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Scalarizations For Adaptively Solving Multi-Objective Optimization Problems

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Abstract In this paper several parameter dependent scalarization approaches for solving nonlinear multi-objective optimization problems are discussed and it is shown that they can be considered as special cases of a scalarization problem by Pascoletti and Serafini (or a modification of this problem).

Based on these connections theoretical results as well as a new algorithm for adaptively controlling the choice of the parameters for generating almost equidistant approximations lately developed for the Pascoletti-Serafini scalarization can be applied to these problems. For example for such well-known problems as the e-constraint or the normal boundary intersection method algorithms for adaptively generating high quality approximations are derived.

Keywords multiobjective optimization · scalarization · approximation · sensitivity · adaptive parameter control

Mathematics Subject Classification (2000) 90C29 · 90C31 · 90C59

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1 Introduction

Multi-objective optimization has become an important tool for supporting decision makers because the problems arising nowadays in applications are getting more and more complex. There several competing objectives are aspired at the same time. A very common approach for solving these problems is a reformulation as a parameter dependent scalar optimization problem like it is done in the well-known weighted sum or the ε-constraint method (for surveys of methods see [6], [16], [18], [23], [26]).

With the scalar problems not only one optimal solution of the multi-objective optimization problem can be found but approximations of the whole solution set can be generated by a variation of the parameters. Information about the entire so called efficient set are of great interest for the decision maker as this delivers a valuable insight in the problem and especially in engineering all optimal design alternatives are of interest. But the quality of the information depends mainly on the quality of the approximation and thus a good choice of parameters is important. A high quality with almost equidistant approximation points is achieved with the adaptive parameter control introduced in [8] for the Pascoletti-Serafini scalarization.

In this paper we will show that many important and wide spread scalarization problems can be seen as a special case of the formulation by Pascoletti and Serafini. Because of this we cannot only apply theoretical results gained for the general method but we can do an adaptive parameter control for many of these special methods, too. With that it is then possible to generate almost equidistant approximations with other problems as the ε-constraint or the normal boundary intersection problem, too.

We start in section 2 with a short recall of the basic concepts of multi-objective optimization followed in section 3 by a presentation of the Pascoletti-Serafini method and the mentioned adaptive parameter control. In the main part of this paper in section 4 we discuss several scalarization approaches and their connection to the Pascoletti-Serafini method. For some of the problems we also present an algorithm for an adaptive parameter control.

2 Basic notations

In this work we present scalarization approaches for solving multi-objective optimization problems given by

$$\begin{array}{c} \min f(x) = (f_1(x), \ldots, f_m(x))^\top \\ \text{under the constraint} \\ x \in \Omega \subset \boldsymbol{R}^n, \end{array} \tag{2.1}$$

with $m, n \in \boldsymbol{N},\ m \geq 2$. We consider K-minimal points with $K \subset \boldsymbol{R}^m$ a closed pointed convex cone introducing an antisymmetric partial ordering $\leq_K$ by $\leq_K := \{(x,y) \in \boldsymbol{R}^m \times \boldsymbol{R}^m \mid y - x \in K\}$.

Definition 2.1 A point $\bar{x} \in \Omega$ is called K-minimal of (2.1) if $(f(\bar{x}) - K) \cap f(\Omega) = \{f(\bar{x})\}$. Additionally for $\mathrm{int}(K) \neq \emptyset$ a point $\bar{x} \in \Omega$ is called weakly K-minimal point of (2.1) if $(f(\bar{x}) - \mathrm{int}(K)) \cap f(\Omega) = \emptyset$.

We denote the set of all K-minimal points as $\mathcal{M}(f(\Omega), K)$ and the set of all weakly K-minimal points as $\mathcal{M}_w(f(\Omega), K)$. The set $\mathcal{E}(f(\Omega), K) := \{f(x) \in \boldsymbol{R}^m \mid x \in \mathcal{M}(f(\Omega), K)\}$ is called efficient set and the set $\mathcal{E}_w(f(\Omega), K) := \{f(x) \in \boldsymbol{R}^m \mid x \in \mathcal{M}_w(f(\Omega), K)\}$ weakly efficient set. For $K = \boldsymbol{R}^m_+$ the K-minimal points are denoted as Edgeworth-Pareto (EP)-minimal points, too. The ordering cone $K = \boldsymbol{R}^m_-$ corresponds to multi-objective maximization problems and the $\boldsymbol{R}^m_-$-minimal points are also called EP-maximal points.

3 Pascoletti-Serafini problem and adaptive parameter control

For solving the multi-objective optimization problem (2.1) we formulate a parameter dependent scalar optimization problem and solve this for a variety of parameters. Our focus lies on the scalarization introduced by Pascoletti and Serafini in [24] named (SP(a,r)) which is defined by

$$\begin{array}{c} \min t \\ \text{under the constraints} \\ a + t\,r - f(x) \in K, \\ t \in \boldsymbol{R}, \\ x \in \Omega, \end{array} \tag{3.1}$$

with parameters $a, r \in \boldsymbol{R}^m$. Here the ordering cone $-K$ is moved on the line $a + t\,r,\ t \in \boldsymbol{R}$, in direction $-r$ starting in point a till the set $(a + t\,r - K) \cap f(\Omega)$ is reduced to the empty set. The smallest $\bar{t}$ for which we have $(a + \bar{t}\,r - K) \cap f(\Omega) \neq \emptyset$ is the minimal value of (3.1), see Fig. 3.1.

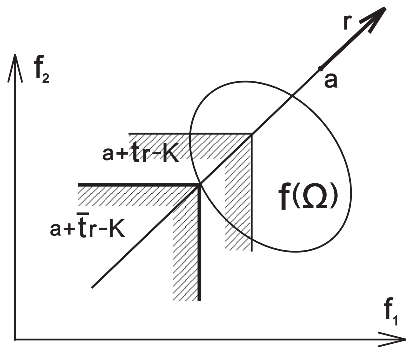

Fig. 3.1 Visualization of the Pascoletti-Serafini problem.

