that $T(t)$ is exponentially stable if and only if there exists a positive $P \in \mathcal{L}(Z)$ which satisfies the Lyapunov inequality (7.34).

b. Show that the above assertion no longer holds if Q is only positive, but not coercive.

7.13. In this exercise we show that even if the Lyapunov inequality

$$\langle Az, Lz\rangle + \langle Lz, Az\rangle < 0$$

has a coercive solution, the semigroup generated by A need not be strongly stable. Take the Hilbert space

$$Z = \left\{ f : [0,\infty) \mapsto \mathbb{C} \mid \int_0^\infty |f(x)|^2 \left[e^{-x} + 1\right] dx < \infty \right\}$$

with the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)} \left[e^{-x} + 1\right] dx.$$

a. Show that

$$\frac{1}{2}\|f\|_Z^2 \le \|f\|^2_{\boldsymbol{L}_2(0,\infty)} \le \|f\|_Z^2.$$

b. Consider the modified shift semigroup

$$\begin{aligned}(T(t)f)(x) &= \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t) \end{cases} \\ &= f(x-t)\mathbb{1}_{[0,\infty]}(x-t).\end{aligned}$$

Show that its infinitesimal generator is given by

$$Af = -\frac{df}{dx}$$

with domain

$$\begin{aligned} D(A) = \{ f \in Z \mid & f \text{ is absolutely continuous} \\ & \left.\frac{df}{dx} \in Z, \text{ and } f(0) = 0 \right\}. \end{aligned}$$

Hint: See Exercise 2.7.

c. Prove that the semigroup is not strongly stable by showing that

$$\|T(t)f\|_Z^2 \geq \frac{1}{2}\|f\|_Z^2.$$

d. Show that

$$\langle Az, z\rangle + \langle z, Az\rangle < 0, \qquad z \in D(A), z \neq 0.$$

7.14. Consider the difference equation

$$z(n+1) = Az(n), \tag{7.35}$$

where $A \in \mathcal{L}(Z)$, and Z is a Hilbert space. As for the continuous-time system we introduce several stability concepts for (8.45).

Definition 7.3.1 The operator A is

a. *power stable* if there exists a $M \geq 1$ and a $\gamma \in (0, 1)$ such that

$$\|A^n\| \leq M\gamma^n \qquad \text{for all } n \in \mathbb{N}. \tag{7.36}$$

b. *strongly stable* if $A^n z \to 0$ as $n \to \infty$ for all $z \in Z$

c. *weakly stable* if $\langle z_1, A^n z_2\rangle \to 0$ as $n \to \infty$ for all $z_1, z_2 \in Z$. ■

We examine the relationships between these stability properties.

a. Show that in the above definition $a. \Rightarrow b. \Rightarrow c.$

b. Prove that if $\|A\| < 1$, then A is power stable. Show that the converse is not true.
Hint: Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

c. Show that A is power stable if and only if the spectral radius of A is less than 1.

d. Assume that (7.35) is the discretization of the continuous-time system

$$\dot{\tilde{z}}(t) = \tilde{A}\tilde{z}(t),$$

where $\tilde{A}$ is the infinitesimal generator of a C_0-semigroup $\tilde{T}(t)$. Hence $z(n) = \tilde{z}(n\tau)$, where τ is the discretization time.
Prove that $\tilde{T}(t)$ is exponentially stable if and only if its discretization is power stable.

7.15. Consider the difference equation (7.35). Assume that for every $z_0 \in Z$ A satisfies

$$\sum_{n=0}^{\infty} \|A^n z_0\|^2 < \infty. \tag{7.37}$$

We shall prove that this implies that A is power stable.

a. Consider the operator $Q_N : Z \to \ell_2(Z)$ defined by

$$Q_N z = (z, Az, \ldots, A^N z, 0, \ldots),$$

where $\ell_2(Z) := \{(z_0, z_1, \dots) \mid z_n \in Z \text{ and } \sum_{n=0}^{\infty} \|z_n\|^2 < \infty\}$ with norm

$$\|(z_0, z_1, \dots)\|^2 := \sum_{n=0}^{\infty} \|z_n\|^2.$$

Prove that $Q_N \in \mathcal{L}(Z, \ell_2(Z))$ and that $\{Q_N, N \geq 0\}$ is uniformly bounded in norm with respect to N. Deduce that there exists a $\gamma > 0$ such that

$$\|A^N\| < \gamma \qquad \text{for some } \gamma \text{ independent of } N.$$

b. Prove that there exists a constant C such that

$$n\|A^n z\|^2 \leq C\|z\|^2,$$

where C is independent of n.
Hint: See the proof of Lemma 7.1.2.

c. Prove that A is power stable if and only if (7.37) holds.

d. Assume that A is power stable. Show that the operator P defined by

$$Pz = \sum_{n=0}^{\infty} A^{n^*} A^n z \tag{7.38}$$

is well defined, $P \in \mathcal{L}(Z)$, $P > 0$, and P satisfies the discrete-time Lyapunov equation

$$A^* P A - P = -I. \tag{7.39}$$

e. Prove that A is power stable if and only if there exists a positive solution to equation (7.39).
Hint: Consider $V(n, z) := \langle PA^n z, A^n z\rangle$ and show that

$$\sum_{n=0}^{N-1} \|A^n z\|^2 = V(0, z) - V(N, z).$$

7.4 Notes and references

There are many concepts of stability for infinite-dimensional systems, and the most important one is exponential stability. Its relation to the existence of a positive solution to the Lyapunov equation has been proved in Datko [44], as was the useful necessary and sufficient condition of Lemma 7.1.2. The latter has been generalized from Hilbert to Banach spaces and from $\boldsymbol{L}_2$- to $\boldsymbol{L}_p$-spaces in Pazy [123]. Indeed, many of the results in this chapter also hold on Banach spaces; for more details on this aspect we refer to Bensoussan et al. [19] and Pazy [125]. The fact that the spectrum of the generator need not determine the growth bound of the semigroup was demonstrated in an indirect way by an example in Hille and Phillips [79, p. 665] and by a more straightforward example in Zabczyk [174].

The counterexamples given in Example 7.1.6 and Exercise 7.9 are adapted from examples in Greiner, Voigt, and Wolff [73]. Of course, it is useful to know which generators have the property that the spectrum of the generator determines the growth bound of the semigroup; the term *spectrum determined growth assumption* for this property was introduced in Triggiani [163]. In this chapter we showed that generators that are bounded or of the Riesz-spectral have this property. The proofs presented in Theorems 7.1.5 and 7.1.7 are new and are used to show that our delay class satisfy the spectrum determined growth assumption. Of course, the latter result was well known (see Hale [77]). In Triggiani [163] it was shown that when $T(t)z$ is differentiable for all $z \in Z$ and $t > 0$ this also holds. In in Zabczyk [174] it was shown that when $T(\tau)$ is compact for some $\tau > 0$, the spectrum determined growth assumption is satisfied. Unfortunately, the spectrum determined growth assumption is not preserved under very simple perturbations (see Zabczyk [174]). More recently, in Prüss [129] and Huang [81] necessary and sufficient conditions for the spectrum determined growth assumption to hold were given. The weaker concept of strong stability has been studied in Benchimol [16]–[18], Huang [82], Balakrishnan [6], Batty [10], Arendt and Batty [1] and Batty and Phong [11] (see also Russell [142] and Slemrod [151]). Recently, Tomilov [162] and Guo, Zwart, and Curtain [76] derived new characterizations of strong stability. These results resemble those obtained for exponential stability in Section 7.1. The characterization of strong stability of Theorem 7.2.5 was given in Curtain, Iftime and Zwart [37]. The sufficiency proof as presented here is taken from Bayazit and Heymann [12].

Sylvester equations with bounded operators A, B were first studied in Rosenblum [135], Lumer and Rosenblum [101], Putnam [131], and Daleckiĭ and Kreĭn [43]. The case where A and B are unbounded generators of C_0-semigroups was considered in Freeman [62], Shaw and Lin [150] and Phóng [126]. This has since been generalized to more general unbounded operators A, B in Arendt *et al* [2].

8
Stabilizability and Detectability

8.1 Exponential stabilizability and detectability

One of the most important aspects of systems theory is that of stability and the design of feedback controls to stabilize or to enhance stability. First we define the concepts of stabilizability and detectability which are natural generalizations of the finite-dimensional concepts.

Definition 8.1.1 Suppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z and that $B \in \mathcal{L}(U, Z)$, where U is a Hilbert space. If there exists an $F \in \mathcal{L}(Z, U)$ such that $A + BF$ generates an exponentially stable C_0 -semigroup, $T_{BF}(t)$, then we say that $\Sigma(A, B, -, -)$ is *exponentially stabilizable*. If $T_{BF}(t)$ is β-exponentially stable we say that $\Sigma(A, B, -, -)$ is *β-exponentially stabilizable*. Such an operator $F \in \mathcal{L}(Z, U)$ will be called a *feedback operator*.

If $C \in \mathcal{L}(Z, Y)$ for a Hilbert space Y and there exists a $L \in \mathcal{L}(Y, Z)$ such that $A + LC$ generates an exponentially stable C_0-semigroup $T_{LC}(t)$, then we say that $\Sigma(A, -, C)$ is *exponentially detectable*. If $T_{LC}(t)$ is β-exponentially stable we say that $\Sigma(A, -, C, -)$ is *β-exponentially detectable*. Such an operator $L \in \mathcal{L}(Y, Z)$ will be called an *output injection operator*. ■

As in the definitions of controllability and observability, the state linear system $\Sigma(A, B, C, D)$ is exponentially stabilizable if $\Sigma(A, B, -, -)$ is exponentially stabilizable and similarly for β-exponential stabilizability, exponential detectability, and β-exponential detectability.

It is clear that the concepts of exponential detectability and stabilizability are dual ones; $\Sigma(A, -, C, -)$ is detectable if and only if $\Sigma(A^*, C^*, -, -)$ is stabilizable. Consequently, we need only investigate the one concept of exponential stabilizability.

We have seen that approximate controllability is a nice property for infinite-dimensional systems, which holds under very mild conditions, in contrast to exact controllability, which rarely holds. Consequently, one would expect that approximate controllability should play the role in infinite dimensions that controllability does in finite dimensions. In particular, one would expect that approximate controllability would imply stabilizability. Unfortunately, this is not true, as the following counterexample shows.

Example 8.1.2 Let $Z = \ell_2(\mathbb{N})$, the space of square-summable infinite sequences $z = (z_1, z_2, \ldots)$ with the norm $\|z\| = \sqrt{\sum_{n=1}^{\infty} |z_n|^2}$, and $U = \mathbb{C}$. Define the system operators (A, B) by

$$Az = (z_1, \frac{1}{2}z_2, \ldots, \frac{1}{n}z_n, \ldots),$$

$$Bu = (b_1 u, b_2 u, \ldots, b_n u, \ldots),$$

where $b_n \neq 0$, $\sum_{n=1}^{\infty} |nb_n|^2 < \infty$ and so $b = (b_1, b_2, \ldots, b_n, \ldots) \in \ell_2(\mathbb{N})$. Notice that A is a Riesz-spectral operator by Example 2.1.16 and Corollary 3.2.9 and since $b_n \neq 0$, $\Sigma(A, B, -)$ is approximately controllable. In Example A.3.23, it is shown that A is a compact operator. Furthermore, since B has one-dimensional range, it is also compact. So, for any $F \in \mathcal{L}(\ell_2(\mathbb{N}), \mathbb{C})$ $A + BF$ is a compact operator. Now we show that $0 \in \sigma(A + BF)$, and hence $\Sigma(A, B, -, -)$ is not exponentially stabilizable. From the Riesz representation Theorem A.3.53, all $F \in \mathcal{L}(\ell_2(\mathbb{N}), \mathbb{C})$ have the form

$$Fz = \langle z, f \rangle \qquad \text{for some } f \in \ell_2(\mathbb{N}).$$

Consider now the solutions of

$$x = (A + BF)z = Az + B\langle z, f \rangle.$$

Considering the components, we see that

$$z_n = nx_n - nb_n\langle z, f \rangle.$$

Since $(nb_n) \in \ell_2(\mathbb{N})$, and not all $(x_n) \in \ell_2(\mathbb{N})$ have the property that $(nx_n) \in \ell_2(\mathbb{N})$, we see that $A + BF$ is not boundedly invertible in $\ell_2(\mathbb{N})$. Thus $0 \in \sigma(A + BF)$. From (7.12), we conclude that the growth bound of $T_{BF}(t)$ (the semigroup generated by $A + BF$) is positive for any $F \in \mathcal{L}(\ell_2(\mathbb{N}), \mathbb{C})$, and so $\Sigma(A, B, -, -)$ is not exponentially stabilizable. ■

In the previous example, we saw that approximate controllability does not necessarily imply exponential stabilizability even for a compact A; the closed-loop operator $A + BF$ was also compact, and $0 \in \sigma(A + BF)$. A natural question that one may pose is: ”which conditions does the property of exponential stabilizability impose on the original system?” This question motivates the following theorems, which establish fundamental properties of exponentially stabilizable systems.

Throughout this section, we shall suppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space, Z, $B \in \mathcal{L}(U, Z)$ and $C \in \mathcal{L}(Z, Y)$. For a real δ, we decompose of the spectrum of A into two distinct parts of the complex plane

$$\sigma_\delta^+(A) := \sigma(A) \cap \overline{\mathbb{C}_\delta^+};\ \mathbb{C}_\delta^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > \delta\}, \tag{8.1}$$

$$\sigma_\delta^-(A) := \sigma(A) \cap \mathbb{C}_\delta^-;\ \mathbb{C}_\delta^- = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < \delta\}. \tag{8.2}$$

The first theorem shows that if $\Sigma(A, B, -, -)$ is stabilizable and B is compact, then the unstable part of the spectrum of A consists only of eigenvalues.

Theorem 8.1.3 *Let the state linear system $\Sigma(A, B, -, -)$ be β-exponentially stabilizable and let $B \in \mathcal{L}(U, Z)$ be a compact operator. Then there exists a constant $\delta < \beta$ such that $\sigma_\delta^+(A)$ comprises a pure point spectrum. Furthermore, for every eigenvalue $\lambda \in \sigma_\delta^+(A)$ and every $\nu > 0$ $\dim\ker(\lambda I - A)^\nu < \infty$.*

Proof Let F be a stabilizing feedback operator and denote the C_0-semigroup generated by $A + BF$ by $T_{BF}(t)$. Then there exist $M > 0$ and $\gamma < \beta$ such that $\|T_{BF}(t)\| \le Me^{\gamma t}$. From Theorem 2.1.15, we know that $(sI - A - BF)$ is invertible for all $s \in \overline{\mathbb{C}_\delta^+}$, where δ is such that $\gamma < \delta < \beta$. We now investigate the spectrum of A in $\overline{\mathbb{C}_\delta^+}$. Notice first that the following identity holds for every $s \in \overline{\mathbb{C}_\delta^+}$

$$(sI - A) = [I + BF(sI - A - BF)^{-1}](sI - A - BF). \tag{8.3}$$

So $(sI - A)$ is invertible as a bounded operator in $\mathcal{L}(Z)$ for s in $\overline{\mathbb{C}_\delta^+}$ if and only if $[I + BF(sI - A - BF)^{-1}]$ is invertible for s in $\overline{\mathbb{C}_\delta^+}$. Since F and $(sI - A - BF)^{-1}$ are bounded operators, we see that $BF(sI - A - BF)^{-1}$ is compact, and so all its spectrum comprises eigenvalues except possibly for 0 (see Theorem A.4.19). Hence $I + BF(sI - A - BF)^{-1}$ is not invertible for $s = s_0$ in $\overline{\mathbb{C}_\delta^+}$ if and only if there exists a $z_0 \in Z$ such that

$$BF(s_0 I - A - BF)^{-1} z_0 = -z_0.$$

Equation (8.3) shows that $(s_0 I - A - BF)^{-1} z_0$ is an eigenvector of A corresponding to the eigenvalue s_0.

Let λ be an eigenvalue of A. First we shall prove that $\ker(\lambda I - A)$ is finite-dimensional. From (8.3) it follows that z_0 is in the kernel of $(\lambda I - A)$ if and only if $(\lambda I - A - BF)z_0$ is in the kernel of $[I + BF(\lambda I - A - BF)^{-1}]$. Since $(\lambda I - A - BF)$ is invertible, this implies that $\dim\ker(\lambda I - A) = \dim\ker(I + BF(\lambda I - A - BF)^{-1})$. The latter equals the number of eigenvectors of $BF(\lambda I - A - BF)^{-1}$ corresponding to the eigenvalue -1, and since $BF(\lambda I - A - BF)^{-1}$ is compact, this is finite. The remark after Definition A.4.5 gives that $\dim\ker(\lambda I - A)^\nu < \infty$ for all $\nu > 1$. ■

The above theorem tells us that systems for which $\sigma_0^+(A)$ contains residual or continuous spectra cannot be exponentially stabilized by means of a compact input operator B. This explains Example 8.1.2, where zero is in the continuous spectrum of A (see Example A.4.6). In the case of finite-rank input operators, we can characterize the eigenvalues of A in $\mathbb{C}_\delta^+$ as the zeros of a holomorphic function.