3.1 Scalarization properties

We summarize the main properties of this approach which can be found in [24], [7], too. Problem (3.1) is also extensively discussed in [30] and further in [14], [15], [28], [29].

Theorem 3.1 *Let $K \subset \boldsymbol{R}^m$ be a closed pointed convex cone.*

a) Let $\bar{x} \in \mathcal{M}(f(\Omega), K)$, then $(0, \bar{x})$ is a minimal solution of (3.1) *with $a = f(\bar{x})$ and $r \in K \setminus \{0_m\}$. Let $\bar{x} \in \mathcal{M}_w(f(\Omega), K)$, then $(0, \bar{x})$ is a minimal solution of* (3.1) *with $a = f(\bar{x})$ and $r \in int(K)$.*
b) Let $(\bar{t}, \bar{x})$ be a minimal solution of (3.1)*, then $\bar{x} \in \mathcal{M}_w(f(\Omega), K)$ and $a + \bar{t}\,r - f(\bar{x}) \in \partial K$ with ∂K the boundary of the cone K.*

Especially the set of parameters for which we solve (3.1) is of interest.

Theorem 3.2 *Let $K \subset \boldsymbol{R}^m$ be a closed pointed convex cone with $int(K) \neq \emptyset$, let the set $f(\Omega) + K$ be closed and convex and let $\mathcal{E}(f(\Omega), K) \neq \emptyset$. Then there exists a minimal solution of* (3.1) *for all parameters $(a, r) \in \boldsymbol{R}^m \times int(K)$.*

As a consequence if we solve (3.1) for any choice of parameters $(a,r) \in \boldsymbol{R}^m \times \operatorname{int}(K)$ with $f(\Omega)+K$ closed and convex and there exists no minimal solution of (3.1) then we know that it is $\mathcal{E}(f(\Omega),K)=\emptyset$. This important property is e. g. not fulfilled for the ε-constraint method as we will see later.

In [8], Theorem 3.3, it is shown that we can restrict the parameter set from which we have to choose the parameters a and r to find all EP-minimal points in the bicriteria case by the following. Assume $\bar{x}^i,\ i=1,2$, are the minimal solutions of $\min_{x\in\Omega} f_i(x),\ i=1,2$, and choose a parameter $r \in K$ and a hyper plane $H=\{y \in \boldsymbol{R}^m \mid b^\top y = \beta\}$ with $b \in \boldsymbol{R}^m,\ b^\top r \neq 0,\ \beta \in \boldsymbol{R}$. Set $\bar{t}^i := \frac{b^\top f(\bar{x}^i)-\beta}{b^\top r}$ and $\bar{a}^i := f(\bar{x}^i) - \bar{t}^i\, r$ for $i=1,2$. Then it is sufficient to consider parameters $a \in H^a$ with

$$H^a = \{y \in H \mid y = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2,\ \lambda \in [0,1]\}.$$

Theorem 3.3 *We consider the multi-objective optimization problem* (2.1) *with* $m=2$ *and* $K=\boldsymbol{R}^2_+$. *For any* $\bar{x} \in \mathcal{M}(f(\Omega),K)$ *there exists a parameter* $a \in H^a \subset H$ *and a scalar* $\bar{t} \in \boldsymbol{R}$ *with* $(\bar{t},\bar{x})$ *minimal solution of* (3.1).

In [8], Theorem 3.3, a generalization for ordering cones $K \neq \boldsymbol{R}^2_+$ is presented and in Theorem 3.5 a restriction of the parameter set for the case $m \geq 3$ is given, too.

For our comparisons in the following chapter we need a modified version of (3.1) named $(\overline{\mathrm{SP}}(a,r))$ and given by

$$\begin{array}{c} \min t \\ \text{under the constraints} \\ a + t\,r - f(x) = 0_m, \\ t \in \boldsymbol{R}, x \in \Omega, \end{array} \tag{3.2}$$

too. For this scalarization the property b) of Theorem 3.1 is no longer valid, which means that minimal solutions of (3.2) are not necessarily weakly K-minimal points of (2.1). But we can still find all K-minimal points only by varying the parameter a on a hyper plane.

Theorem 3.4 *Let the hyper plane* $H=\{y \in \boldsymbol{R}^m \mid b^\top y = \beta\}$ *with* $b \in \boldsymbol{R}^m,\ \beta \in \boldsymbol{R}$ *be given an let* $\bar{x} \in \mathcal{M}(f(\Omega),K)$ *and* $r \in K \setminus \{0_m\}$ *with* $b^\top r \neq 0$. *Then there is a parameter* $a \in H$ *and a* $\bar{t} \in \boldsymbol{R}$ *with* $(\bar{t},\bar{x})$ *minimal solution of* (3.2).

Proof According to Theorem 3.2 in [8] there is a parameter $\bar{a} \in H$ and a $t \in \boldsymbol{R}$ with $(t,\bar{x})$ minimal solution of (3.1) for the parameters $(\bar{a},r)$. According to Theorem 3.6 in [8] there is then a $\bar{k} \in K$ with $\bar{a}+t\,r-f(\bar{x})=\bar{k}$ and a point $a \in H$ and a $\bar{t} \in \boldsymbol{R}$ with $(\bar{t},\bar{x})$ minimal solution of (3.1) with

$$a + \bar{t}\,r - f(\bar{x}) = 0_m.$$

But then $(\bar{t},\bar{x})$ is a minimal solution of (3.2), too.

3.2 Adaptive parameter control

In [7], [8] an algorithm for an adaptive parameter control for generating an almost equidistant approximation (e. g. w. r. t. the Euclidean norm $\|\cdot\|_2$) of the efficient set based on the Pascoletti-Serafini problem was developed. We present here only the algorithm for bicriteria problems, i. e. for $m=2$. Further we restrict ourselves to the special case of the natural ordering represented by the ordering cone $K=\boldsymbol{R}^2_+$ and we assume the following:

Assumption 1 *Let the constraint set* Ω *be given by* $\Omega=\{x \in \boldsymbol{R}^n \mid g(x) \geq_p 0_p,\ h(x)=0_q\}$ *with* $\geq_p$ *the natural ordering in* $\boldsymbol{R}^p$, *let the functions* $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^m$, $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^p$, *and* $h\colon \boldsymbol{R}^n \to \boldsymbol{R}^q$ $(p,q \in \boldsymbol{N})$ *be continuously differentiable on* $\boldsymbol{R}^n$.

Further let the functions f, g, and h be even two times continuously differentiable and let the assumptions of Theorem 4.2 in [8] be fulfilled for all considered minimal solutions $(\bar{t},\bar{x})$ of (SP(a,r)), i. e. among others let $(\bar{t},\bar{x})$ be a regular point fulfilling a strict second order sufficient condition. We assume the hyper plane $H=\{y\in \boldsymbol{R}^2 \mid b^\top y=\beta\}$ to be given.

Algorithm 1 (Genral algorithm for bicriteria problems)

Input: *Choose* $r\in\boldsymbol{R}^2_+$ *with* $r_1>0$ *and the desired distance* $\alpha\in\boldsymbol{R},\ \alpha>0$, *between the approximation points. Deliver* $M^1\in\boldsymbol{R}$ *with* $M^1>f_2(x)-f_1(x)\frac{r_2}{r_1}$ *for all* $x\in\Omega$.

Step 1: *Solve (SP*$(\tilde{a}^1,r)$*) for* $\tilde{a}^1=(0,M^1)^\top$ *with minimal solution* $(\tilde{t}^1,x^1)$ *and Lagrange multiplier* $\mu^1\in\boldsymbol{R}^2_+$ *to the constraint* $\tilde{a}^1+t\,r-f(x)\geq_2 0_2$. *Calculate* $t^1:=\frac{b^\top f(x^1)-\beta}{b^\top r}$ *and* $a^1:=f(x^1)-t^1\,r$. *Set* $k^1:=0_2,\ l:=1$.

Step 2: *Solve* $\min_{x\in\Omega} f_2(x)$ *with minimal solution* x^E *and calculate* $t^E:=\frac{b^\top f(x^E)-\beta}{b^\top r}$ *and* $a^E:=f(x^E)-t^E\,r$. *Set* $v:=a^E-a^1$.

Step 3: *Determine* a^{l+1} *by:*
If $k_2^l=0$ *set*

$$a^{l+1}:=a^l+\frac{\alpha}{\|v+((-\mu^l)^\top v)\,r\|_2}\cdot v. \tag{3.3}$$

Else *(*$k_2^l\neq 0$*) calculate* $\tilde{t}^l:=\frac{b^\top f(x^l)-\beta}{b^\top r}$ *and* $\tilde{a}^l:=f(x^l)-\tilde{t}^l\,r$ *and set*

$$a^{l+1}:=\tilde{a}^l+\frac{\alpha}{\|v+((-\mu^l)^\top v)\,r\|_2}\cdot v.$$

Step 4: *Set l:=l+1. If* $a^l=a^1+\lambda\cdot v$ *for a* $\lambda\in[0,1]$ *solve (SP*(a^l,r)*) with minimal solution* (t^l,x^l) *and Lagrange multiplier* μ^l *to the constraint* $a^l+t\,r-f(x)\geq_2 0_2$, *set* $k^l:=a^l+t^l\,r-f(x^l)$ *and go to step 3. Else stop.*

Output: *The set* $A:=\{x^1,\cdots,x^{l-1},x^E\}$ *is an approximation of the set* $\mathcal{M}_w(f(\Omega),\boldsymbol{R}^2_+)$.

The scalar optimization problem in Step 1 equals the problem $\min_{x\in\Omega} f_1(x)$. In Step 1 and 2 the set H^a (see Theorem 3.3) is determined. The vector v is a direction in $\boldsymbol{R}^2$ for which we have $a^l+s\cdot v\in H$ for a parameter $a^l\in H$ and $s\in\boldsymbol{R}$. In Step 3 a first order approximation of the weakly efficient point $f(x(a))$ depending on the parameter a based on sensitivity information for the minimal value $t(a)$ (see Corollary 4.5 in [8]) is used for determining s with $a^{l+1}=a^l+s\cdot v$, $s\in\boldsymbol{R}$, and with $\|f(x(a^{l+1}))-f(x(a^l))\|\approx\alpha$. The sensitivity information $\nabla_a t(a^l)=-\mu^l$ is delivered by the negative of the Lagrange multiplier μ^l to the constraint $a^l+t\,r-f(x)\geq 0_2$.

With this algorithm almost equidistant approximations of the efficient set are generated. But because the efficient set can be non-connected there can also be gaps in the efficient set with non-equal distances between the approximation points.

We demonstrate this algorithm on the following test problem. It is a convex problem but the algorithm works on non-convex problems, too.

Problem 3.5 We consider the bicriteria optimization problem

$$\min\begin{pmatrix}\sqrt{1+x_1^2}\\ x_1^2-4x_1+x_2+5\end{pmatrix}$$
under the constraints
$$x_1^2-4x_1+x_2+5\leq 3.5,$$
$$x_1\geq 0,\ x_2\geq 0$$

w. r. t. the ordering cone $K = \boldsymbol{R}^2_+$.

Applying Algorithm 1 with $r = (0.1, 1)^\top$, $\alpha = 0.2$ and $b = r$, $\beta = 1$ we get the parameters and the approximation shown in Fig. 3.2a). For comparison an approximation generated with an equidistant parameter choice for the parameter a without the adaptive parameter control with a less good quality is shown in Fig. 3.2b).