Lemma 8.1.4 *Suppose that B has finite rank, $F \in \mathcal{L}(Z, U)$, and $(A + BF)$ generates the β-exponentially stable C_0-semigroup, $T_{BF}(t)$. For every δ larger than the growth bound γ of $T_{BF}(t)$ s lies in the set $\rho_\delta^+(A) = \overline{\mathbb{C}_\delta^+} \cap \rho(A)$ if and only if $I + F(sI - A - BF)^{-1}B$ is invertible in $\mathcal{L}(U)$. Moreover, $\sigma_\delta^+(A)$ equals the set of elements s in $\overline{\mathbb{C}_\delta^+}$ for which $\det(I + F(sI - A - BF)^{-1}B) = 0$ and the multiplicity of every eigenvalue of A in $\mathbb{C}_\delta^+$ is finite.*

Proof *a*. If $s \in \overline{\mathbb{C}_\delta^+}$, $\delta > \gamma$, then by Lemma 2.1.14 $sI - A - BF$ is invertible in $\mathcal{L}(Z)$. An easy calculation gives

$$I = (sI - A)(sI - A - BF)^{-1} - BF(sI - A - BF)^{-1} \tag{8.4}$$

and

$$B = (sI - A)(sI - A - BF)^{-1}B - BF(sI - A - BF)^{-1}B,$$

and hence we obtain the following identity for $s \in \overline{\mathbb{C}_\delta^+}$:

$$B[I + F(sI - A - BF)^{-1}B] = (sI - A)(sI - A - BF)^{-1}B. \tag{8.5}$$

b. Necessity: Suppose now that $I + F(sI - A - BF)^{-1}B$ is invertible. Then from (8.5), we obtain

$$B = (sI - A)(sI - A - BF)^{-1}B[I + F(sI - A - BF)^{-1}B]^{-1}$$

and substituting this in the second term in (8.4) gives

$$\begin{aligned} I \quad = \quad & (sI - A)(sI - A - BF)^{-1} \cdot \\ & \left[I - B[I + F(sI - A - BF)^{-1}B]^{-1}F(sI - A - BF)^{-1}\right]. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} & (sI - A - BF)^{-1} \cdot \\ & \quad \left[I - B[I + F(sI - A - BF)^{-1}B]^{-1}F(sI - A - BF)^{-1}\right](sI - A) \\ & \qquad = \quad (sI - A - BF)^{-1} \cdot \\ & \qquad\qquad \left[sI - A - B[I + F(sI - A - BF)^{-1}B]^{-1} \cdot \right. \\ & \qquad\qquad \left. F[I + (sI - A - BF)^{-1}BF]\right] \\ & \qquad = \quad (sI - A - BF)^{-1} \cdot \\ & \qquad\qquad \left[sI - A - B[I + F(sI - A - BF)^{-1}B]^{-1} \cdot \right. \\ & \qquad\qquad \left. [I + F(sI - A - BF)^{-1}B]F\right] \\ & \qquad = \quad I_{\boldsymbol{D}(A)}. \end{aligned}$$

The linear operator $\left[I - B[I + F(sI - A - BF)^{-1}B]^{-1}F(sI - A - BF)^{-1}\right]$ is bounded and so is $(sI - A - BF)^{-1}$ for $s \in \overline{\mathbb{C}_\delta^+}$. Thus $s \in \rho_\delta^+(A)$.

c. Sufficiency. Suppose now that $(sI - A)$ is invertible in $\mathcal{L}(Z)$ for $s \in \overline{\mathbb{C}_\delta^+}$. Equation (8.5) yields

$$(sI - A)^{-1}B[I + F(sI - A - BF)^{-1}B] = (sI - A - BF)^{-1}B \tag{8.6}$$

and so

$$F(sI - A)^{-1}B[I + F(sI - A - BF)^{-1}B] = F(sI - A - BF)^{-1}B. \tag{8.7}$$

Suppose now that $[I + F(sI - A - BF)^{-1}B]$ is not invertible in $\mathcal{L}(U)$. Since $F(sI - A - BF)^{-1}B$ is compact, there must exist an eigenvector $u \neq 0$ such that

$$F(sI - A - BF)^{-1}Bu = -u.$$

Substituting this in (8.7) implies that $u = 0$, which is a contradiction. Consequently, $I + F(sI - A - BF)^{-1}B$ must be invertible.

d. Since F has finite rank, $I + F(sI - A - BF)^{-1}B$ is a square matrix, and this is invertible in $\mathcal{L}(U)$ if and only if $\det(I + F(sI - A - BF)^{-1}B) \neq 0$. Furthermore, since $\det(I + F(sI - A - BF)^{-1}B)$ is a holomorphic function on $\mathbb{C}_\delta^+$ we have that the order of every zero is finite. Hence if s_0 is a zero of $\det(I + F(sI - A - BF)^{-1}B)$, then there exists a $\nu_0 > 0$ such that $\lim_{s \to s_0}(s - s_0)^{\nu_0}(I + F(sI - A - BF)^{-1}B)^{-1}$ exists. From the first part of this proof, we know that s_0 is an eigenvalue of A, and we now show that the order is less than or equal to ν_0. From equation (8.4), we have that for an arbitrary $z \in Z$ and $s \in \mathbb{C}_\delta^+$

$$
\begin{aligned}
&\lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A)^{-1}z \\
&\quad = \lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A - BF)^{-1}z - {} \\
&\qquad \lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A)^{-1}BF(sI - A - BF)^{-1}z \\
&\qquad \text{by equation (8.4)} \\
&\quad = -\lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A)^{-1}BF(sI - A - BF)^{-1}z \\
&\qquad \text{since } (sI - A - BF)^{-1} \text{ is holomorphic on } \mathbb{C}_\delta^+ \text{ by} \\
&\qquad \text{Lemma 2.1.14} \\
&\quad = -\lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A - BF)^{-1}B \cdot \\
&\qquad [I + F(sI - A - BF)^{-1}B]^{-1} \cdot \\
&\qquad F(sI - A - BF)^{-1}z \qquad \text{by equation (8.6).}
\end{aligned}
$$

By the definition of ν_0 and the fact that $(sI - A - BF)^{-1}$ is holomorphic on $\mathbb{C}_\delta^+$, this last limit exists. Hence the order of s_0 as an eigenvalue of A cannot be larger than ν_0 (see Definition A.4.5). From Theorem 8.1.3, it follows that the multiplicity of the eigenvalue s_0 is finite. ■

So we see that in the case that B has finite rank and the state linear system $\Sigma(A, B, -, -)$ is exponentially stabilizable, we can decompose the spectrum of A into a δ-stable part and a δ-unstable part that comprises eigenvalues with finite multiplicity. In fact, we shall show that A can have at most finitely many eigenvalues in $\overline{\mathbb{C}_\delta^+}$. Since such a separation of the spectrum is an important property of the generator, we give it a special name.

Definition 8.1.5 A satisfies the *spectrum decomposition assumption at* δ if $\sigma_\delta^+(A)$ is bounded and separated from $\sigma_\delta^-(A)$ in such a way that a rectifiable, simple, closed curve, Γ_δ, can be drawn so as to enclose an open set containing $\sigma_\delta^+(A)$ in its interior and $\sigma_\delta^-(A)$ in its exterior. ■

Classes of operators that satisfy the spectrum decomposition assumption are the Riesz-spectral class with a pure point spectrum and only finitely many eigenvalues in $\sigma_\delta^+(A)$ and the class of retarded differential equations.

In Lemma **??**, we showed that such a decomposition of the spectrum induces a corresponding decomposition of the state space Z and of the operator, A. Summarizing, the spectral projection P_δ defined by

$$P_\delta z = \frac{1}{2\pi \jmath} \int_{\Gamma_\delta} (\lambda I - A)^{-1} z d\lambda, \tag{8.8}$$

where Γ_δ is traversed once in the positive direction (counterclockwise), induces the following decomposition:

$$Z = Z_\delta^+ \oplus Z_\delta^-, \qquad \text{where } Z_\delta^+ := P_\delta Z \text{ and } Z_\delta^- := (I - P_\delta)Z. \tag{8.9}$$

In view of this decomposition, it is convenient to use the notation

$$A = \begin{pmatrix} A_\delta^+ & 0 \\ 0 & A_\delta^- \end{pmatrix}, \quad T(t) = \begin{pmatrix} T_\delta^+(t) & 0 \\ 0 & T_\delta^-(t) \end{pmatrix}, \tag{8.10}$$

$$B = \begin{pmatrix} B_\delta^+ \\ B_\delta^- \end{pmatrix}, \quad C = \begin{pmatrix} C_\delta^+ & C_\delta^- \end{pmatrix}, \tag{8.11}$$

where $B_\delta^+ = P_\delta B \in \mathcal{L}(U, Z_\delta^+)$, $B_\delta^- = (I - P_\delta)B \in \mathcal{L}(U, Z_\delta^-)$ $C_\delta^+ = CP_\delta \in \mathcal{L}(Z_\delta^+, Y)$, and $C_\delta^- = C(I - P_\delta) \in \mathcal{L}(Z_\delta^+, Y)$. In fact, we have decomposed our system $\Sigma(A, B, C)$ as the vector sum of the two subsystems: $\Sigma(A_\delta^+, B_\delta^+, C_\delta^+)$ on Z_δ^+ and $\Sigma(A_\delta^-, B_\delta^-, C_\delta^-)$ on Z_δ^-.

The following theorem reveals that the concept of exponential stabilizability is a very strong one for state linear systems with a finite-rank input operator; it implies that A satisfies a spectrum decomposition assumption and has, at most, finitely many unstable eigenvalues. In particular, systems with infinitely many eigenvalues on the imaginary axis cannot be exponentially stabilized by a finite rank input operator.

Theorem 8.1.6 *If the state linear system $\Sigma(A, B, -, -)$ on the state space Z is such that B has finite rank, then the following assertions are equivalent:*

a. $\Sigma(A, B, -, -)$ *is β-exponentially stabilizable;*

b. $\Sigma(A, B, -, -)$ *satisfies the spectrum decomposition assumption at* β*,* Z_β^+ *is finite-dimensional,* $T_\beta^-(t)$ *is* β*-exponentially stable, and the finite-dimensional system* $\Sigma(A_\beta^+, B_\beta^+, -, -)$ *is controllable, where we have used the notation introduced in equations (8.9), (8.10), and (8.11).*

If $\Sigma(A, B, -, -)$ *is* β*-exponentially stabilizable, then a* β*-stabilizing feedback operator is given by* $F = F_0 P_\beta$*, where* F_0 *is a* β*-stabilizing feedback operator for* $\Sigma(A_\beta^+, B_\beta^+, -, -)$*.*

Proof $b \Rightarrow a$. Since the finite-dimensional system $\Sigma(A_\beta^+, B_\beta^+, -, -)$ is controllable, there exists a feedback operator $F_0 \in \mathcal{L}(Z_\beta^+, U)$ such that the spectrum of $A_\beta^+ + B_\beta^+ F_0$ lies in $\mathbb{C}_\beta^-$. Choose the feedback operator $F = (F_0, 0) \in \mathcal{L}(Z, U)$ for the system $\Sigma(A, B, -, -)$. The perturbed operator $A + BF = \begin{pmatrix} A_\beta^+ + B_\beta^+ F_0 & 0 \\ B_\beta^- F_0 & A_\beta^- \end{pmatrix}$ generates a C_0-semigroup by Lemma 4.2.2. Furthermore, its growth bound is the maximum of that of the semigroups generated by $A_\beta^+ + B_\beta^+ F_0$ and A_β^-. By construction and assumption this is a $\beta_1 < \beta$.
$a \Rightarrow b$. By Definition 7.1.1, there exist constants M and $\gamma < \beta$, such that

$$\|T_{BF}(t)\| \leq M e^{\gamma t}. \tag{8.12}$$

From Lemma 8.1.4, for every $\varepsilon > 0$, $s \in \sigma_{\gamma+\varepsilon}^+(A)$ if and only if $\det(I + F(sI - A - BF)^{-1}B) = 0$. Now the determinant is holomorphic on $\mathbb{C}_\gamma^+$ and so there cannot be an accumulation point of zeros in $\overline{\mathbb{C}_{\gamma+\varepsilon}^+}$ unless the determinant is identically zero (Theorem A.1.9).

From (8.12), it follows that for all $\varepsilon > 0$

$$\int_0^\infty e^{2(-\gamma-\varepsilon)t} \|F T_{BF}(t) B\|^2 dt < \infty$$

and by the Paley-Wiener Theorem A.6.21, we deduce that

$$F((s + \gamma + \varepsilon)I - A - BF)^{-1} B \in \boldsymbol{H}_2(\mathcal{L}(U)). \tag{8.13}$$

Since U is finite-dimensional, this implies that

$$\lim_{\rho \to \infty} \sup_{s \in \mathbb{C}_{\gamma+\varepsilon}^+, |s| \geq \rho} \|F(sI - A - BF)^{-1} B\| = 0$$

(see Lemma A.6.18).

Consequently, $\det(I + F(sI - A - BF)^{-1}B)$ cannot be identically zero in $\overline{\mathbb{C}_\beta^+}$, and it has no finite accumulation point there. Moreover, we can always find a sufficiently large ρ such that

$$\|F(sI - A - BF)^{-1} B\| \leq \frac{1}{2} \text{ in } \overline{\mathbb{C}_\beta^+} \setminus D(\rho, \beta), \tag{8.14}$$

where $D(\rho, \beta) = \{s \in \overline{\mathbb{C}_\beta^+} \mid |s| \leq \rho\}$ and $I + F(sI - A - BF)^{-1}B$ is invertible for all $s \in \overline{\mathbb{C}_\beta^+} \setminus D(\rho, \beta)$. Inside the compact set, $D(\rho, \beta)$, a holomorphic function has, at most, finitely many zeros (Theorem A.1.9), and applying Lemma 8.1.4 we see that $\sigma_\beta^+(A)$ comprises, at most, finitely many points. Theorem 8.1.3 shows that these points are all eigenvalues with finite multiplicity and the spectrum decomposition assumption holds at β. From Lemma **??**.c and e we have that $Z_\beta^+ = \operatorname{ran} P_\beta$ is finite-dimensional and $\sigma(A_\beta^+) = \sigma_\beta^+(A) \subset \overline{\mathbb{C}_\beta^+}$. Thus it remains to show that $T_\beta^-(t)$ is β-exponentially stable and that $\Sigma(A_\beta^+, B_\beta^+, -)$ is controllable.

By Lemma **??** and **??**, we have that A_β^- is the infinitesimal generator of the C_0-semigroup $T_\beta^-(t)$ on Z_β^-, and $(sI - A)^{-1}|_{Z_\beta^-} = (sI - A_\beta^-)^{-1}$. Further, since $\sigma(A_\beta^-) \subset \mathbb{C}_\beta^-$, $(sI - A_\beta^-)^{-1}$ is holomorphic on $\overline{\mathbb{C}_\beta^+}$. We now proceed to show that $((s+\beta)I - A_\beta^-)^{-1}z$ is in $\boldsymbol{H}_2(Z)$ for every $z \in Z$. From Lemma 8.1.4, $[I + F(sI - A - BF)^{-1}B]^{-1}$ is invertible in $\rho_\beta^+(A)$, and using (8.6) we obtain

$$(sI - A)^{-1}B = (sI - A - BF)^{-1}B[I + F(sI - A - BF)^{-1}B]^{-1}. \quad (8.15)$$

Using the properties of the spectral projection P_β from Lemma **??**, we obtain

$$\begin{aligned}(sI - A_\beta^-)^{-1}B_\beta^- &= (sI - A)^{-1}(I - P_\beta)B \\ &= (I - P_\beta)(sI - A)^{-1}B \\ &= (I - P_\beta)(sI - A - BF)^{-1}B\cdot \\ &\quad\ [I + F(sI - A - BF)^{-1}B]^{-1} \quad \text{from (8.15).}\end{aligned} \quad (8.16)$$

The left-hand side of this equation is holomorphic on $\mathbb{C}_\beta^+$, since A_β^- has no eigenvalues there. In addition, $(sI - A - BF)^{-1}B$ is holomorphic on $\mathbb{C}_\beta^+$ from (8.12) and from (8.14) $\|(sI - A - BF)^{-1}B\| \leq \frac{1}{2}$ for $s \in \mathbb{C}_\beta^+ \setminus D(\rho, \beta)$. Thus, for sufficiently large ρ, $[I + F(sI - A - BF)^{-1}B]^{-1}$ is uniformly bounded in norm in $\overline{\mathbb{C}_\beta^+} \setminus D(\rho, \beta)$ and inside the half-circle $D(\rho, \beta)$, it has finitely many poles. However, (8.16) shows that the product in the right-hand side expression can have no poles in $\overline{\mathbb{C}_\beta^+}$. So $(sI - A_\beta^-)^{-1}B_\beta^-$ is uniformly bounded in norm on $\overline{\mathbb{C}_\beta^+}$. For $z \in Z_\beta^-$ and $s \in \mathbb{C}_\beta^+$, from Lemma **??** we obtain

$$\begin{aligned}(sI - A_\beta^-)^{-1}z &= (sI - A)^{-1}(I - P_\beta)z = (I - P_\beta)(sI - A)^{-1}z \\ &= (I - P_\beta)[(sI - A - BF)^{-1} - \\ &\quad\ (sI - A)^{-1}BF(sI - A - BF)^{-1}]z \qquad \text{by (8.4)} \\ &= (I - P_\beta)(sI - A - BF)^{-1}z - \\ &\quad\ (sI - A_\beta^-)^{-1}B_\beta^- F(sI - A - BF)^{-1}z.\end{aligned}$$

Now from (8.12) and the Paley-Wiener Theorem A.6.21 it follows that $((s+\beta)I - A - BF)^{-1}z \in \boldsymbol{H}_2(Z)$. Notice that we already showed that $(sI - A_\beta^-)^{-1}B_\beta^-$ is uniformly bounded in norm on $\overline{\mathbb{C}_\beta^+}$. Thus for $z \in Z_\beta^-$ $((s+\beta)I - A_\beta^-)^{-1}z \in \boldsymbol{H}_2(Z)$ (see Theorem A.6.26.b) as claimed. Corollary A.6.23 then implies that $\int_0^\infty \|e^{-\beta t}T_\beta^-(t)z\|^2 dt < \infty$ and Lemma 7.1.2 shows that $T_\beta^-(t)$ is β-exponentially stable.

Finally, we prove that the system $\Sigma(A_\beta^+, B_\beta^+, -, -)$ is controllable. Suppose on the contrary that $\Sigma(A_\beta^+, B_\beta^+, -, -)$ is not controllable. Then there must exist a $v \in Z_\beta^+$ such that v is orthogonal to the reachable subspace of Definition **??**, i.e., $v \perp \int_0^t T_\beta^+(t-s)B_\beta^+u(s)ds$ for all $u \in \boldsymbol{L}_2([0,t];U)$. From Theorem 4.2.1, it follows that

$$P_\beta T_{BF}(t)z = P_\beta T(t)z + \int_0^t P_\beta T(t-s)BFT_{BF}(s)zds$$

and if $z \in Z_\beta^+$ we have

$$P_\beta T_{BF}(t)z = T_\beta^+(t)z + \int_0^t T_\beta^+(t-s)B_\beta^+ FT_{BF}(s)zds$$

and taking the inner product with this v gives

$$\langle v, P_\beta T_{BF}(t)z\rangle = \langle v, T_\beta^+(t)z\rangle.$$

Now $\langle v, P_\beta e^{-\beta t}T_{BF}(t)z\rangle \to 0$ as $t \to \infty$, since $T_{BF}(t)$ has the growth bound $\gamma < \beta$. So $\langle v, e^{-\beta t}T_\beta^+(t)z\rangle \to 0$ as $t \to \infty$ for every $z \in Z_\beta^+$. However, since A_β^+ is the infinitesimal generator of $T_\beta^+(t)$ on the finite-dimensional state space Z_β^+ and $\sigma(A_\beta^+)$ is contained in $\overline{\mathbb{C}_\beta^+}$, this can only happen if $v = 0$. ■

The duality between the concepts of stabilizability and detectability lead immediately to the following results on β-exponential detectability.

Theorem 8.1.7 *If the system $\Sigma(A, -, C, -)$ on the state space Z is such that C has finite rank, then the following assertions are equivalent:*

a. $\Sigma(A, -, C, -)$ is β-exponentially detectable;

b. A satisfies the spectrum decomposition assumption at β, Z_β^+ is finite-dimensional, $T_\beta^-(t)$ is β-exponentially stable, and $\Sigma(A_\beta^+, -, C_\beta^+, -)$ is observable, where we have used the notation of (8.9), (8.10), and (8.11).

If $\Sigma(A, -, C, -)$ is β-exponentially detectable, then a β-stabilizing output injection operator L is given by $L = i_\beta L_0$, where L_0 is such that $A_\beta^+ + L_0 C_\beta^+$ is β-exponentially stable and i_β is the injection operator from Z_β^+ to Z.

Proof We have that $\Sigma(A, -, C, -)$ is β-exponentially detectable if and only if $\Sigma(A^*, C^*, -)$ is β-exponentially stabilizable. By Theorem 8.1.6, it follows that A^* satisfies the spectrum decomposition assumption at β. The corresponding spectral projection is given by

$$\tilde{P}_\beta z = \frac{1}{2\pi j}\int_{\Gamma_\beta}(\lambda I - A^*)^{-1}zd\lambda,$$

where Γ_β is traversed once in the positive direction (counterclockwise). Without loss of generality, we may assume that Γ_β is symmetric with respect to the real axis. So we have that $\tilde{P}_\beta = P_\beta^*$, and so the decomposition of the system $\Sigma(A^*, C^*, -)$ is the adjoint of the decomposition of the system $\Sigma(A, -, C, -)$. Now the results follow easily by duality arguments. ■

We remark that Theorems 8.1.6 and 8.1.7 have been proved under the special assumptions that B and C have *finite rank* and they are *bounded*.

Example 8.1.8 Consider the heat equation example discussed in Examples 5.1.3 and **??**. In Example 3.2.11, we showed that it has a state linear system realization $\Sigma(A, B, C, -)$ on the state space $Z = \boldsymbol{L}_2(0,1)$, where

$$A = \frac{d^2}{dx^2} \qquad \text{with } \boldsymbol{D}(A) = \{h \in \boldsymbol{L}_2(0,1) \mid h, \tfrac{dh}{dx} \text{ are absolutely continuous, } \tfrac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,1) \text{ and } \tfrac{dh}{dx}(0) = 0 = \tfrac{dh}{dx}(1)\},$$

$$Bu = bu \qquad \text{with } b(x) = \frac{1}{2\varepsilon}\mathbb{1}_{[x_0-\varepsilon, x_0+\varepsilon]}(x),$$

and

$$Ch = \langle h, c\rangle \qquad \text{with } c(x) = \frac{1}{2\nu}\mathbb{1}_{[x_1-\nu, x_1+\nu]}(x).$$

A has the eigenvalues $0, -n^2\pi^2$, $n \geq 1$ and the corresponding eigenvectors $\{1, \sqrt{2}\cos(n\pi x),\ n \geq 1\}$, A is self-adjoint and has the following spectral decomposition on Z:

$$Az = \sum_{n=1}^{\infty} -(n\pi)^2 \langle z, \sqrt{2}\cos(n\pi\cdot)\rangle \sqrt{2}\cos(n\pi\cdot) \qquad \text{for } z \in \boldsymbol{D}(A).$$

It generates the C_0-semigroup $T(t)$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-(n\pi)^2 t}\langle z, \sqrt{2}\cos(n\pi\cdot)\rangle \sqrt{2}\cos(n\pi\cdot) + \langle z, 1\rangle 1,$$

and so it follows that A satisfies the spectrum decomposition assumption for any β. Suppose we choose $\beta = -2$. Then Z_β^+ has dimension 1 and the subsystem, $\Sigma(A_\beta^+, B_\beta^+, C_\beta^+, -) = \Sigma(0, 1, 1)$, is controllable and observable. So by Theorem 8.1.6, $\Sigma(A, B, -, -)$ is exponentially stabilizable by the feedback $u = Fz$, where $F = (-3, 0)$. Thus $Fz = -3\langle z, \phi_0\rangle = -3\langle z, 1\rangle$. $A + BF$ then has the eigenvalues $-3, -(n\pi)^2$, $n \geq 1$. Similarly, $\Sigma(A, -, C, -)$ is exponentially detectable by the output injection operator $L = \binom{-3}{0}$; hence $Ly = -3y\phi_0 = -3y1$. The system $A + LC$ has the eigenvalues $-3, -(n\pi)^2$, $n \geq 1$. The decay constants of the semigroups generated by $A + BF$ and $A + LC$ are both 3.

In a similar manner, it can be shown that $\Sigma(A, B, -, -)$ is β-exponentially stabilizable for $\beta \in \mathbb{R}$ provided that $\cos(n\pi x_0)\sin(n\pi\varepsilon) \neq 0$ for those $n \geq 1$ with $\beta \leq -n^2\pi^2$ (see Exercise 8.4). ■

8.2 Tests for exponential stabilizability and detectability

In general it is difficult to establish exponential stabilizability and detectability. However, in this section we derive verifiable criteria for exponential stabilizability and detectability for spatially invariant systems, Riesz-spectral systems and delay systems.

For the class of spatially invariant systems it is possible to give simple conditions for exponential stabilizability and exponential detectability by analyzing its Fourier transformed system.

Theorem 8.2.1 *Let $\Sigma(A_{cv}, B_{cv}, C_{cv}, -)$ be a spatially invariant system on the state space $\ell_2(\mathbb{Z}; \mathbb{C}^n)$. Define $\check{A} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times n})$, $\check{B} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{n\times m})$, and $\check{C} \in \boldsymbol{L}_\infty(\partial\mathbb{D}; \mathbb{C}^{p\times n})$ as in Example 5.1.2. Suppose further that $\check{A}(\phi)$, $\check{B}(\phi)$ and $\check{C}(\phi)$ are continuous in $\phi \in \partial\mathbb{D}$.*

a. The following equivalences hold:

(i) $\Sigma(A_{cv}, B_{cv}, -, -)$ is exponentially stabilizable;
(ii) $\Sigma(\check{A}(\phi), \check{B}(\phi), -, -)$ is stabilizable for all $\phi \in \partial\mathbb{D}$;
(iii) For all $\phi \in \partial\mathbb{D}$ and for all $\lambda \in \mathbb{C}_0^+$ the following rank condition holds

$$\operatorname{rank}\begin{pmatrix} (\lambda I - \check{A}(\phi)) & \check{B}(\phi) \end{pmatrix} = n. \tag{8.17}$$

b. The following equivalences hold:

(i) $\Sigma(A_{cv}, -, C_{cv}, -)$ is exponentially detectable;
(ii) $\Sigma(\check{A}(\phi), -, \check{C}(\phi), -)$ is detectable for all $\phi \in \partial\mathbb{D}$;
(iii) For all $\phi \in \partial\mathbb{D}$ and for all $\lambda \in \mathbb{C}_0^+$ the following rank condition holds

$$\operatorname{rank}\begin{pmatrix} \lambda I - \check{A}(\phi) \\ \check{C}(\phi) \end{pmatrix} = n. \tag{8.18}$$

Proof First recall from Example 5.1.2 that $\Sigma(A_{cv}, B_{cv}, C_{cv}, -)$ is isomorphic to $\Sigma(\Lambda_{\check{A}}, \Lambda_{\check{B}}, \Lambda_{\check{C}}, -)$. So $\Sigma(A_{cv}, B_{cv}, C_{cv}, -)$ will be exponentially stabilizable (detectable) if and only if $\Sigma(\check{A}, \check{B}, \check{C}, -)$ is exponentially stabilizable (detectable) with respect to the state space $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$. Moreover from the finite-dimensional theory we know that (ii) and (iii) are equivalent. So we need to show that $\Sigma(\check{A}, \check{B}, \check{C}, -)$ is exponentially stabilizable (detectable) with respect to the state space $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$ if and only if $\Sigma(\check{A}(\phi), \check{B}(\phi), \check{C}(\phi), -)$ is stabilizable (detectable) for all $\phi \in \partial\mathbb{D}$. Since stabilizability and detectability are dual statements, it suffices to prove the stabilizability.

Suppose that $\Sigma(\check{A}(\phi), \check{B}(\phi), -, -)$ is stabilizable for all $\phi \in \partial\mathbb{D}$. Let $\mathbb{J}$ be a countable dense subset of $\partial\mathbb{D}$, for instance, all ϕ for which the argument, $\arg(\phi)$, is rational. By assumption, for $\phi_0 \in \mathbb{J}$ there exists a F_0 such that the eigenvalues of $\check{A}(\phi_0) + \check{B}(\phi_0)F_0$ are in the left half-plane. Thus there exists an $\varepsilon_0 > 0$ such that the real part of the eigenvalues is less than $-\varepsilon_0$. Now by the continuity of $\check{A}$ and $\check{B}$ there exists a $\delta_0 > 0$ such that the real part of the eigenvalues of $\check{A}(\phi) + \check{B}(\phi)F_0$ is less than $-\varepsilon_0/2$ for all $\phi \in \{\phi \in \partial\mathbb{D} \mid \arg(\phi_0) - \delta_0 \leq \arg(\phi) \leq \arg(\phi_0) + \delta_0\} =: V(\phi_0)$.

Since $\mathbb{J}$ is dense in $\partial\mathbb{D}$, we have that $\cup_{\phi_0\in\mathbb{J}} V(\phi_0) = \partial\mathbb{D}$. Furthermore, since $\partial\mathbb{D}$ is a compact subset of $\mathbb{C}$, there exists a finite sub-covering. Hence there exists a $K > 0$ such that $\partial\mathbb{D} = \cup_{k=1,\dots,K} V(\phi_k)$. We can restrict the arcs $V(\phi_k)$ such that they intersect at finitely many points on $\partial\mathbb{D}$, and they still cover $\partial\mathbb{D}$. Hence for every $\phi \in \partial\mathbb{D}$, there exists an F_k such that the real part of the eigenvalues of $\breve{A}(\phi) + \breve{B}(\phi)F_k$ are less than $-\varepsilon_k/2 \le \max_{k=1,\dots,K}\{-\varepsilon_k/2\} < 0$. Thus by choosing $\breve{F}(\phi)$ to be piecewise constant with one value at the points of intersection, the eigenvalues of $\breve{A}(\phi) + \breve{B}(\phi)\breve{F}(\phi)$ will be bounded away from the imaginary axis. From Theorem 3.1.6 we conclude that $\breve{A} + \breve{B}\breve{F}$ generates an exponentially stable semigroup on $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$, which implies that $\Sigma(A_{cv}, B_{cv}, -, -)$ is exponentially stabilizable (see Definition 8.1.1).

Now suppose that $\Sigma(\breve{A}, \breve{B}, -, -)$ is exponentially stabilizable with respect to the state space $\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$. From Definition 8.1.1 there exists an $F \in \mathcal{L}(\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n), \boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^m))$ such that $\breve{A} + \breve{B}F$ generates an exponentially stable semigroup on $Z = \boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$. Thus by Lemma A.3.61 the dual semigroup is exponentially stable as well. By Theorem 2.3.6 and Theorem 7.1.3 there exists a positive $P \in \mathcal{L}(Z)$ such that for all $f \in Z = \boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$

$$\langle(\breve{A} + \breve{B}F)^* f, Pf\rangle + \langle Pf, (\breve{A} + \breve{B}F)^* f\rangle = -\langle f, f\rangle. \tag{8.19}$$

Suppose that $\Sigma(\breve{A}(\phi_0), \breve{B}(\phi_0), -, -)$ is not stabilizable. Then there exists a $v \in \mathbb{C}^n$ of norm one and a $\lambda \in \mathbb{C}_0^+$ such that

$$\breve{A}(\phi_0)^* v = \lambda v \quad \text{and} \quad B(\phi_0)^* v = 0. \tag{8.20}$$

Let $V(\phi_0, \delta)$ be an arc containing ϕ_0, i.e., $V(\phi_0, \delta) = \{\phi \in \partial\mathbb{D} \mid \arg(\phi_0) - \delta \le \arg(\phi) \le \arg(\phi_0) + \delta\}$. Define f_δ on $\partial\mathbb{D}$ as

$$f_\delta(\phi) = \begin{cases} \sqrt{\frac{\pi}{\delta}}\, v & \text{for } \phi \in V(\phi_0, \delta), \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f_\delta \in \boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n)$ and its norm equals one. Define $\breve{A}_0 \in \mathcal{L}(\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n))$ as $(\breve{A}_0 f)(\phi) = \breve{A}(\phi_0) f(\phi)$, and $\breve{B}_0 \in \mathcal{L}(\boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^m), \boldsymbol{L}_2(\partial\mathbb{D}; \mathbb{C}^n))$ as $(\breve{B}_0 g)(\phi) = \breve{B}(\phi_0) g(\phi)$.