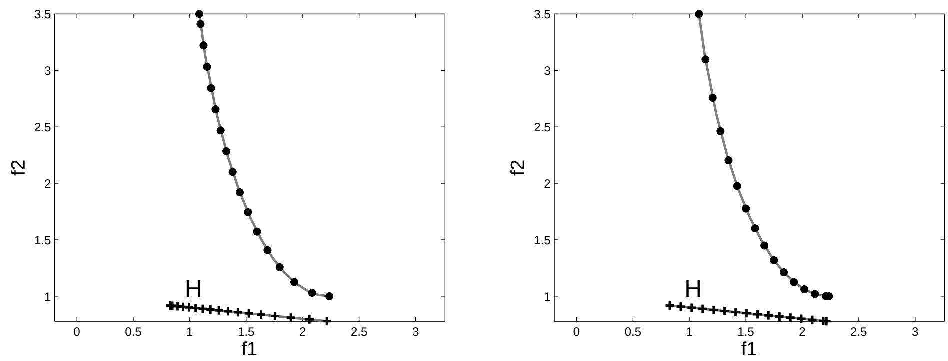

Fig. 3.2 Numerical results for the test problem 3.5 with a) an adaptive parameter control and b) with equidistant parameters.

4 Correlations between scalarizations

In this section we will see that many well known and wide spread scalarization approaches can be considered as a special case or as a modification of (3.1) or (3.2).

4.1 ε-constraint method

The ε-constraint method ([13], [22]) is a very wide spread method especially in engineering design for finding EP-minimal points because the method is very intuitive and the parameters as upper bounds are easy to interpret. For an arbitrary $k \in \{1, \ldots, m\}$ and parameters $\varepsilon_i \in \boldsymbol{R}$, $i \in \{1, \ldots, m\} \setminus \{k\}$, the scalarization called $(P_k(\varepsilon))$ reads as follows:

$$\begin{array}{c} \min f_k(x) \\ \text{under the constraints} \\ f_i(x) \le \varepsilon_i, \quad i \in \{1, \ldots, m\} \setminus \{k\}, \\ x \in \Omega. \end{array} \tag{4.1}$$

It is easy to see that this is just a special case of the Pascoletti-Serafini scalarization for the ordering cone $K = \boldsymbol{R}^m_+$. We even get a connection w. r. t. the Lagrange multipliers.

Theorem 4.1 *Let Assumption 1 hold. A point $\bar{x}$ is a minimal solution of* (4.1) *with Lagrange multipliers $\bar{\mu}_i \in \boldsymbol{R}_+$ for $i \in \{1, \ldots, m\} \setminus \{k\}$, $\bar{\nu} \in \boldsymbol{R}^p_+$, and $\bar{\xi} \in \boldsymbol{R}^q$, if and only if $(f_k(\bar{x}), \bar{x})$ is a minimal solution of* (3.1) *with $K = \boldsymbol{R}^m_+$, Lagrange multipliers $(\bar{\mu}, \bar{\nu}, \bar{\xi})$ with $\bar{\mu}_k = 1$, and*

$$a_i = \varepsilon_i, \quad \forall i \in \{1, \ldots, m\} \setminus \{k\}, \quad a_k = 0 \text{ and } r = e_k \tag{4.2}$$

with e_k the k-th unit vector.

Proof By introducing the additional variable $t \in \boldsymbol{R}$ the scalar problem (4.1) can be formulated as

$$
\begin{array}{c}
\min t \\
\text{under the constraints} \\
\begin{array}{lcll}
\varepsilon_i - f_i(x) & \geq & 0, & i \in \{1,\ldots,m\} \setminus \{k\}, \\
t - f_k(x) & \geq & 0, & \\
g_j(x) & \geq & 0, & j = 1,\ldots,p, \\
h_l(x) & = & 0, & l = 1,\ldots,q, \\
t \in \boldsymbol{R},\ x \in \boldsymbol{R}^n. & & &
\end{array}
\end{array}
\tag{4.3}
$$

If $\bar{x}$ is a minimal solution of (4.1) then $(\bar{t},\bar{x}) := (f_k(\bar{x}),\bar{x})$ is a minimal solution of the problem (4.3). But problem (4.3) is equivalent to the Pascoletti-Serafini problem (3.1) with a and r as in (4.2).

Because $\bar{\mu}_i$, $i \in \{1,\ldots,m\} \setminus \{k\}$, $\bar{\nu}_j$, $j = 1,\ldots,p$, $\bar{\xi}_l$, $l = 1,\ldots,q$, are Lagrange multipliers to $\bar{x}$ for (4.1), we have

$$
\begin{aligned}
\bar{\mu}_i(\varepsilon_i - f_i(\bar{x})) = 0 \qquad & \text{for all } i \in \{1,\ldots,m\} \setminus \{k\}, \\
\bar{\nu}_j(g_j(\bar{x})) = 0 \qquad & \text{for all } j \in \{1,\ldots,p\},
\end{aligned}
$$

and

$$
\nabla f_k(\bar{x}) + \sum_{\substack{i=1\\ i\neq k}}^{m} \bar{\mu}_i \nabla f_i(\bar{x}) - \sum_{j=1}^{p} \bar{\nu}_j \nabla g_j(\bar{x}) - \sum_{l=1}^{q} \bar{\xi}_l \nabla h_l(\bar{x}) = 0_n. \tag{4.4}
$$