Since $\breve{A}$, $\breve{B}$ are continuous functions of ϕ, F is a bounded operator, and f_δ has support located around $\arg(\phi_0)$, given ε, there exists $\delta > 0$ such that

$$\|(\breve{A}^* - A_0^*) f_\delta\| \le \varepsilon, \qquad \|F^*(\breve{B}^* - B_0^*) f_\delta\| \le \varepsilon. \tag{8.21}$$

Substituting f_δ in (8.19) gives

$$\begin{aligned}
-\|f_\delta\|^2 &= \langle(\check{A}^* + F^*\check{B}^*) f_\delta, Pf_\delta\rangle + \langle Pf_\delta, (\check{A}^* + F^*\check{B}^*) f_\delta\rangle \\
&= \langle(\check{A}^* - A_0^*) f_\delta + F^*(\check{B}^* - B_0^*) f_\delta, Pf_\delta\rangle + \\
&\quad \langle Pf_\delta, (\check{A}^* - A_0^*) f_\delta + F^*(\check{B}^* - B_0^*) f_\delta\rangle + \\
&\quad \langle(A_0^* + F^*B_0^*) f_\delta, Pf_\delta\rangle + \langle Pf_\delta, (A_0^* + F^*B_0^*) f_\delta\rangle \\
&= \langle(\check{A}^* - A_0^*) f_\delta + F^*(\check{B}^* - B_0^*) f_\delta, Pf_\delta\rangle + \\
&\quad \langle Pf_\delta, (\check{A}^* - A_0^*) f_\delta + F^*(\check{B}^* - B_0^*) f_\delta\rangle + \\
&\quad \langle\lambda f_\delta, Pf_\delta\rangle + \langle Pf_\delta, \lambda f_\delta\rangle,
\end{aligned}$$

where we have used (8.20). So using the self-adjointness of P and (8.21) gives

$$\begin{aligned}
&\left|\|f_\delta\|^2 + 2\,\mathrm{Re}(\lambda)\langle f_\delta, Pf_\delta\rangle\right| \\
&\qquad \le 2\left[\|(\check{A}^* - A_0^*) f_\delta\| + \|F^*(\check{B}^* - B_0^*) f_\delta\|\right]\|Pf_\delta\| \\
&\qquad \le 4\varepsilon\|Pf_\delta\| \le 4\varepsilon\|P\|,
\end{aligned}$$

since $\|f_\delta\| = 1$. But P is positive, and $\lambda \in \mathbb{C}_0^+$. Hence for sufficiently small ε this provides a contradiction. Thus $\Sigma(\check{A}(\phi_0), \check{B}(\phi_0), -, -)$ is stabilizable, and so (ii) holds. ∎

We remark that a consequence of this lemma is that if $\Sigma(A_{cv}, B_{cv}, C_{cv}, -)$ is exponentially stabilizable (detectable), then one can always choose a spatially invariant operator to achieve exponential stability.

Example 8.1.8 is a special case of the large class of Riesz-spectral operators for which we have the following general results.

Theorem 8.2.2 *Suppose that $\Sigma(A, B, C, -)$ is the self-adjoint Riesz-spectral system of Theorem* **??** *and that A has compact resolvent. The following condition is necessary and sufficient for $\Sigma(A, B, -, -)$ to be β-exponentially stabilizable:*

$$\mathrm{rank}\begin{pmatrix} \langle b_1, \phi_{n_1}\rangle & \cdots & \langle b_m, \phi_{n_1}\rangle \\ \vdots & & \vdots \\ \langle b_1, \phi_{n_{r_n}}\rangle & \cdots & \langle b_m, \phi_{n_{r_n}}\rangle \end{pmatrix} = r_n \tag{8.22}$$

for all n such that $\lambda_n \in \sigma_\beta^+(A)$.

Similarly, $\Sigma(A, -, C, -)$ is β-exponentially detectable if and only if

$$\mathrm{rank}\begin{pmatrix} \langle c_1, \phi_{n_1}\rangle & \cdots & \langle c_k, \phi_{n_1}\rangle \\ \vdots & & \vdots \\ \langle c_1, \phi_{n_{r_n}}\rangle & \cdots & \langle c_k, \phi_{n_{r_n}}\rangle \end{pmatrix} = r_n \tag{8.23}$$

for all n such that $\lambda_n \in \sigma_\beta^+(A)$.

Proof From Theorem A.4.26, A has the representation

$$Az = \sum_{n=1}^{\infty} \lambda_n \sum_{j=1}^{r_n} \langle z, \phi_{n_j} \rangle \phi_{n_j}$$

for $z \in \boldsymbol{D}(A) = \{z \in Z \mid \sum_{n=1}^{\infty} |\lambda_n|^2 \sum_{j=1}^{r_n} |\langle z, \phi_{n_j} \rangle|^2 < \infty\}$, where we have assumed that $\lambda_n \neq \lambda_m$ for $n \neq m$. Since A is self-adjoint, $\lambda_n \in \mathbb{R}$ by Lemma A.4.16. Furthermore, since A is the infinitesimal generator of a C_0-semigroup we must have that $\lambda_n < \omega$, for some $\omega \in \mathbb{R}$ (see equation (7.12)). The spectrum of A cannot have an accumulation point, since A^{-1} is compact (see Theorems A.4.26 and A.4.19). Combining these results shows that for any $\beta \in \mathbb{R}$ there are only finitely many eigenvalues in $\mathbb{C}_\beta^+$ and by assumption, these eigenvalues have finite multiplicity r_n.

From Example A.5.41 and equations (2.26) and (8.8), we obtain that

$$P_\beta z = \sum_{n=1}^{N} \sum_{j=1}^{r_n} \langle z, \phi_{n_j} \rangle \phi_{n_j} \qquad \text{with } \{\lambda_1, \ldots, \lambda_N\} = \sigma_\beta^+(A)$$

is the projection on the finite-dimensional subspace

$$Z_\beta^+ = \operatorname{span}\{\phi_{1_1}, \ldots, \phi_{1_{r_1}}, \phi_{2_1}, \ldots, \phi_{N_{r_N}}\}.$$

An easy calculation gives

$$T_\beta^-(t) = \sum_{n=N+1}^{\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle \cdot, \phi_{n_j} \rangle \phi_{n_j},$$

$$A_\beta^+ = \sum_{n=1}^{N} \lambda_n \sum_{j=1}^{r_n} \langle \cdot, \phi_{n_j} \rangle \phi_{n_j},$$

$$B_\beta^+ u = \sum_{n=1}^{N} \sum_{j=1}^{r_n} \langle Bu, \phi_{n_j} \rangle \phi_{n_j} \qquad \text{and}$$

$$C_\beta^+ z = \sum_{n=1}^{N} \sum_{j=1}^{r_n} \langle z, \phi_{n_j} \rangle C\phi_{n_j}.$$

Equation (2.28) from Example 2.1.16 shows that $T_\beta^-(t)$ is β-exponentially stable. From Theorems 8.1.6 and 8.5, we have that $\Sigma(A, B, C, -)$ is β-exponentially stabilizable or β-exponentially detectable if and only if the system $\Sigma(A_\beta^+, B_\beta^+, C_\beta^+, -)$ is controllable or observable, respectively.

We shall prove that (8.23) is equivalent to the observability of the system $\Sigma(A_\beta^+, -, C_\beta^+, -)$. The state linear system $\Sigma(A_\beta^+, -, C_\beta^+, -)$ is not observable if and only if there exists a nonzero $z \in Z_\beta^+$ such that $CT_\beta^+ z = 0$ for all $t \geq 0$. Since $\lambda_n \neq \lambda_m$ for $n, m \in \{1, \ldots, N\}$, this is equivalent to

$$\sum_{j=1}^{r_n} \langle z, \phi_{n_j} \rangle C\phi_{n_j} = 0 \qquad \text{for } n = 1, \ldots, N.$$

Using the definition of C, this equation gives

$$\sum_{j=1}^{r_n} \langle z, \phi_{n_j} \rangle \begin{pmatrix} \langle \phi_{n_j}, c_1 \rangle \\ \vdots \\ \langle \phi_{n_j}, c_k \rangle \end{pmatrix} = 0 \qquad \text{for } n = 1, \ldots, N,$$

which is equivalent to

$$\begin{pmatrix} \langle z, \phi_{n_j} \rangle, & \cdots & , \langle z, \phi_{n_j} \rangle \end{pmatrix} C_n = 0 \qquad \text{for } n = 1, \ldots, N. \tag{8.24}$$

Since $z \neq 0$, this implies that for some n such that $\lambda_n \in \sigma_\beta^+$ the rank of $C_n < r_n$. On the other hand, if the rank of $C_n < r_n$, then we have the existence of a $z \neq 0$ in Z_β^+ such that (8.24) holds, and hence $\Sigma(A_\beta^+, -, C_\beta^+, -)$ is not observable.

The equivalence between (8.22) and the controllability of $\Sigma(A_\beta^+, B_\beta^+, -, -)$ is easily shown by using duality arguments. ■

Conditions (8.22) and (8.23) are clearly weaker than those for approximate controllability we obtained in Theorem **??**, where (8.22) was required to hold for all n. Similar results hold for the Riesz-spectral systems of Theorem **??**.

Theorem 8.2.3 *Suppose that $\Sigma(A, B, C, -)$ is the Riesz-spectral system of Theorem* **??**. *Necessary and sufficient conditions for $\Sigma(A, B, -, -)$ to be β-exponentially stabilizable are that there exists an $\varepsilon > 0$ such that $\sigma_{\beta-\varepsilon}^+(A)$ comprises, at most, finitely many eigenvalues and*

$$\text{rank}(\langle b_1, \psi_n \rangle, \cdots, \langle b_m, \psi_n \rangle) = 1 \tag{8.25}$$

for all n such that $\lambda_n \in \sigma_{\beta-\varepsilon}^+(A)$.

Necessary and sufficient conditions for $\Sigma(A, -, C, -)$ to be β-exponentially detectable are that there exists an $\varepsilon > 0$ such that $\sigma_{\beta-\varepsilon}^+(A)$ comprises, at most, finitely many eigenvalues and

$$\text{rank}(\langle c_1, \phi_n \rangle, \cdots, \langle c_k, \phi_n \rangle) = 1 \tag{8.26}$$

for all n such that $\lambda_n \in \sigma_{\beta-\varepsilon}^+(A)$.

Proof We shall only prove the necessity and sufficiency for β-exponential stabilizability. The conditions for β-exponential detectability can be proved analogously or by duality arguments. We recall that the multiplicity of all eigenvalues of a Riesz-spectral operator is one.

Sufficiency for β-exponential stabilizability: Since $\sigma_{\beta-\varepsilon}^+(A)$ comprises, at most, finitely many eigenvalues, $Z_{\beta-\varepsilon}^+$ is finite-dimensional and A satisfies the spectrum decomposition assumption at $\beta-\varepsilon$. Using equation (3.28) and Cauchy's Theorem A.5.41, we see that

$$P_{\beta-\varepsilon}z = \sum_{\lambda_n \in \sigma^+_{\beta-\varepsilon}} \langle z, \psi_n \rangle \phi_n.$$

Hence we have that

$$Z^+_{\beta-\varepsilon} = \operatorname*{span}_{\lambda_n \in \sigma^+_{\beta-\varepsilon}} \{\phi_n\},$$

$$Z^-_{\beta-\varepsilon} = \overline{\operatorname*{span}}_{\lambda_n \in \sigma^-_{\beta-\varepsilon}} \{\phi_n\},$$

$$T^-_{\beta-\varepsilon}(t)z = \sum_{\lambda_n \in \sigma^-_{\beta-\varepsilon}} e^{\lambda_n t} \langle z, \psi_n \rangle \phi_n,$$

$$A^+_{\beta-\varepsilon}z = \sum_{\lambda_n \in \sigma^+_{\beta-\varepsilon}} \lambda_n \langle z, \psi_n \rangle \phi_n, \qquad \text{and} \tag{8.27}$$

$$B^+_{\beta-\varepsilon}u = \sum_{\lambda_n \in \sigma^+_{\beta-\varepsilon}} \langle Bu, \psi_n \rangle \phi_n. \tag{8.28}$$

From this, we see that $T^-_{\beta-\varepsilon}(t)$ is a C_0-semigroup corresponding to a Riesz-spectral operator on $Z^-_{\beta-\varepsilon}$. Hence it satisfies the spectrum determined growth assumption and it is β-exponentially stable. Now we shall show that the finite-dimensional system $\Sigma(A^+_{\beta-\varepsilon}, B^+_{\beta-\varepsilon}, -, -)$ is controllable. The reachability subspace of $\Sigma(A^+_{\beta-\varepsilon}, B^+_{\beta-\varepsilon}, -, -)$ is the smallest $A^+_{\beta-\varepsilon}$-invariant subspace of $Z^+_{\beta-\varepsilon}$ that contains ran $B^+_{\beta-\varepsilon}$ (see Lemma **??**). The finite-dimensional version of Lemma **??** shows that this subspace is the span of eigenvectors of $A^+_{\beta-\varepsilon}$. Hence if this subspace does not equal the state space $Z^+_{\beta-\varepsilon}$, then there is a $\lambda_i \in \sigma^+_{\beta-\varepsilon}(A)$ such that $\phi_i \in Z^+_{\beta-\varepsilon}$ and $\phi_i \notin \operatorname{ran} B^+_{\beta-\varepsilon}$. Thus its biorthogonal element ψ_i is orthogonal to the reachability subspace, and in particular $\langle \psi_i, Bu \rangle = 0$ for every $u \in \mathbb{C}^m$. This is in contradiction to (8.25), and so $\Sigma(A^+_{\beta-\varepsilon}, B^+_{\beta-\varepsilon}, -, -)$ is controllable. Theorem 8.1.6 now shows that $\Sigma(A, B, -, -)$ is β-exponentially stabilizable.

Necessity for β-exponential stabilizability: By Definitions 8.1.1 and 7.1.1, we know that if $\Sigma(A, B, -, -)$ is β-exponentially stabilizable, then there exists an $\varepsilon > 0$ such that it is also $(\beta-\varepsilon)$-exponentially stabilizable. From Theorem 8.1.6, we have that $\Sigma(A, B, -, -)$ satisfies the spectrum decomposition assumption at $\beta-\varepsilon$. Furthermore, the subspace $Z^+_{\beta-\varepsilon}$ is $T(t)$-invariant (see Lemma **??**). Lemma **??** implies that

$$Z^+_{\beta-\varepsilon} = \overline{\operatorname*{span}}_{n \in \mathbb{J}} \{\phi_n\}.$$

Since $Z^+_{\beta-\varepsilon}$ is finite-dimensional, $\mathbb{J}$ contains at most finitely many elements. The spectrum of $A^+_{\beta-\varepsilon} = A|_{Z^+_{\beta-\varepsilon}}$ is contained in $\mathbb{C}^+_{\beta-\varepsilon}$ and the spectrum of $A^-_{\beta-\varepsilon}$ is contained in $\mathbb{C}^-_{\beta-\varepsilon}$. From this, we conclude that the index set $\mathbb{J}$ equals the set $\{n|\lambda_n \in \sigma^+_{\beta-\varepsilon}(A)\}$, and so $\sigma^+_{\beta-\varepsilon}(A)$ comprises at most finitely many eigenvalues.

As in the sufficiency part of this proof, we have that $A^+_{\beta-\varepsilon}$ and $B^+_{\beta-\varepsilon}$ are given by (8.27) and (8.28), respectively, and from Theorem 8.1.6 $\Sigma(A^+_{\beta-\varepsilon}, B^+_{\beta-\varepsilon}, -, -)$ is controllable. Suppose now that (8.25) does not hold. Then there exists a $\lambda_n \in \sigma^+_{\beta-\varepsilon}(A)$ such that $\langle \psi_n, Bu\rangle = 0$ for all $u \in \mathbb{C}^m$. Corollary **??** shows that the reachable subspace of $\Sigma(A^+_{\beta-\varepsilon}, B^+_{\beta-\varepsilon}, -, -)$ is not equal to $Z^+_{\beta-\varepsilon}$ and hence $\Sigma(A^+_{\beta-\varepsilon}, B^+_{\beta-\varepsilon}, -, -)$ is not controllable. This provides the contradiction. ■

We remark that the above theorem does not hold for $\varepsilon = 0$. As a counterexample, take $B = 0$ and A defined by $Ae_n = (\beta - \jmath\frac{1}{n})e_n$, where $\{e_n\}$ is an orthonormal basis for ℓ_2.

We now derive necessary and sufficient conditions for exponential stabilizability and detectability that are reminiscent of the finite-dimensional Hautus conditions (1.19) and (1.20) discussed in Chapter 1.

Theorem 8.2.4 *Consider the linear system $\Sigma(A, B, C, -)$ with B and C finite rank operators. Suppose that A satisfies the spectrum decomposition assumption at β, $T^-_\beta(t)$ is β-exponentially stable, and $\sigma^+_\beta(A)$ comprises, at most, finitely many eigenvalues with finite multiplicity. $\Sigma(A, B, -, -)$ is β-exponentially stabilizable if and only if*

$$\operatorname{ran}(sI - A) + \operatorname{ran} B = Z \qquad \text{for } s \in \overline{\mathbb{C}^+_\beta}. \tag{8.29}$$

$\Sigma(A, -, C)$ is β-exponentially detectable if and only if

$$\ker(sI - A) \cap \ker C = \{0\} \qquad \text{for } s \in \overline{\mathbb{C}^+_\beta}. \tag{8.30}$$

Proof *a*. Since A satisfies the spectrum decomposition assumption at β, we may assume the spectral decomposition $A = \begin{pmatrix} A^+_\beta & 0 \\ 0 & A^-_\beta \end{pmatrix}$, $B = \begin{pmatrix} B^+_\beta \\ B^-_\beta \end{pmatrix}$, $C = (C^+_\beta, C^-_\beta)$, where $\sigma(A^-_\beta) \subset \mathbb{C}^-_\beta$, $\sigma(A^+_\beta) \subset \mathbb{C}^+_\beta$ and the system $\Sigma(A^+_\beta, B^+_\beta, C^+_\beta, -)$ is a finite-dimensional system.

b. Assume that $\Sigma(A, B, -, -)$ is β-exponentially stabilizable. If $F \in \mathcal{L}(Z, U)$ is a β-stabilizing feedback operator, then the growth bound is smaller than β. Hence there exists an $\varepsilon > 0$ such that for every $z \in Z$ and every $s \in \mathbb{C}^+_{\beta-\varepsilon}$ there holds

$$\begin{aligned} z &= (sI - A - BF)(sI - A - BF)^{-1}z \\ &= (sI - A)(sI - A - BF)^{-1}z - BF(sI - A - BF)^{-1}z. \end{aligned}$$

Hence (8.29) is necessary for β-exponential stabilizability.

Conversely, suppose that for every z there exists a $\tilde{z} \in \boldsymbol{D}(A)$ and a $u \in U$ such that

$$z = (sI - A)\tilde{z} + Bu \qquad s \in \overline{\mathbb{C}^+_\beta}.$$

Then

$$\begin{aligned} P_\beta z &= P_\beta(sI - A)\tilde{z} + P_\beta B u \\ &= (sI - A_\beta^+)P_\beta^+\tilde{z} + B_\beta^+ u \qquad \text{for } s \in \overline{\mathbb{C}_\beta^+}. \end{aligned}$$

From (1.19), we conclude that the finite-dimensional system $\Sigma(A_\beta^+, B_\beta^+, -, -)$ is β-exponentially stabilizable, and since $\sigma(A_\beta^+, -) \subset \mathbb{C}_\beta^+$, this implies that $\Sigma(A_\beta^+, B_\beta^+, -, -)$ is controllable. Theorem 8.1.6 then implies that $\Sigma(A, B, -, -)$ is β-exponentially stabilizable.

c. From *a* it easily follows that (8.30) holds if and only if

$$\ker(sI - A_\beta^+) \cap \ker C_\beta^+ = \{0\} \qquad \text{for } s \in \overline{\mathbb{C}_\beta^+}. \tag{8.31}$$

If $\Sigma(A, -, C, -)$ is β-exponentially detectable, then $\Sigma(A_\beta^+, -, C_\beta^+, -)$ is an observable, finite-dimensional system and so (8.31) holds. Conversely, suppose that (8.31) holds for the finite-dimensional system $\Sigma(A_\beta^+, -, C_\beta^+, -)$. Since $\sigma(A_\beta^+)$ is in $\mathbb{C}_\beta^+$, we deduce that (8.31) holds for all $s \in \mathbb{C}$ and this means that $\Sigma(A_\beta^+, -, C_\beta^+, -)$ is observable. Theorem 8.1.7 then shows that $\Sigma(A, -, C, -)$ is β-exponentially detectable. ∎

8.3 Compensator design

In the last section, we considered the problem of stabilizing by state feedback, i.e., $u = Fz$. This assumes that one can measure the whole state, which is not possible for an infinite-dimensional system. A more realistic assumption is that we can measure an output that contains information about a part of the state, as, for example, in Example 5.1.3, where we assume that we can measure an average of the temperature around a certain point. The problem that naturally arises is how to stabilize the system using only partial information about the state, as is schematically shown in Figure 8.1. The second system, which has as its input the

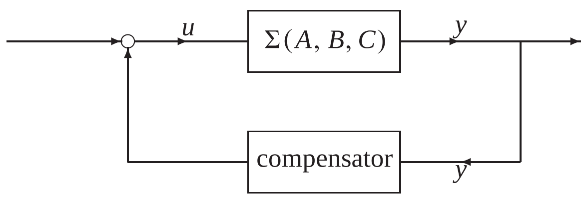

Figure 8.1. General closed-loop system

output y of the original system and as its output the input u of the original system is called a *compensator*, and the overall system as given in Figure 7.3 is called the *closed-loop system*. A fundamental question is how to design a compensator.

One answer we present here is to use the measurements (partial information) to estimate the full state (the construction of an observer) and to apply state feedback on the estimated state. First we consider the problem of estimating the full state.

Definition 8.3.1 Consider the state linear system $\Sigma(A, B, C, -)$ with state space Z, input space U, and output space Y. A *Luenberger observer* for this system is given by

$$\begin{aligned} \dot{\hat{z}}(t) &= A\hat{z}(t) + Bu(t) + L(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C\hat{z}(t), \end{aligned} \tag{8.32}$$

where $L \in \mathcal{L}(Y, Z)$. ■

Given the system $\Sigma(A, B, C, -)$, we would like to design an observer of the form (8.32) with the property that $\hat{z}$ is a good estimate of the state z. The following lemma proves that this is possible provided that $\Sigma(A, B, C, -)$ is exponentially detectable.

Lemma 8.3.2 *Consider the state linear system $\Sigma(A, B, C, -)$ and a corresponding Luenberger observer given by (8.32). If L is such that $A + LC$ generates an exponentially stable C_0-semigroup, then the* approximation error $e(t) := \hat{z}(t) - z(t)$ *converges exponentially to zero as $t \to \infty$.*

Proof Using the fact that $y = Cz$, the solution of (8.32) is given by

$$\hat{z}(t) = T_{LC}(t)\hat{z}_0 + \int_0^t T_{LC}(t-s)Bu(s)ds - \int_0^t T_{LC}(t-s)LCz(s)ds. \tag{8.33}$$

We now formulate the solution of $\Sigma(A, B, C)$ using Theorem 4.2.1:

$$\begin{aligned} z(t) &= T(t)z_0 + \int_0^t T(t-s)Bu(s)ds \qquad (8.34) \\ &= T_{LC}(t)z_0 - \int_0^t T_{LC}(t-s)LCT(s)z_0 ds + \\ &\quad \int_0^t \left[T_{LC}(t-s)B - \int_0^{t-s} T_{LC}(t-s-\tau)LCT(\tau)Bd\tau \right] u(s)ds \\ &\quad \text{from (4.10)} \\ &= T_{LC}(t)z_0 - \\ &\quad \int_0^t T_{LC}(t-s)LCT(s)z_0 ds + \int_0^t T_{LC}(t-s)Bu(s)ds - \\ &\quad \int_0^t \int_0^{t-s} T_{LC}(t-s-\tau)LCT(\tau)Bu(s)d\tau ds \\ &= T_{LC}(t)z_0 - \\ &\quad \int_0^t T_{LC}(t-s)LCT(s)z_0 ds + \int_0^t T_{LC}(t-s)Bu(s)ds - \\ &\quad \int_0^t \int_s^t T_{LC}(t-\alpha)LCT(\alpha-s)Bu(s)d\alpha ds \end{aligned}$$

$$
\begin{aligned}
& \text{substituting } \tau = \alpha - s \\
= \quad & T_{LC}(t)z_0 - \\
& \int_0^t T_{LC}(t-s)LCT(s)z_0 ds + \int_0^t T_{LC}(t-s)Bu(s)ds - \\
& \int_0^t \int_0^\alpha T_{LC}(t-\alpha)LCT(\alpha - s)Bu(s)dsd\alpha \\
& \text{changing the order of integration using Fubini's Theorem A.5.27} \\
= \quad & T_{LC}(t)z_0 + \int_0^t T_{LC}(t-s)Bu(s)ds - \\
& \int_0^t T_{LC}(t-\alpha)LCz(\alpha)d\alpha \qquad \text{using (8.34).} \qquad (8.35)
\end{aligned}
$$

Subtracting (8.35) from (8.33) yields the error

$$
e(t) = \hat{z}(t) - z(t) = T_{LC}(t)(\hat{z}_0 - z_0) = T_{LC}(t)e_0,
$$

where $e_0 = \hat{z}_0 - z_0$.

Since $T_{LC}(t)$ is an exponentially stable semigroup, $e(t)$ converges exponentially to zero, as $t \to \infty$. ■

So we see that a Luenberger observer (8.32) gives a good estimate of the state of $\Sigma(A, B, C, -)$ provided that $A + LC$ is exponentially stable.

If we knew the state $z(t)$, then in order to stabilize the system we would apply the feedback $u(t) = Fz(t)$, with F such that $A + BF$ is exponentially stable. However, we only have partial information of the state $z(t)$ through the measurement $y(t) = Cz(t)$. In the following theorem, we shall show that the feedback $u(t) = F\hat{z}(t)$ based on the estimated state has the same effect, provided that the estimation error converges to zero, as $t \to \infty$.

Theorem 8.3.3 *Consider the state linear system $\Sigma(A, B, C, -)$ and assume that it is exponentially stabilizable and exponentially detectable. If $F \in \mathcal{L}(Z, U)$ and $L \in \mathcal{L}(Y, Z)$ are such that $A + BF$ and $A + LC$ generate exponentially stable semigroups, then the controller $u = F\hat{z}$, where $\hat{z}$ is the Luenberger observer with output injection L, stabilizes the closed-loop system. The stabilizing compensator is given by*

$$
\begin{aligned}
\dot{\hat{z}}(t) &= (A + LC)\hat{z}(t) + Bu(t) - Ly(t) \\
u(t) &= F\hat{z}(t)
\end{aligned} \qquad (8.36)
$$

and it is depicted in Figure 8.2.

Proof Since $\Sigma(A, B, C, -)$ is exponentially stabilizable and detectable, there exist operators F and L such that $T_{BF}(t)$ and $T_{LC}(t)$ are exponentially stable.

Combining the abstract differential equations, we see that the closed-loop system is given by the dynamics of the extended state $z^e := \binom{z}{\hat{z}}$

$$
\begin{pmatrix} \dot{z} \\ \dot{\hat{z}} \end{pmatrix}(t) = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix} \begin{pmatrix} z \\ \hat{z} \end{pmatrix}(t), \quad t \geq 0. \qquad (8.37)
$$

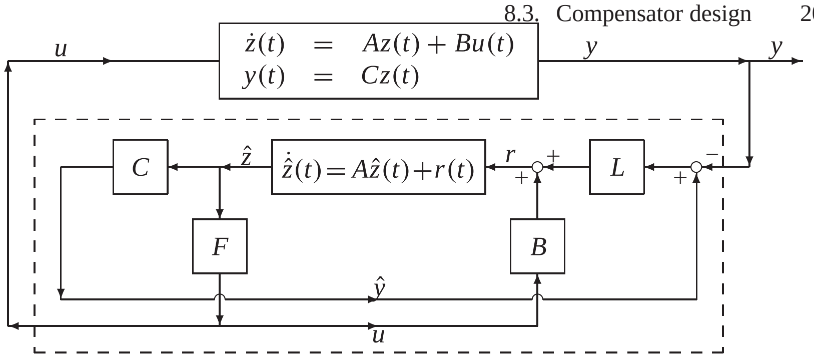

Figure 8.2. $\Sigma(A, B, C, -)$ with compensator (8.36)

This is the infinitesimal generator of a C_0-semigroup, since it is a bounded perturbation of an infinitesimal generator $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ (see Theorem 4.2.1). We shall prove that it is exponentially stable. We easily see that the following identity holds on $\boldsymbol{D}(A) \oplus \boldsymbol{D}(A)$:

$$\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} A & BF \\ -LC & A+BF+LC \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} A+LC & 0 \\ -LC & A+BF \end{pmatrix}. \tag{8.38}$$

From Exercise 2.4, we conclude that the system operator of equation (8.37) generates a C_0-semigroup with the same growth constant as the operator on the right-hand side of equation (8.38), and Lemma 4.2.2 shows that the latter has a growth bound equal to the maximum of the growth bounds of $T_{BF}(t)$ and $T_{LC}(t)$. By construction, these are negative, and hence the system (8.37) is exponentially stable. ∎

Example 8.3.4 Consider the system

$$\begin{aligned} \dot{z}_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t), \qquad t \geq 0 \\ y_r(t) &= z_{r-1}(t), \quad r \in \mathbb{Z}, \end{aligned}$$

where β is a positive real constant. As in Example 5.1.2 this system is isomorphic to the state linear system $\Sigma(\check{A}, \check{B}, \check{C}, 0)$ where

$$\check{A}(\phi) = 1, \quad \check{B}(\phi) = 1 + \beta e^{-\phi}, \quad \check{C}(\phi) = e^{-\phi}, \phi \in \partial\mathbb{D}.$$

It is clearly unstable, but it is exponentially detectable (see Theorem 8.2.1). Using this theorem once more, we see that the system is exponentially stabilizable if and only if $\beta \neq 1$. For $\beta \neq 1$, $\beta \geq 0$ and $\alpha > 0$ the following is a stabilizing feedback

$$\check{F}(\phi) = -\frac{(1+\beta e^{\phi})(1+\alpha)}{(1-\beta)^2},$$

since

$$\check{A}(\phi) + \check{B}(\phi)\check{F}(\phi) = 1 - \frac{(1+2\beta\cos\phi+\beta^2)(1+\alpha)}{(1-\beta)^2} \leq -\alpha.$$

In the case that $\beta = -\gamma^2 < 0$ we can choose

$$\alpha = \frac{8\gamma^2}{(1-\gamma^2)^2}$$

so that

$$\breve{A}(\phi) + \breve{B}(\phi)\breve{F}(\phi) = 1 - \frac{(1-2\gamma^2\cos\phi + \gamma^4)(1+\alpha)}{(1+\gamma^2)^2} \leq -\frac{4\gamma^2}{(1+\gamma^2)^2} < 0.$$

Similarly $\breve{L}(\phi) = -e^{\phi}(1+\alpha)$ is a stabilising output injection, since $\breve{A}(\phi) + \breve{L}(\phi)\breve{C}(\phi) = -\alpha$. From Theorem 8.3.3 we conclude that the following compensator stabilizes the original system:

$$\begin{aligned}
\dot{\hat{z}}_r(t) &= -\alpha\hat{z}_r(t) + u_r(t) + \beta u_{r-1}(t) - (1+\alpha)y_{r+1}(t)\\
u_r(t) &= -\frac{1+\alpha}{(1-\beta)^2}\left(\hat{z}_r(t) + \beta\hat{z}_{r+1}(t)\right), \quad r \in \mathbb{Z}.
\end{aligned}$$

■

Example 8.3.5 Consider the metal rod of Example 5.1.3 again. As in Example 8.1.8, we want to stabilize this system. However, now we want to use a compensator instead of a state feedback. The model is given by (see also (5.4) and (5.5))

$$\begin{aligned}
&\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + \frac{1}{2\varepsilon}\mathbb{1}_{[x_0-\varepsilon, x_0+\varepsilon]}(x)u(t), \quad z(x,0) = z_0(x),\\
&\frac{\partial z}{\partial x}(0,t) = \frac{\partial z}{\partial x}(1,t) = 0,\\
&y(t) = \frac{1}{2\nu}\int_{x_1-\nu}^{x_1-\nu} z(x,t)dx.
\end{aligned}$$

From Example 8.1.8, we have a stabilizing feedback given by

$$Fz = -3\langle z, 1\rangle$$

and a stabilizing output injection given by

$$Ly = -3y\,1.$$

From Theorem 8.3.3, we conclude that a stabilizing compensator is given by

$$\begin{aligned}
\frac{\partial \hat{z}}{\partial t}(x,t) &= \frac{\partial^2 \hat{z}}{\partial x^2}(x,t) - 3\frac{1}{2\nu}\int_{x_1-\nu}^{x_1+\nu}\hat{z}(x,t)dx +\\
&\qquad \frac{1}{2\varepsilon}\mathbb{1}_{[x_0-\varepsilon, x_0+\varepsilon]}(x)u(t) + 3y(t)
\end{aligned}$$

$$\hat{z}(x,0) = \hat{z}_0(x),$$

$$\frac{\partial \hat{z}}{\partial x}(0,t) = \frac{\partial \hat{z}}{\partial x}(1,t) = 0,$$

$$u(t) = -3 \int_0^1 \hat{z}(x,t) dx.$$

■

The compensator design outlined above is of theoretical interest only. It entails implementing an infinite-dimensional observer, which for Example 8.3.5 is a partial differential equation. In practice, one would only be able to implement finite-dimensional approximations of the given observer.