The derivative of the Lagrange function $\mathcal{L}(t,x,\mu,\nu,\xi,a,r)$ to (3.1) with a,r as in (4.2) in the point $(f_k(\bar{x}),\bar{x})$ reads as follows:

$$
\begin{aligned}
&\nabla_{(t,x)} \mathcal{L}(f_k(\bar{x}),\bar{x},\mu,\nu,\xi,a,r) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu_k \begin{pmatrix} 1 \\ -\nabla f_k(\bar{x}) \end{pmatrix} \\
&- \sum_{\substack{i=1\\ i\neq k}}^{m} \mu_i \begin{pmatrix} 0 \\ -\nabla f_i(\bar{x}) \end{pmatrix} - \sum_{j=1}^{p} \nu_j \begin{pmatrix} 0 \\ \nabla g_j(\bar{x}) \end{pmatrix} - \sum_{l=1}^{q} \xi_l \begin{pmatrix} 0 \\ \nabla h_l(\bar{x}) \end{pmatrix}.
\end{aligned}
$$

By choosing $\bar{\mu}_k = 1$ and applying (4.4) we get

$$
\nabla_{(t,x)} \mathcal{L}(f_k(\bar{x}),\bar{x},\bar{\mu},\bar{\nu},\bar{\xi},a,r) = 0_{n+1},
$$

and hence $(\bar{\mu},\bar{\nu},\bar{\xi})$ are Lagrange multipliers to the point $(f_k(\bar{x}),\bar{x})$ to problem (3.1), too.

The statement of Theorem 4.1 is visualized in Fig. 4.1 on a bicriteria optimization problem with $k = 2$ in the ε-constraint method.

It can be followed from Theorem 3.1a) that any EP-minimal solution $\bar{x}$ of (2.1) can be found by solving $(P_k(\varepsilon))$ for the parameters $\varepsilon_i = f_i(\bar{x}),\ i \in \{1,\ldots,m\} \setminus \{k\}$. In opposition to the Pascoletti-Serafini method in general not any weakly EP-minimal solution can be found by solving $(P_k(\varepsilon))$ because we choose $r \in \partial K = \partial \boldsymbol{R}^m_+$ for the ε-constraint method. But weakly EP-minimal points which are not also EP-minimal are not of practical interest. Besides it follows from Theorem 3.1b) that any minimal solution of (4.1) is an at least weakly EP-minimal solution of (2.1).

The ε-constraint method has a big drawback against the more general Pascoletti-Serafini method. According to Theorem 3.2 if we have $\mathcal{E}(f(\Omega),\boldsymbol{R}^m_+) \neq \emptyset$ and $f(\Omega) + K$ is closed and convex for any choice of the parameters $(a,r) \in \boldsymbol{R}^m \times \operatorname{int}(K)$ there exists a minimal solution of problem (3.1). This is no longer true for the ε-constraint method as the following example demonstrates.

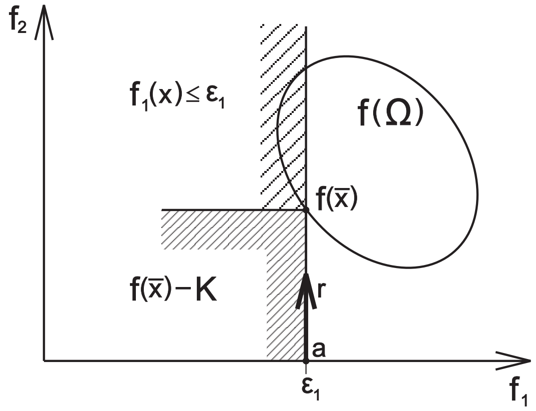

Fig. 4.1 Connection between the ε-constrain and the Pascoletti-Serafini method.

Example 4.2 Let $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $f(x) = x$ for all $x \in \boldsymbol{R}^2$ be given. We consider the bicriteria optimization problem

$$\begin{gathered} \min f(x) = x \\ \text{under the constraints} \\ \|x\|_2 \leq 1, \\ x \in \boldsymbol{R}^2 \end{gathered}$$

w. r. t. the natural ordering. It is $f(\Omega)$ convex and the efficient set is

$$\mathcal{E}(f(\Omega), \boldsymbol{R}^2_+) = \{x = (x_1, x_2)^\top \in \boldsymbol{R}^2 \mid \|x\|_2 = 1,\ x_1 \leq 0,\ x_2 \leq 0\} \neq \emptyset.$$

The ε-constraint scalarization for $k = 2$ is given by

$$\begin{gathered} \min f_2(x) \\ \text{under the constraints} \\ f_1(x) \leq \varepsilon_1, \\ \|x\|_2 \leq 1, \\ x \in \boldsymbol{R}^2, \end{gathered}$$

but for $\varepsilon_1 < -1$ there exists no feasible point and thus no minimal solution.

Hence it can happen that the ε-constraint method is solved for a high number of parameters without getting any solution and with that weakly EP-minimal points or at least the information that it is $\mathcal{M}(f(\Omega), \boldsymbol{R}^m_+) = \emptyset$.

We can restrict the set of parameters which we have to consider such that we can find any arbitrary EP-minimal point applying Theorem 3.3.

Corollary 4.3 *Let $m = 2$, $K = \boldsymbol{R}^2_+$, and $\bar{x} \in \mathcal{M}(f(\Omega), K)$. Let $\bar{x}^1$ be a minimal solution of $\min_{x\in\Omega} f_1(x)$ and $\bar{x}^2$ a minimal solution of $\min_{x\in\Omega} f_2(x)$. Then there is a parameter $\varepsilon \in \{y \in \boldsymbol{R} \mid f_1(\bar{x}^1) \leq y \leq f_1(\bar{x}^2)\}$ with $\bar{x}$ minimal solution of $(P_2(\varepsilon))$.*

The same for $(P_1(\varepsilon))$.

We have seen that we can restrict the parameter set. Now it is of great interest to know how to choose the parameters ε to generate approximations of the efficient set with a high quality with almost equidistant points. Hence we formulate Algorithm 1 for this important special case:

Algorithm 2 (Algorithm for the ε-constraint method)

Input: *Choose the desired distance $\alpha \in \boldsymbol{R}$, $\alpha > 0$, between the approximation points and $M > f_2(x)$ for all $x \in \Omega$.*

Step 1: *Solve $(P_1(M))$ with minimal solution x^1 and Lagrange multiplier μ^1 to the constraint $f_2(x) \leq M$. Set $\varepsilon^1 = f_2(x^1)$ and $l := 1$.*

Step 2: *Solve $\min_{x \in \Omega} f_2(x)$ with minimal solution x^E and set $\varepsilon^E = f_2(x^E)$.*

Step 3: *Set $\varepsilon^{l+1} := f_2(x^l) - \frac{\alpha}{\sqrt{1+(\mu^l)^2}}$ and set $l := l + 1$.*

Step 4: *For $\varepsilon^l \geq \varepsilon^E$ solve $(P_1(\varepsilon^l))$ with minimal solution x^l and Lagrange multiplier μ^l to the constraint $f_2(x^l) \leq \varepsilon^l$ and go to step 3. Else stop.*

Output: *The set $A := \{x^1, \cdots, x^{l-1}, x^E\}$ is an approximation of the set $\mathcal{M}_w(f(\Omega), \boldsymbol{R}^2_+)$.*

Applying this algorithm for determining an approximation of the test problem 3.5 leads to the parameters and approximation points shown in Fig. 4.2.

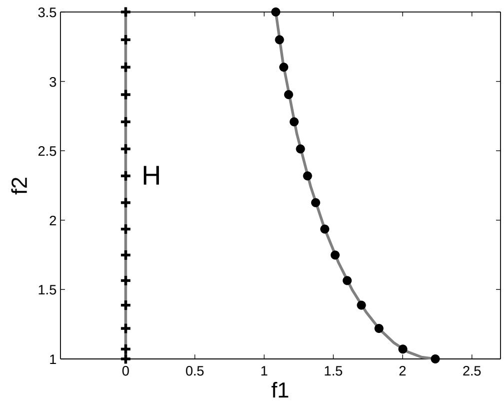

Fig. 4.2 Numerical results with Algorithm 2.

4.2 Normal boundary intersection method

We start with a short recapitulation of this method introduced by Das and Dennis in [3], [4]. For determining EP-minimal points the scalar optimization problems

$$\begin{array}{c} \max s \\ \text{under the constraints} \\ \Phi\beta + s\,\bar{n} = f(x) - f^*, \\ x \in \Omega, s \in \boldsymbol{R}, \end{array} \tag{4.5}$$

named (NBI(β)) for parameters $\beta \in \boldsymbol{R}^m_+$, $\sum_{i=1}^m \beta_i = 1$, are solved. Here f^* denotes the so called ideal point defined by $f_i^* := f_i(x^i) =: \min_{x \in \Omega} f_i(x),\ i = 1, \ldots, m$. The matrix $\Phi \in \boldsymbol{R}^{m \times m}$ consists of the columns $f(x^i) - f^*$ $(i = 1, \ldots, m)$ and the set

$$f^* + \{\Phi\beta \mid \beta \in \boldsymbol{R}^m_+,\ \sum_{i=1}^m \beta_i = 1\} \tag{4.6}$$

is then the set of all convex combinations of the extremal points $f(x^i),\ i = 1, \ldots, m$, the so called CHIM (convex hull of individual minima). The vector $\bar{n}$ is defined as normal unit vector to the hyper plane extending the CHIM and directing to the negative orthant.

The idea of this method is that by solving the problem (4.5) for an equidistant choice of parameters β an equidistant approximation of the efficient set is generated. But already for the case $m \geq 3$ in general not all EP-minimal points can be found as a solution of (4.5) (see [3], Fig. 3) and what is more, the maximal solutions of (4.5) are not necessarily weakly EP-minimal (see [30], Ex. 7.9.1).

There is a direct connection between the normal boundary intersection (NBI) method and the modified version (3.2) of the Pascoletti-Serafini problem.

Theorem 4.4 *A point $(\bar{s}, \bar{x})$ is a maximal solution of* (4.5) *with $\beta \in \boldsymbol{R}^m$, $\sum_{i=1}^m \beta_i = 1$, if and only if $(-\bar{s}, \bar{x})$ is a minimal solution of* (3.2) *with $a = f^* + \Phi\beta$ and $r = -\bar{n}$.*

Proof By setting $a = f^* + \Phi\beta$ and $r = -\bar{n}$ we see immediately that solving problem (4.5) is equivalent to solve

$$\begin{gathered} -\min t \\ \text{under the constraints} \\ a + t\,r - f(x) = 0_m, \\ x \in \Omega, t \in \boldsymbol{R}, \end{gathered}$$

being again equivalent to solve (3.2).

Hence, the NBI method is a restriction of the modified Pascoletti-Serafini method as the parameter a is chosen only from the CHIM and the parameter $r = -\bar{n}$ is chosen constant. That is the reason why in general not all EP-minimal points can be found with the NBI scalarization for the case $m \geq 3$. But by allowing the parameter β to vary arbitrarily and with that the parameter a to vary arbitrarily on the hyper plane including the CHIM all EP-minimal points of (2.1) can be found by solving (3.2) and (4.5) respectively (see Theorem 3.4).