8.4 Exercises

8.1. In this exercise we show that even when A has stable spectrum, the stable subspace may be the zero set.
Consider the multiplication operator A on $Z = \boldsymbol{H}_2(\mathbb{D})$ defined by

$$(Af)(s) = sf(s), \qquad s \in \mathbb{D}.$$

a. Show that A is a bounded operator.
b. Show that the C_0-semigroup e^{At} is not (uniformly) bounded.
c. Show that the stable part of the spectrum of A, i.e., $\sigma(A) \cap \mathbb{C}_0^-$ is non-empty.
d. Show that if $z_0 \in Z$ is such that

$$\lim_{t\to\infty} e^{At} z_0 \to 0$$

then $z_0 = 0$.
e. Show that this system does not satisfy the spectrum decomposition assumption.

8.2. Let the system $\Sigma(A, B, -)$ be β-exponentially stabilizable, and let $F \in \mathcal{L}(Z, U)$, where U is finite-dimensional. Prove that the C_0-semigroup generated by $A + BF$ is β-exponentially stable if and only if $\sigma_p(A + BF) \subset \mathbb{C}_\beta^-$.

8.3. Prove that if $A + BF$ is exponentially stable, then $A + B\tilde{F}$ is also exponentially stable, provided that $\|F - \tilde{F}\|$ is sufficiently small.

8.4. Consider the following system on $\boldsymbol{L}_2(0, 1)$:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + \frac{1}{2\varepsilon} \mathbb{1}_{[x_0-\varepsilon, x_0+\varepsilon]}(x) u(t),$$

$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t),$$

$$y(t) = \frac{1}{\nu} \int_{x_1-\nu}^{x_1+\nu} z(x,t) dx.$$

a. Show that this system is β-exponentially stabilizable if and only if $\cos(n\pi x_0)\sin(n\pi\varepsilon) \neq 0$ for those $n \geq 1$ with $\beta \leq -n^2\pi^2$.
b. Show that this system is β-exponentially detectable if and only if $\cos(n\pi x_1)\sin(n\pi\nu) \neq 0$ for those $n \geq 1$ with $\beta \leq -n^2\pi^2$.

8.5. Consider the following system on $\boldsymbol{L}_2(0,1)$:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[0,\frac{1}{\pi}]}(x)u(t),$$

$$z(0,t) = 0 = z(1,t),$$

$$y(t) = \int_0^1 z(x,t)dx.$$

a. For which $\beta \in \mathbb{R}$ is this system β-exponentially stabilizable?
b. For which $\beta \in \mathbb{R}$ is this system β-exponentially detectable?
c. Design a feedback operator such that the closed-loop system is (-12)-exponentially stable.

8.6. Consider the following model of a flexible beam as considered in Exercise 3.10 with the following control input:

$$\frac{\partial^2 f}{\partial t^2}(x,t) + \frac{\partial^4 f}{\partial x^4}(x,t) - 2\alpha\frac{\partial^3 f}{\partial t \partial x^2}(x,t) = b(x)u(t),$$

$$f(0,t) = f(1,t) = 0 = \frac{\partial^2 f}{\partial x^2}(0,t) = \frac{\partial^2 f}{\partial x^2}(1,t),$$

$$f(x,0) = f_1(x),\ \frac{\partial f}{\partial t}(x,0) = f_2(x),$$

where $\alpha > 0$, $\alpha \neq 1$, and $b(x) = \frac{1}{2\varepsilon}\mathbb{1}_{[x_0-\varepsilon,x_0+\varepsilon]}(x)$. As the observation, we take

$$y(t) = \frac{1}{2\nu}\int_{x_1-\nu}^{x_1+\nu} f(x,t)dt.$$

Let us formulate this as the state linear system $\Sigma(A,B,C)$ on the state space Z defined in Exercise 3.10 (see Exercise **??**).

a. Give conditions on ε such that the system $\Sigma(A,B,-)$ is β-exponentially stabilizable. Distinguish between the cases $0 < \alpha < 1$ and $\alpha > 1$.
b. Give conditions on ν such that the system $\Sigma(A,-,C)$ is β-exponentially detectable. Distinguish between the cases $0 < \alpha < 1$ and $\alpha > 1$.
c. Let $x_0 = x_1 = \frac{1}{2}$, $\varepsilon = \frac{1}{4}$, $\nu = \frac{1}{2}$, and $\alpha = \frac{5}{3}$. Design a (-8)-exponentially stabilizing feedback operator and output injection operator. Use these to construct a compensator such that the closed-loop system is (-8)-exponentially stable.

8.7. Consider the shift semigroup on $\boldsymbol{L}_2(0,\infty)$ of Example 2.1.4. Let B be a finite-rank input operator. Prove that this system is not exponentially stabilizable.
Hint: Show that every $\lambda \in (-\infty, 0)$ is in the point spectrum of A using the results of Exercise 2.7.

8.8. In this exercise, we shall use the graphical test of Exercise 6.7 to show that the following transfer function is unstable;

$$g(s) = \frac{s+2}{s-1-e^{-s}}.$$

Show that g satisfies the conditions of Exercise 6.7, and use that exercise to show that g has one pole in $\mathbb{C}_0^+$. Conclude that Δ has one zero in $\mathbb{C}_0^+$.

8.9. *Instability Due to Delay*: Consider the finite-dimensional, single-input, single-output system

$$\dot{y}(t) = y(t) + u(t).$$

In order to stabilize this system we want to apply the output feedback

$$u(t) = -2y(t).$$

However, due to computational delay we actually apply the feedback

$$u(t) = -2y(t-h),$$

where h is a small positive number.

a. Formulate the closed-loop dynamics as an abstract differential equation.
b. We call $h_{\max}$ the largest h such that the system is stable for $0 \leq h < h_{\max}$.
Use Exercise 6.7 to show that $1 < h_{\max} < 1.5$.
c. Find $h_{\max}$ within an accuracy of 2 percent.

8.10. *Stabilizability by High Gain Feedback*:
Consider the system of Exercise 6.8 defined on $\boldsymbol{L}_2(0,1)$:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + b(x)u(t);$$
$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t);$$
$$y(t) = \int_0^1 z(x,t)c(x)dx,$$

where $b(x) = c(x) = \mathbb{1}_{[1/2,1]}(x)$.

a. For what values of β is the system β-exponentially stable?
b. For what values of β is the system β-exponentially stabilizable?
c. Consider the feedback $u(t) = ky(t)$.
   i. Prove that for every k in $\mathbb{R}$ the eigenvalues of $A + kBC$ are real, where $Bu = b(x)u$ and $Cz = \int_0^1 z(x)c(x)dx$.

ii. Calculate the largest eigenvalue of the closed-loop system i.e., for $A + kBC$, for $k = 0, -1, -10, -100$, all correct to 5 percent.
Hint: See Exercise 6.9.

iii. For which of the above mentioned values of k is the system (-5)-exponentially stable, and why?
Hint: See Exercise 8.2.

iv. For what values of β does there exist a $k < 0$ such that $A + kBC$ generates a β-exponentially semigroup?
Hint: See Exercise 6.9.

8.11. In Section 8.3, we have only considered compensators that stabilize the closed-loop system. However, one would often like to ensure stability with a given decay rate. Let $\Sigma(A, B, C)$ be a state linear system on the Hilbert space Z. Show that for this system the following holds:
There exists a compensator such that the closed-loop system is β-exponentially stable if $\Sigma(A, B, -)$ is β-exponentially stabilizable and $\Sigma(A, -, C)$ is β-exponentially detectable.

8.12. In Theorem 8.3.3, we considered the stabilizing compensator (8.36) without an external input. However, in many applications it is very natural to allow for an external input. Instead of (8.36) we apply the compensator

$$\begin{aligned} \dot{\hat{z}}(t) &= (A + LC)\hat{z}(t) + Bu(t) - Ly(t) \\ u(t) &= F\hat{z}(t) + v(t), \end{aligned} \tag{8.39}$$

where $F \in \mathcal{L}(Z, U)$ and $L \in \mathcal{L}(Y, Z)$ are such that $A + BF$ and $A + LC$ are β-exponentially stable and $v(t)$ is the external input.

a. Formulate the closed-loop system as a linear system $\Sigma(A^e, B^e, C^e)$ on the state space $Z^e = Z \oplus Z$, where $v(t)$ is the input, $z^e(t) = \begin{pmatrix} z(t) \\ \hat{z}(t) \end{pmatrix}$ is the state, and $y(t)$ is the output.
b. Prove that the infinitesimal generator A^e is β-exponentially stable.
c. Calculate the transfer function of the closed-loop system $\Sigma(A^e, B^e, C^e)$ from v to y.

8.13. Consider the infinite-dimensional linear system $\Sigma(A_1, B_1, C_1, D_1)$ connected in feedback with the linear system $\Sigma(A_2, B_2, C_2, D_2)$ as shown in Figure 8.3. The inputs to the closed-loop system are u_1 and u_2, and the

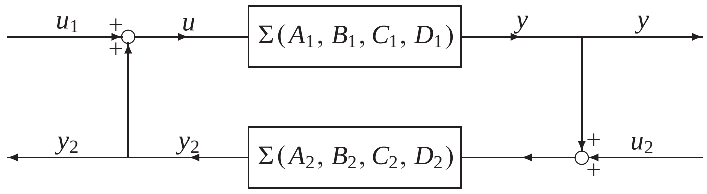

Figure 8.3. Closed-loop system of Exercise 8.13

outputs are y and y_2. Assume that $I + D_1 D_2$ is invertible.

a. Find the resulting closed-loop state-space formulation. Show that it is a state linear system $\Sigma(A^e, B^e, C^e, D^e)$ and give the expressions for A^e, B^e, C^e, and D^e.
b. Prove that if $\Sigma(A_1, B_1, C_1, D_1)$ and $\Sigma(A_2, B_2, C_2, D_2)$ are exponentially stabilizable, then $\Sigma(A^e, B^e, C^e, D^e)$ is also exponentially stabilizable.
c. Prove that if $\Sigma(A_1, B_1, C_1, D_1)$ and $\Sigma(A_2, B_2, C_2, D_2)$ are exponentially detectable, then $\Sigma(A^e, B^e, C^e, D^e)$ is also exponentially detectable.
d. Express the transfer function of $\Sigma(A^e, B^e, C^e, D^e)$ in terms of the transfer functions of the systems $\Sigma(A_1, B_1, C_1, D_1)$ and $\Sigma(A_2, B_2, C_2, D_2)$.
e. Assume that the input and output spaces are finite-dimensional. Prove that the conditions in b and c are necessary and sufficient.
Hint: Prove first that the state linear system

$$\Sigma(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, -)$$

is exponentially stabilizable if and only if the state linear systems $\Sigma(A_1, B_1, -)$ and $\Sigma(A_2, B_2, -)$ are exponentially stabilizable.

8.14. In Section 8.3, we designed an infinite-dimensional compensator. In this and the next exercise, we shall show one way to design a finite-dimensional compensator.
Consider the state linear system $\Sigma(A, B, C)$ with finite-dimensional input and output spaces. Assume that $F \in \mathcal{L}(Z, U)$ and $L \in \mathcal{L}(Y, Z)$ are such that $A + BF$ and $A + LC$ generate exponentially stable semigroups. Assume further that there exists a finite-dimensional subspace V such that V is $T_{BF}(t)$-invariant and ran $L \subset V$.

a. Show that the system

$$\begin{aligned} \dot{\hat{z}}(t) &= N\hat{z}(t) + L(\hat{y} - y(t)) \\ \hat{y}(t) &= C|_V \hat{z}(t) \\ u(t) &= F|_V \hat{z}(t), \end{aligned} \tag{8.40}$$

where $N = (A + BF)|_V$, is a well defined finite-dimensional system with state space V.
b. Show that (8.40) is a stabilizing compensator for the state linear system $\Sigma(A, B, C)$.

8.15. In this exercise, we shall show that if the state linear system $\Sigma(A, B, C)$ is such that A is a Riesz-spectral operator and B and C have finite rank, then it is stabilizable by a finite-dimensional compensator.
Let A be a Riesz-spectral operator, and let B be the finite-rank input operator. Assume further that $\Sigma(A, B, -)$ is exponentially stabilizable. As

stabilizing feedback operator we choose the feedback $F = F_0 P_0$ of Theorem 8.1.6, where F_0 is such that $A_0^+ + B_0^+ F_0$ is exponentially stable and P_0 is the projection on the unstable eigenvalues of A. Assume that $\sigma(A_0^+ + B_0^+ F_0) \cap \sigma(A) = \emptyset$ and $A_0^+ + B_0^+ F_0$ has no generalized eigenvectors. Notice that we can find always an F_0 such that this is satisfied.

a. Prove that $\sigma(A + BF) = \sigma(A_0^+ + B_0^+ F_0) \cup \sigma(A_0^-)$.
b. With the notation of (8.9)–(8.11) show that the eigenvectors of $A + BF$ are given by

$$\varphi_i = \begin{pmatrix} v_i \\ (\lambda_i I - A_0^-)^{-1} B_0^- F_0 v_i \end{pmatrix}, \tag{8.41}$$

for $\lambda_i \in \sigma(A_0^+ + B_0^+ F_0)$ with eigenvector v_i and

$$\varphi_i = \begin{pmatrix} 0 \\ w_i \end{pmatrix}, \qquad \text{for } \lambda_i \in \sigma(A_0^-). \tag{8.42}$$

c. Show that $\{\varphi_i, i \geq 1\}$ is maximal.
d. If L is a stabilizing output-injection operator, show that for any $\varepsilon > 0$ there exist N and $L_N \in \mathcal{L}(Y, Z)$ such that $\operatorname{ran} L_N \subset \underset{1 \leq i \leq N}{\operatorname{span}} \{\varphi_i\}$ and $\|L - L_N\| < \varepsilon$.
e. Prove that if the state linear system is exponentially stabilizable and detectable, then there exists a finite-dimensional compensator.
Hint: Use Exercises 8.3 and 8.14 and notice that $\underset{1 \leq i \leq N}{\operatorname{span}} \{\varphi_i\}$ is $T_{BF}(t)$-invariant.

8.16. Consider the single-input, single-output system $\Sigma(A, B, C)$ described by

$$\dot{z}(t) = Az(t) + Bu(t), \ y(t) = Cz(t). \tag{8.43}$$

In this chapter, we have seen how we can apply state feedback such that the state, and hence the output, converges to zero. In this exercise, we shall see that we can apply similar ideas to steer the output to an arbitrary constant q. This is the problem of *tracking*. We shall use a feedback of the form

$$u(t) = Fz(t) + v, \tag{8.44}$$

where v is an element of $\mathbb{C}$.

a. Assume that F is a stabilizing feedback operator. Show that state trajectory of the closed-loop system

$$\dot{z}(t) = (A + BF)z(t) + Bv$$

converges to $-(A + BF)^{-1} Bv$ as $t \to \infty$.
b. Assume that $C(A + BF)^{-1} B \neq 0$. Find v such that $\lim_{t \to \infty} y(t) = q$.
c. Assume that $\Sigma(A, B, C)$ is also exponentially detectable. Design a compensator such that the output of the closed-loop system converges to q as t goes to infinity.

8.17. Consider the simple supported beam of Exercise 5.1.iv with the same control term, but the observation

$$y(t) = \int_0^1 b(x) \frac{\partial f}{\partial t}(x,t) dx.$$

a. Show that this can be formulated as a state linear system of the form $\Sigma(A, B, B^*)$ on a suitable state space Z, using the results of Exercise 3.9

b. Prove that with the static output feedback $u(t) = -y(t) + v(t)$, the closed loop system becomes strongly system stable. Is it exponentially stable?