We can apply the results from section 3.2 for determining an algorithm for choosing the parameters β for the normal boundary intersection method adaptively, such that the resulting points are almost equidistant, too. We consider here again only the bicriteria case. We have for $\beta = (\beta_1, 1 - \beta_1)^\top$

$$\begin{aligned} f^* + \Phi\,\beta &= \begin{pmatrix} f_1(x^1) \\ f_2(x^2) \end{pmatrix} + \begin{pmatrix} 0 & f_1(x^2) - f_1(x^1) \\ f_2(x^1) - f_2(x^2) & 0 \end{pmatrix} \beta \\ &= \beta_1 \begin{pmatrix} f_1(x^1) \\ f_2(x^1) \end{pmatrix} + (1 - \beta_1) \begin{pmatrix} f_1(x^2) \\ f_2(x^2) \end{pmatrix} \end{aligned}$$

and hence the hyper plane H including the CHIM is given by

$$\begin{aligned} H &:= \{f^* + \Phi\,\beta \mid \beta \in \boldsymbol{R}^2,\ \beta_1 + \beta_2 = 1\} \\ &= \{\beta\, f(x^1) + (1 - \beta)\, f(x^2) \mid \beta \in \boldsymbol{R}\} \end{aligned}$$

and the set H^a according to Theorem 3.3 is given by

$$H^a = \{\beta\, f(x^1) + (1 - \beta)\, f(x^2) \mid \beta \in [0, 1]\}$$

which equals (only here in the bicriteria case) the proposed set by Das and Dennis in (4.6). We assume again that all necessary assumptions as mentioned in section 3.2 are fulfilled. We have according to Algorithm 1 $v = a^E - a^1$. Together with $a^E = f^* + \Phi\,\beta^E$, $a^1 = f^* + \Phi\,\beta^1$ and $\beta^1 = (1, 0)^\top$, $\beta^E = (0, 1)^\top$ we get

$$v = \Phi\,(\beta^E - \beta^1) = \Phi \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Using $a^l = f^* + \Phi\,\beta^l$ for arbitrary $l \in \boldsymbol{N}$ and $r = -\bar{n}$ we can transform equation (3.3) to

$$f^* + \Phi\,\beta^{l+1} = f^* + \Phi\,\beta^l + \frac{\alpha}{\|v + ((\mu^l)^\top v)\bar{n}\|_2}\Phi\begin{pmatrix}-1\\1\end{pmatrix}$$

resulting in

$$\beta^{l+1} = \beta^l + \frac{\alpha}{\|v + ((\mu^l)^\top v)\bar{n}\|_2}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Then we get the following algorithm for controlling the parameter $\beta = (\beta_1, 1-\beta_1) \in \boldsymbol{R}^2$. We assume the matrix Φ, the point f^* and the normal unit vector $\bar{n}$ to be given.

Algorithm 3 (Algorithm for the NBI method)

Input: *Choose the desired distance* $\alpha \in \boldsymbol{R}$, $\alpha > 0$, *between the approximation points.*

Step 1: *Set* $\beta^1 := (1,0)^\top$ *and solve (NBI(β)) with solution* $(\bar{s}^1, \bar{x}^1) = (0, x^1)$ *and Lagrange multiplier* μ^1 *to the constraint* $\Phi\,\beta^1 + s\,\bar{n} = f(x) - f^*$. *Set* $l = 1$.

Step 2: *Set* $\beta^E := (0,1)^\top$ *and set* $v := \Phi\begin{pmatrix}-1\\1\end{pmatrix}$.

Step 3: *Set*

$$\beta^{l+1} := \beta^l + \frac{\alpha}{\|v + ((\mu^l)^\top v)\bar{n}\|_2}\cdot\begin{pmatrix}-1\\1\end{pmatrix}$$

ans set $l := l+1$.

Step 4: *If* $\beta_1^l \geq 0$ *solve (NBI(β)) with solution* (s^l, x^l) *and Lagrange multiplier* μ^l *to the constraint* $\Phi\,\beta^l + s\,\bar{n} = f(x) + f^*$ *and go to step 3. Else stop.*

Step 5: *Determine the set* $\tilde{A} := \{x^1, \ldots, x^{l-1}, x^E\}$ *and the set* $A := \mathcal{M}(\tilde{A}, \boldsymbol{R}^2_+)$ *of non-dominated points of* $\tilde{A}$.

Output: *The set* A *is an approximation of the set* $\mathcal{M}_w(f(\Omega), \boldsymbol{R}^2_+)$.

Remark that the approximation points gained with this algorithm are not necessarily weakly EP-minimal points. Using the normal boundary intersection problem and Algorithm 3 leads for $\alpha = 0.2$ to the approximation of test problem 3.5 presented in Fig. 4.3.

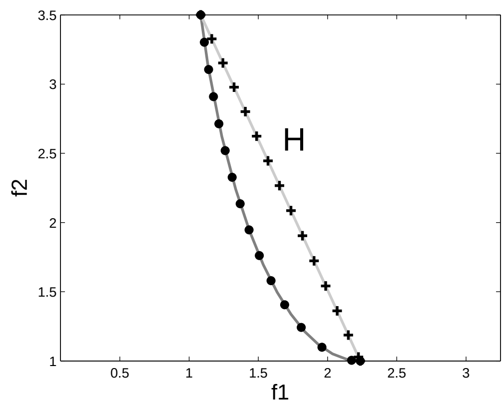

Fig. 4.3 Numerical results with Algorithm 3.

4.3 Modified Polak method

The modified Polak method ([16], [17], [25]) has a similar connection to the Pascoletti-Serafini problem as the normal boundary intersection method. We restrict the presentation of the modified Polak method here to the bicriteria case. Then, for different values of the parameter $y_1 \in \boldsymbol{R}$, the scalar optimization problems called (MP(y_1))

$$\begin{array}{c} \min f_2(x) \\ \text{under the constraints} \\ f_1(x) = y_1, \\ x \in \Omega, \end{array} \tag{4.7}$$

are solved. Here the objectives are transformed to constraints with an equality constraint, as in the normal boundary intersection problem, instead of to inequality constraints as in the general Pascoletti-Serafini method. Besides the constraint $f_1(x) = y_1$ shows a similarity to the ε-constraint method with the constraint $f_1(x) \leq \varepsilon_1$.