8.18. Consider the discrete-time system

$$\begin{aligned} z(n+1) &= Az(n) + Bu(n) \\ y(n) &= Cz(n), \end{aligned} \tag{8.45}$$

where $A \in \mathcal{L}(Z)$, $B \in \mathcal{L}(U, Z)$, $C \in \mathcal{L}(Z, Y)$, and Z, U, Y are Hilbert spaces. We say that the discrete-time system (8.45) is *power stabilizable* if there exists an $F \in \mathcal{L}(Z, U)$ such that $A + BF$ is power stable.
Let $\mathbb{D}_\delta$ denote the disk $\{s \in \mathbb{C} \mid |s| < \delta\}$ and let $\mathbb{E}_\delta$ denote its complement.

a. Prove that if (8.45) is power stabilizable and B is compact, then there exists a $\delta < 1$ such that $\sigma(A) \cap \mathbb{E}_\delta$ is a pure point spectrum.

b. Suppose that B has finite rank and that the system (8.45) is power stabilizable.

   i. Prove that there exists a $\delta < 1$ such that $s \in \sigma(A) \cap \mathbb{E}_\delta$ if and only if $I + F(sI - A - BF)^{-1}B$ is not invertible.
   Hint: See the proof of Lemma 8.1.4 and notice that equalities (8.4)–(8.7) hold here also.

   ii. Show that $\sigma(A) \cap \mathbb{E}_\delta$ consists of, at most, finitely many eigenvalues with finite multiplicity.
   Hint: Use the fact that $\sigma(A) \subset \mathbb{D}_{\|A\|}$.

c. Formulate and prove the discrete-time version of Theorem 8.1.6.

8.5 Notes and references

There are many concepts of stability for infinite-dimensional systems, and the most important one is exponential stability. Its relation to the existence of a positive solution to the Lyapunov equation has been proved in Datko [44], as was the useful necessary and sufficient condition of Lemma 7.1.2. The latter has been generalized from Hilbert to Banach spaces and from $\boldsymbol{L}_2$- to $\boldsymbol{L}_p$-spaces in Pazy [123]. Indeed, many of the results in this chapter also hold on Banach spaces; for more details on this aspect we refer to Bensoussan et al. [19] and Pazy [125]. The fact that the spectrum of the generator need not determine the growth bound of the semigroup was demonstrated in an indirect way by an example in Hille and Phillips [79, p. 665] and by a more straightforward example in Zabczyk [174]. The counterexamples given in Example 7.1.6 and Exercise 7.9 are adapted from examples in Greiner, Voigt, and Wolff [73]. Of course, it is useful to know which generators have the property that the spectrum of the generator determines the growth bound of the semigroup; the term *spectrum determined growth assumption* for this property was introduced in Triggiani [163]. Sufficient conditions for this to hold are any of the following:

a. A is bounded;

b. $T(t)z$ is differentiable for all $z \in Z$ and $t > 0$;

c. $T(\tau)$ is compact for some $\tau > 0$;

d. A is a Riesz-spectral operator.

Conditions a and b were proved in Triggiani [163], c was proved in Zabczyk [174] and d is proved in Theorem 3.2.8.c. Parabolic partial differential equations typically satisfy b or d and retarded equations satisfy c. Unfortunately, the spectrum determined growth assumption is not preserved under very simple perturbations (see Zabczyk [174]). More recently, in Prüss [129] and Huang [81] necessary and sufficient conditions for the spectrum determined growth assumption to hold were given. The proofs presented in Theorems 7.1.5 and 7.1.7 are new and are used to show that retarded systems satisfy the spectrum determined growth assumption. Of course, the latter result was well known (see Hale [77]). We also mention the existence of a Lyapunov stability theory for proving the stability of nonlinear equations via a Lyapunov functional (see Walker [168] and Pazy [124]).

The literature on stabilizability is extensive, and we refer to the useful surveys by Pritchard and Zabczyk [128] and by Russell [143]. Sufficient conditions for exponential stabilizability were obtained fairly early in 1975 by Triggiani [163] (see also Wang [169] and Bhat [22]), but it was only in 1985 by Desch and Schappacher [52] that it was shown that these conditions were also necessary for finite-rank inputs (see Theorem 8.1.6). Other proofs of this important result can be found in Jacobson and Nett [85] and Nefedov and Sholokhovich [111]; the proof in Section 8.1 was inspired by the latter proof. In Rebarber [134] it was shown that this result also holds if B is a compact operator. This explains the previous result in Gibson [66] on the lack of exponential stabilizability of oscillatory systems by compact feedback (see also Triggiani [164]). For unbounded control operators a similar result holds, see [83]. Althought for unbounded input operators the unstable part of the state space may be infinite-dimensional, it still consists of only eigenvectors. In this book, we have concentrated on deriving simple tests for exponential stabilizability (and detectability) for finite-rank systems of either the Riesz-spectral type or the retarded delay type. The tests for self-adjoint generators were proven in Curtain and Pritchard [39], but the proof for the general Riesz-spectral case is new. The Hautus tests in Theorem 8.2.4 for retarded systems first appeared in Bhat [22], and easily verifiable necessary and sufficient conditions for stabilizability of the type derived in Theorem **??** can be found in Olbrot [115], Pandolfi [119], and Manitius and Triggiani [104, 105]. In infinite dimensions, it is not possible to achieve arbitrary eigenvalue assignment, but interesting results on partial assignment can be found in Clarke and Williamson [31], Russell [141], Sun [155], and Rebarber [133]. Exercise **??** on boundary control systems was inspired by Zabczyk [175], but for stabilizability of more general boundary control systems we refer to Bensoussan et al. [20], for neutral systems to Salamon [146], and for abstract linear systems see Weiss and Rebarber [171].

The theory of compensator design is a straightforward extension of the finite-dimensional theory and has been used as a starting point in many control designs for distributed parameter systems (see Orner and Foster [118], Kitamura et al. [90], Sakawa and Matsushita [145], Balas [7], Gressang and Lamont [74], and Fuji [64]). These compensators are infinite dimensional and hence are not implementable. In Exercises 8.14 and 8.15, a direct design of finite-dimensional compensators is developed that is an adaptation of the approach by Schumacher in [149] and [148]. Alternative direct state-space finite-dimensional compensator designs can be found in Bernstein and Hyland [21], Curtain [34], Kamen et al. [87], and Sakawa [144] (see Exercise **??**). For extensions to systems with unbounded input and output operators see Curtain [35] and Curtain and Salamon [41], and for a comparison of various finite-dimensional control designs see Curtain [36]. For more recent results on dynamic compensators for abstract linear systems see Weiss and Curtain [170].

The weaker concept of strong stability has been studied in Benchimol [16]–[18], Huang [82], Balakrishnan [6], Batty [10], Arendt and Batty [1] and Batty and Phong [11] (see also Russell [142] and Slemrod [151]). Recently, Tomilov [162] and Guo, Zwart, and Curtain [76] derived new characterizations of strong stability. These results resemble those obtained for exponential stability in Section 7.1. However, the useful concept of strongly system stable is of more recent origin in Staffans [153]. Our treatment follows the thesis by Oostveen [116] which also contains a theory for dynamic compensator design and many physical examples of such systems.

The stabilizability concepts for discrete-time systems investigated in Exercises 7.14 and 7.15 were first studied in Przyluski [130]. Exercise 8.18 is taken from Logemann [98].

References

[1] W. Arendt and C.J.K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Transactions of the American Mathematical Society*, 306:837–841, 1988.

[2] W. Arendt, F. Räbiger, and A. Sourour. Spectral properties of the operator equation $AX + XB = Y$. *Quart. J. Math. Oxford Ser. (2)*, 45(178):133–149, 1994.

[3] J.P. Aubin. *Applied Functional Analysis*. John Wiley & Sons, New York, 1979.

[4] J. Bak and D.J. Newman. *Complex Analysis*. Springer, New York, second edition, 1997.

[5] A.V. Balakrishnan. *Applied Functional Analysis*. Springer Verlag, Berlin, 1976.

[6] A.V. Balakrishnan. Strong stabilizability and the steady state Riccati equation. *Applied Mathematics and Optimization*, 7:335–345, 1981.

[7] M. Balas. Towards a (more) practical control theory for distributed parameter systems, control and dynamic systems. In C.T. Leondes, editor, *Advances in Theory and Applications*, volume 18. Academic Press, New York, 1980.

[8] J. Ball. Strongly continuous semigroups, weak solutions and the variation of constants formula. *Proc. Amer. Math. Soc.*, 63:370–373, 1977.

[9] B. Bamieh, F. Paganini, and M.A. Dahleh. Distributed control of spatially invariant systems. *IEEE Trans. Automat. Control*, 47(7):1091–1107, jul 2002.

[10] C.J.K. Batty. Tauberian theorems for the laplace-stieltjes transform. *Trans. Amer. Math. Soc.*, 322:783–804, 1990.

[11] C.J.K. Batty and V.Q. Phong. Stability of individual elements under one-parameter semigroups. *Trans. Amer. Math. Soc.*, 322:805–818, 1990.

[12] F. Bayazit and R. Heymann. Stability of multiplication operators and multiplication semigroups. *arXiv*, math.FA, Feb 2012.

[13] A. Bellini-Morante. *Applied Semigroups and Evolution Equations*. Clarendon Press, Oxford, 1979.

[14] R. Bellman and K.L. Cooke. Asymptotic behavior of solutions of differential-difference equations. *Mem. Amer. Math. Soc.*, 35, 1959.

[15] R. Bellman and K.L. Cooke. *Differential-Difference Equations*. Academic Press, 1963.

[16] C.D. Benchimol. *The Stabilizability of Infinite Dimensional Linear Time Invariant Systems*. PhD thesis, Thesis, UCLA, 1977.

[17] C.D. Benchimol. Feedback stabilizability in Hilbert spaces. *J. App. Math. and Opt.*, 4:225–248, 1978.

[18] C.D. Benchimol. A note on weak stabilizability of contraction semigroup. *SIAM J. Control and Optim.*, 16:373–379, 1978.

[19] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems*, volume 1 of *Systems & Control: Foundations & Applications*. Birkhäuser, Boston, 1992.

[20] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems*, volume 2 of *Systems & Control: Foundations & Applications*. Birkhäuser, Boston, 1993.

[21] D.S. Bernstein and D.C. Hyland. The optimal projection equations for finite-dimensional fixed-order dynamic compensation of infinite-dimensional systems. *SIAM J. Contr. and Optim.*, 24:122–151, 1986.

[22] K.P.M. Bhat. *Regulator Theory for Evolution Systems*. PhD thesis, University of Toronto, 1976.

[23] H. Blomberg and R. Ylinen. *Algebraic Theory for Multivariable Systems*. Academic Press, 1983.

[24] S. Bochner and K. Chandrasekharan. *Fourier Transforms*. Number 19 in Annals of Mathematics Studies. Princeton University Press, Princeton, 1949.

[25] H. Bohr. *Almost Periodic Functions*. Chelsea Publishing Company, New York, 1947.

[26] A.G. Butkovskiy. *Theory of Optimal Control of Distributed Parameter Systems*. American Elsevier, 1969.

[27] P.L. Butzer and H. Berens. *Semigroups of Operators and Approximations*. Springer Verlag, 1967.

[28] F.M. Callier and C.A. Desoer. An algebra of transfer functions for distributed linear time-invariant systems. *IEEE Trans. Circuits and Systems*, CAS-25:651–663, 1978. (Corrections: CAS-26, p. 320, 1979).

[29] F.M. Callier and C.A. Desoer. Stabilization, tracking and distributed rejection in multivariable convolution systems. *Ann. Soc. Sci. Bruxelles*, 94:7–51, 1980.

[30] M. Capiński and E. Kopp. *Measure, integral and probability*. Springer Undergraduate Mathematics Series. Springer-Verlag London Ltd., London, second edition, 2004.

[31] B.M.N. Clarke and D. Williamson. Control canonical forms and eigenvalue assignment by feedback for a class of linear hyperbolic systems. *SIAM J. Control and Optim.*, 19:711–729, 1981.

[32] B.D. Coleman and V.J. Mizel. Norms and semigroups in the theory of fading memory. *Arch. Rational Mech. Anal.*, 23:87–123, 1966.

[33] C. Corduneanu. *Alomst Periodic Functions*. J. Wiley, New York, 1968.

[34] R.F. Curtain. Stabilization of boundary control distributed systems via integral dynamic output feedback of a finite-dimensional compensator. In A. Bensoussan and J.L. Lions, editors, *Analysis and Optimization of Systems*, volume 44 of *Lecture Notes in Control and Information Sciences*, pages 761–776. Springer Verlag, 1982.

[35] R.F. Curtain. Finite-dimensional compensators for parabolic distributed systems with unbounded control and observation. *SIAM J. Control and Optim.*, 22:255–276, 1984.

[36] R.F. Curtain. A comparison of finite-dimensional controller designs for distributed parameter systems. *Control Theory and Advanced Technology*, 9:609–628, 1993.

[37] R.F. Curtain, O.V Iftime, and H. Zwart. System theoretic properties of a class of spatially distributed systems. *Automatica*, 45:1619–1627, 2009.

[38] R.F. Curtain and K. Morris. Transfer functions of distributed parameter systems; a tutorial. *Automatica*, 45:1101–1116, 2009.

[39] R.F. Curtain and A.J. Pritchard. The infinite dimensional Riccati equation. *J. Math. Anal. & Appl.*, 47:43–57, 1974.

[40] R.F. Curtain and A.J. Pritchard. *Infinite-Dimensional Linear Systems Theory*, volume 8 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1978.

[41] R.F. Curtain and D. Salamon. Finite dimensional compensators for infinite dimensional systems with unbounded input operators. *SIAM J. Control and Optim.*, 24:797–816, 1986.

[42] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Number 21 in Texts in Applied Mathematics. Springer-Verlag, New York, 1995.

[43] Ju. L. Dalec′kiĭ and M. G. Kreĭn. *Stability of Solutions of Differential Equations in Banach Space*. American Mathematical Society, Providence, R.I., 1974. Translated from the Russian by S. Smith, Translations of Mathematical Monographs, Vol. 43.

[44] R. Datko. Extending a theorem of A.M. Lyapunov to Hilbert space. *J. Math. Anal. Appl.*, 32:610–616, 1970.

[45] E.B. Davies. *One-Parameter Semigroups*. Academic Press, London, 1980.

[46] M.C. Delfour. State theory for linear hereditary-differential systems. *Anal. Appl.*, 60:8–35, 1977.

[47] M.C. Delfour. The largest class of hereditary systems defining a C_0-semigroup on the product space. *Canadian J. Math.*, XXXII(4):969–978, 1980.

[48] M.C. Delfour and J. Karrakchou. State space theory of linear time invariant systems with delays in state, control and observation variables, part I. *J. Math. Anal. and Appl.*, 125:361–399, 1987.

[49] M.C. Delfour and J. Karrakchou. State space theory of linear time invariant systems with delays in state, control and observation variables, part II. *J. Math. Anal. and Appl.*, 125:400–450, 1987.

[50] M.C. Delfour and S.K. Mitter. Hereditary differential systems with constant delays I: General case. *J. Diff. Eqns.*, 12:213–235, 1972.

[51] M.C. Delfour and S.K. Mitter. Hereditary differential systems with constant delays II: A class of affine systems and the adjoint problem. *J. Diff. Eqns.*, 18:18–28, 1975.

[52] W. Desch and W. Schappacher. Spectral properties of finite-dimensional perturbed linear semigroups. *J. Diff. Eqns.*, 59:80–102, 1985.

[53] J. Diestel and J. J. Uhl, Jr. *Vector Measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[54] G. Doetsch. *Introduction to the Theory and Application of Laplace Transform*. Springer Verlag, Berlin, 1974.

[55] N. Dunford and J.T. Schwartz. *Linear Operators, part 1*. Interscience, 1959.

[56] N. Dunford and J.T. Schwartz. *Linear Operators, part 3*. Interscience, 1971.

[57] H. Dym and H. P. McKean. *Fourier series and integrals*. Academic Press, New York, 1972. Probability and Mathematical Statistics, No. 14.

[58] K.-J. Engel and R. Nagel. *One-parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[59] H.O. Fattorini. Boundary control systems. *SIAM J. Control*, 6:349–388, 1968.

[60] B.A. Francis. *A Course in H_∞-Control Theory*, volume 88 of *LNCIS*. Springer Verlag, Berlin, 1987.

[61] D. Franke. *Systeme mit Ortlich Verteilten Parametern. Eine Einführung in die Modellbildung, Analyse und Regelung*. Springer-Verlag, Berlin, Heidelberg, 1987.

[62] J. M. Freeman. The tensor product of semigroups and the operator equation $SX - XT = A$. *J. Math. Mech.*, 19:819–828, 1969/1970.

[63] P. Fuhrman. *Linear Systems and Operators in Hilbert Space*. McGraw-Hill, New York, 1981.

[64] N. Fuji. Feedback stabilization of distributed parameter systems by a functional observed. *SIAM J. Control and Optim.*, 18:108–121, 1980.

[65] G. Ghu, P.P. Khargonekar, and E.B. Lee. Approximation of infinite-dimensional systems. *IEEE Trans. Autom. Control*, AC-34:610–618, 1989.

[66] J.S. Gibson. A note on stabilization of infinite dimensional linear oscillators by compact linear feedback. *SIAM J. Control and Optim.*, 18:311–316, 1980.

[67] K. Glover, R.F. Curtain, and J.R. Partington. Realisation and approximation of linear infinite dimensional systems with error bounds. *SIAM J. Control and Optim.*, 26:863–898, 1988.

[68] K. Glover, J. Lam, and J.R. Partington. Rational approximation of a class of infinite dimensional systems I: Singular values of Hankel operators. *MCSS*, 3:325–344, 1990.

[69] K. Glover, J. Lam, and J.R. Partington. Rational approximation of a class of infinite-dimensional systems II: Optimal convergence rates of L_∞-approximants. *MCSS*, 4:233–246, 1991.

[70] K. Glover, J. Lam, and J.R. Partington. Rational approximation of a class of infinite dimensional systems III: The L_2-case. In P. Nevai and A. Pinhaus, editors, *Progress in Approximation Theory*, pages 405–440. Academic Press, 1991.

[71] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of Linear Operators. Vol. II*, volume 63 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1993.

[72] J.A. Goldstein. *Semigroups of Linear Operators and Applications*. Oxford Mathematical Monographs. Oxford University Press, New York, 1985.

[73] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, 5:245–256, 1981.

[74] R. Gressang and G. Lamont. Observers for systems characterized by semigroups. *IEEE Trans. Autom. Control*, AC-20:523–528, 1975.

[75] G. Gripenberg, S.O. Londen, and O. Staffans. *Volterra Integral and Functional Equations*. Cambridge University Press, Cambridge, 1990.

[76] B.Z. Guo, H. Zwart, and R.F. Curtain. On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform. Technical Report 1593, University of Twente, the Netherlands, 2001. availible at http://www.math.utwente.nl/publications/.

[77] J.K. Hale. *Theory of Functional Differential Equations*. Springer Verlag, New York, 1977.

[78] H. Helson. *Harmonic Analysis*. Addison-Wesley, London, 1983.

[79] E. Hille and R.S. Phillips. *Functional Analysis and Semigroups*, volume 31. Amer. Math. Soc. Coll. Publ., Providence, R.I., 1957.

[80] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York, 1974.

[81] F. Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. of Diff. Equations*, 1:43–55, 1985.

[82] F. Huang. Strong asymptotic stability of linear dynamical systems in Banach spaces. *J. Diff. Eqns.*, 104:307–324, 1993.

[83] B. Jacob and H. Zwart. Equivalent conditions for stabilizability of infinite-dimensional systems with admissible control operators. *SIAM Journal on Control and Optimization*, 37:1419–1455, 1999.

[84] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, volume 223 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2012.

[85] C.A. Jacobson and C.N. Nett. Linear state space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans. Autom. Control*, AC-33:541–550, 1988.

[86] N. Jacobson. *Lectures in Abstract Algebra, Vol. 1*. Van Nostrand, New York, 1953.

[87] E.W. Kamen, P.P. Khargonekar, and A. Tannenbaum. Stabilization of time-delay systems using finite-dimensional compensators. *IEEE Trans. Autom. Control*, AC-30:75–78, 1985.

[88] T. Kato. *Perturbation Theory of Linear Operators*. Springer Verlag, 1966.

[89] T. Kawata. *Fourier Analysis in Probability Theory*. Academic Press, New York and London, 1972.

[90] S. Kitamura, H. Sakairi, and M. Mishimura. Observers for distributed parameter systems. *Electrical Eng. in Japan*, 92:142–149, 1972.

[91] G. Klambauer. *Real Analysis*. Dover Publication, Inc., Mineola, New York, 2005. Republication of the work published by the American Elsevier Publishing Company, 1973.

[92] E. Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley & Sons, New York, 1978.

[93] B. Ya. Levin. *Lectures on Entire Functions*, volume 150 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.

[94] N. Levison and R.M. Redheffer. *Complex Variables*. Holden-Day, Inc., San Francisco, 1970.

[95] J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problem, I, II, III*. Springer Verlag, Berlin, 1972.

[96] H. Logemann. *Funktionentheoretische Methoden in der Regelungstheorie Unendlichdimensionaler Systeme*. PhD thesis, Institut für Dynamische Systeme, Universitát Bremen, Germany, 1986. Report nr. 156.

[97] H. Logemann. Finitely generated ideals in certain algebras of transfer functions of infinite-dimensional systems. *Int. J. Control*, 45:247–250, 1987.

[98] H. Logemann. Stability and stabilizability of linear infinite-dimensional discrete-time systems. *IMA Journal of Mathematical Control & Information*, 9:255–263, 1992.

[99] L.H. Loomis. *An introduction to abstract harmonic analysis*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1953.

[100] G. Lumer and R.S. Phillips. Dissipative operators in a Banach space. *Pacific J. Math.*, 11:679–698, 1961.

[101] Gunter Lumer and Marvin Rosenblum. Linear operator equations. *Proc. Amer. Math. Soc.*, 10:32–41, 1959.

[102] S. MacLane and G. Birkhoff. *Algebra*. MacMillan, New York, 1965.

[103] P.M. Mäkilä. Laguerre series approximation of infinite-dimensional systems. *Automatica*, 26:985–996, 1990.

[104] A. Manitius and R. Triggiani. Function space controllability of linear retarded systems: A derivation from abstract operator conditions. *SIAM J. Control and Optim.*, 16:599–645, 1978.

[105] A. Manitius and R. Triggiani. Sufficient conditions for function space controllability and feedback stabilizability of linear retarded systems. *IEEE Trans. Autom. Control*, AC-23:659–665, 1978.

[106] M. Marcus. *Introduction to Modern Algebra*. Marcel Dekker Inc., New York, 1978.

[107] S.M. Melzer and B.C. Kuo. Optimal regulation of systems described by a countably infinite number of objects. *Automatica*, 7:359–366, 1971.

[108] A.S. Morse. System invariants under feedback and cascade control. In *Proc. Int. Conf. on Math. Syst. Theory*, Udine, Italy, 1976.

[109] S. Mossaheb. On the existence of right coprime factorizations for functions meromorphic in a half-plane. *IEEE Trans. Autom. Control*, AC-25:550–551, 1980.

[110] A.W. Naylor and G.R. Sell. *Linear Operator Theory in Engineering and Science*, volume 40 of *Applied Mathematical Sciences*. Springer, New York, 1982.

[111] S.A. Nefedov and F.A. Sholokhovich. A criterion for the stabilizability of dynamical systems with finite-dimensional input. *Differential'nye Uravneniya*, 22:163–166, 1986.

[112] C.N. Nett. The fractional representation approach to robust linear feedback design: A self-contained exposition. Master's thesis, Dept. of ECSE, Rensselaer Polytechnic Institute, Troy, NY, USA, 1984.

[113] N. K. Nikol′skiĭ. *Treatise on the Shift Operator*, volume 273 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1986. Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.

[114] R. Ober and S. Montgomery-Smith. Bilinear transformation of infinite-dimensional state-space systems and balanced realisation of nonrational transfer function. *SIAM J. Control and Optim.*, 28:438–465, 1990.

[115] A.W. Olbrot. Stabilizability, detectability and spectrum assignment for linear autonomous systems with general time delays. *IEEE Trans. Autom. Control*, AC-23:887–890, 1978.

[116] J.C. Oostveen. *Strongly Stabilizable Distributed Parameter Systems*. SIAM, Philadelphia, 2000.

[117] M. Opmeer and R.F. Curtain. The suboptimal Nehari problem for well-posed infinite-dimensional systems. *SIAM J. Control and Optimization*, 44, 2005.

[118] P.A. Orner and A.M. Foster. A design procedure for a class of distributed parameter control systems. *Trans. A.S.M.E. Series G. Journal of Dynamical Systems, Measurment & Control*, 93:86–93, 1971.

[119] L. Pandolfi. On feedback stabilization of functional differential equations. *Boll. UHI 4, Il Supplemento al Fascicolo 3*, XI(IV):626–635, 1975.

[120] L. Pandolfi. Generalized control systems, boundary control systems, and delayed control systems. *MCSS*, 3:165–181, 1990.

[121] J. R. Partington. *Linear Operators and Linear Systems*, volume 60 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004. An analytical approach to control theory.

[122] J.R. Partington, K. Glover, H.J. Zwart, and R.F. Curtain. L_∞-approximation and nuclearity of delay systems. *Systems and Control Letters*, 10:59–65, 1988.

[123] A. Pazy. Asymptotic behavior of the solution of an abstract evolution equation and some applications. *J. Diff. Eqns.*, 4:493–509, 1968.

[124] A. Pazy. On the applicability of Lyapunov's theorem in Hilbert spaces. *SIAM J. Math. Anal.*, 3:291–295, 1972.

[125] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer Verlag, New York, 1983.

[126] Vũ Quôc Phóng. The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems. *Math. Z.*, 208(4):567–588, 1991.

[127] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory, A Behavioral Approach*, volume 26 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1998.

[128] A.J. Pritchard and J. Zabczyk. Stability and stabilizability of infinite dimensional systems. *SIAM Review*, 23, 1981.

[129] J. Prüss. On the spectrum of C_0-semigroups. *Trans. Am. Math. Soc.*, 284:847–856, 1984.

[130] K.M. Przyluski. The Lyapunov equations and the problem of stability for linear bounded discrete-time systems in Hilberts space. *Appl. Math. Optim.*, 6:97–112, 1980.

[131] C. R. Putnam. *Commutation Properties of Hilbert Space Operators and Related Topics*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36. Springer-Verlag New York, Inc., New York, 1967.

[132] W.H. Ray. *Advanced Process Control*. McGraw-Hill, New York, London, 1981.

[133] R. Rebarber. Spectral assignability for distributed parameter systems with unbounded scalar control. *SIAM J. Control and Optim.*, 27:148–169, 1989.

[134] R. Rebarber. Necessary conditions for exponential stabilizability of distributed parameter systems with infinite-dimensional unbounded control. *Systems and Control Letters*, 14:241–248, 1990.

[135] M. Rosenblum. On the operator equation $BX - XA = Q$. *Duke Math. J.*, 23:263–269, 1956.

[136] M. Rosenblum and J. Rovnyak. *Hardy Classes and Operator Theory*. Oxford University Press, New York, 1985.

[137] H. L. Royden. *Real Analysis*. The Macmillan Co., New York, second edition, 1963.

[138] W. Rudin. *Functional Analysis*. McGraw-Hill Book Company, New York, 1973.

[139] W. Rudin. *Principals of Mathematical Analysis*. McGraw-Hill Book Company, New York, third edition, 1976.

[140] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, New York, third edition, 1987.

[141] D.L. Russell. Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems. *J. Math. Anal.*, 62:182–255, 1968.

[142] D.L. Russell. Linear stabilization of the linear oscillator in Hilbert space. *J. Math. Anal. Appl.*, 25:663–675, 1969.

[143] D.L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open problems. *SIAM Review*, 20:639–739, 1978.

[144] Y. Sakawa. Feedback control of second order evolution equations with damping. *SIAM J. Control and Optim*, 22:343–361, 1984.

[145] Y. Sakawa and T. Matsushita. Feedback stabilization for a class of distributed systems and construction of a state estimator. *IEEE Trans. Autom. Control*, AC-20:748–753, 1975.

[146] D. Salamon. *Control and Observation of Neutral Systems*. Number 91 in Research Notes in Mathematics. Pitman Advanced Publishing Program, Boston, 1984.

[147] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21:147–164, 1989.

[148] J.M. Schumacher. *Dynamic Feedback in Finite and Infinite-Dimensional Linear Systems*, volume 143 of *Mathematical Centre Tracts*. Mathematical Centrum, Amsterdam, 1981.

[149] J.M. Schumacher. A direct approach to compensator design for distributed parameter systems. *SIAM J. Control and Optim.*, 21:823–836, 1983.

[150] Sen-Yen Shaw and S. C. Lin. On the equations $Ax = q$ and $SX - XT = Q$. *J. Funct. Anal.*, 77(2):352–363, 1988.

[151] M. Slemrod. A note on complete controllability and stabilizability of linear control systems in Hilbert space. *SIAM J. Control*, 12:500–508, 1974.

[152] O. J. Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[153] O.J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM Journal on Control and Optimization*, 36:1268–1292, 1998.

[154] E.M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[155] S.H. Sun. On spectrum distribution of complete controllable systems. *SIAM J. Control and Optim.*, 19:730–743, 1981.

[156] H. Tanabe. *Equations of Evolution*. Pitman, 1979.

[157] A.E. Taylor. *Introduction to Functional Analysis*. John Wiley, 1958.

[158] A.E. Taylor and D.C. Lay. *Introduction to Functional Analysis*. John Wiley & Sons, New York-Chichester-Brisbane, second edition, 1980.

[159] E.G.F. Thomas. Totally summable functions with values in locally convex spaces. In A. Bellow and D. Kölzow, editors, *Measure Theory*, volume 541 of *Lecture Notes in Mathematics*, pages 117–131, Berlin, 1976. Springer Verlag. Proceedings of the Conference held in Oberwolfach, 15-21 June, 1975.

[160] E.G.F. Thomas. Vector-valued integration with applications to the operator valued H_∞ space. *Journal of Mathematical Control and Information*, 14:109–136, 1997.

[161] E. C. Titchmarsh. *The Theory of Functions*. Oxford University Press, 1939. Second edition.

[162] Y. Tomilov. A resolvent approach to stability of operator semigroups. *J. Operator Theory*, 46:63–98, 2001.

[163] R. Triggiani. On the stabilization problem in Banach space. *J. Math. Anal. Appl.*, 52:383–403, 1975.

[164] R. Triggiani. Lack of uniform stabilization for noncontractive semigroups under compact perturbations. *Proc. Amer. Math. Soc.*, 105:375–383, 1989.

[165] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[166] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press, Cambridge, Mass., USA, 1985.

[167] M. Vidyasagar, H. Schneider, and B.A. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Trans. Autom. Control*, AC-27:880–894, 1982.

[168] J.A. Walker. On the application of Lyapunov's direct method to linear dynamical systems. *J. Math. Analysis and Appl.*, 53:187–220, 1976.

[169] P.K.C. Wang. Modal feedback stabilization of a linear distributed system. *IEEE Trans. Autom.*, AC-17:552–553, 1972.

[170] G. Weiss and R.F. Curtain. Dynamic stabilization of regular linear systems. *IEEE Transactions on Automatic Control*, 42:4–21, 1997.

[171] G. Weiss and R. Rebarber. Optimizability and estimatability for infinite-dimensional systems. *SIAM Journal of Control and Optimization*, 39:1204–1232, 2000.

[172] W.M. Wonham. *Linear Multivariable Control; A Geometric Approach*. Springer Verlag, New York, 1974.

[173] K. Yosida. *Functional Analysis*. Springer Verlag, 1966.

[174] J. Zabczyk. A note on C_0-semigroups. *Bull. l'Acad. Pol. de Sc. Serie Math.*, 23:895–898, 1975.

[175] J. Zabczyk. Stabilization of boundary control systems. *J. Diff. Eqns*, 32, 1979.

[176] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms and approximate inverses. *IEEE Trans. Autom. Control*, AC-26:301–320, 1981.

[177] O. Zariski and P. Samuel. *Commutative Algebra, Vol. 1*. Van Nostrand, New York, 1958.

[178] E. Zauderer. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1989.

[179] H. Zwart. Transfer functions for infinite-dimensional systems. *Systems Control Lett.*, 52(3-4):247–255, 2004.

[180] H.J. Zwart. *Geometric Theory for Infinite-Dimensional Systems*, volume 115 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1989.

[181] H.J. Zwart, R.F. Curtain, J.R. Partington, and K. Glover. Partial fraction expansions for delay systems. *Systems & Control Letters*, 10:235–244, 1988.

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