Theorem 4.5 *Let $m = 2$. A point $\bar{x}$ is a minimal solution of* (4.7) *if and only if* $(f_2(\bar{x}), \bar{x})$ *is a minimal solution of* (3.2) *with $a = (y_1, 0)$ and $r = (0, 1)$.*

Proof With the parameters a and r as defined in the theorem problem (3.2) reads as follows:

$$\begin{array}{c} \min t \\ \text{under the constraints} \\ y_1 - f_1(x) = 0, \\ t - f_2(x) = 0, \\ t \in \boldsymbol{R},\ x \in \Omega, \end{array}$$

and it can immediately be seen that solving this problem is equivalent to solve problem (4.7).

Of course a generalization to the case $m \geq 3$ can be done as well.

The modified Polak method is also a restriction of the modified Pascoletti-Serafini method but because the parameter $a = (y_1, 0)$ is allowed to vary arbitrarily in the hyperplane $H = \{y \in \boldsymbol{R}^m \mid y_m = 0\}$ in opposition to the NBI method all EP-minimal points can be found even for the case $m \geq 3$. The disadvantage of the ε-constraint method as shown in Example 4.2 and the results of Corollary 4.3 can be shown for the modified Polak method, too.

The algorithm for this special case reads as follows.

Algorithm 4 (Algorithm for the modified Polak method)

Input: *Choose the desired distance $\alpha \in \boldsymbol{R}$, $\alpha > 0$, between the approximation points.*

Step 1: *Determine the numbers $y_1^1 := f_1(x^1) := \min_{x \in \Omega} f_1(x)$ and $y_1^E := f_1(x^E)$ with $f_2(x^E) := \min_{x \in \Omega} f_2(x)$.*

Step 2: *Solve $(MP(y_1^1))$ with minimal solution x^1 and Lagrange multiplier $\mu^1 \in \boldsymbol{R}$ to the constraint $f_1(x) = y_1^1$. Set $l = 1$.*

Step 3: *Set $y_1^{l+1} := y_1^l + \frac{\alpha}{\sqrt{1+(\mu^l)^2}}$ ans set $l := l + 1$.*

Step 4: *If $y_1^l \leq y_1^E$ solve $(MP(y_1^l))$ with minimal solution x^l and Lagrange multiplier μ^l to the constraint $f_1(x) = y_1^l$ and go to step 3. Else stop.*

Step 5: *Determine the set $\tilde{A} := \{x^1, \ldots, x^{l-1}, x^E\}$ and the set $A := \mathcal{M}(\tilde{A}, \boldsymbol{R}^2_+)$ of non-dominated points of $\tilde{A}$.*

Output: *The set A is an approximation of the set $\mathcal{M}_w(f(\Omega), \boldsymbol{R}^2_+)$.*

An algorithm for the modified Polak method without an adaptive parameter control can be found in [18], Alg. 12.1. Remark again that the approximation points gained with this algorithm are not necessarily weakly EP-minimal. In Step 2 problem $(MP(y_1^1))$ can lead to numerical difficulties because it can happen that the constraint set is reduced to one point only. For avoiding this y_1^1 can be replaced by $y_1^1 + \Delta\varepsilon$ with a small value $\Delta\varepsilon > 0$. Applying Algorithm 4 to test problem 3.5 with this modification and by choosing $\alpha = 0.2$ and $\Delta\varepsilon = \frac{\alpha}{10}$ leads to the parameters and the approximation points shown in Fig. 4.4.

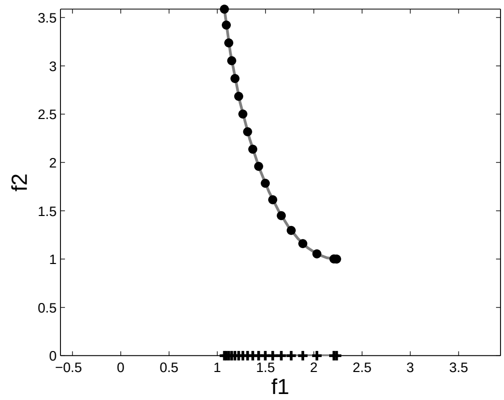

Fig. 4.4 Numerical results with Algorithm 4.

4.4 Weighted Chebyshev norm

In this scalarization method (see [5], [16], p.13, [21]) for determining EP-minimal points we have weights $w_i > 0$, $i = 1, \ldots, m$, and a reference point $a \in \boldsymbol{R}^m$ with $a_i < \min_{x\in\Omega} f_i(x)$, $i = 1, \ldots, m$, (assuming solutions exist), i. e. $f(\Omega) \subset a + \operatorname{int}(\boldsymbol{R}^m_+)$, as parameters. For scalarizing the multi-objective optimization problem the weighted Chebyshev norm of the function $f - a$ is minimized

$$\min_{x\in\Omega} \max_{i\in\{1,\ldots,m\}} w_i(f_i(x) - a_i) \tag{4.8}$$

having the following connection to the Pascoletti-Serafini formulation.

Theorem 4.6 *A point* $(\bar{t}, \bar{x}) \in \boldsymbol{R} \times \Omega$ *is a minimal solution of* (3.1) *with* $K = \boldsymbol{R}^m_+$ *and with parameters* $a \in \boldsymbol{R}^m$, $a_i < \min_{x\in\Omega} f_i(x)$, $i = 1, \ldots, m$, *and* $r \in int(\boldsymbol{R}^m_+)$ *if and only if* $\bar{x}$ *is a solution of* (4.8) *with point* a *and weights* $w_i = \frac{1}{r_i} > 0$, $i = 1, \ldots, m$.

Proof If we set $r_i = \frac{1}{w_i} > 0$ and $K = \boldsymbol{R}^m_+$ problem (3.1) reads as follows:

$$\begin{gathered} \min t \\ \text{under the constraints} \\ a_i + t\,\frac{1}{w_i} - f_i(x) \geq 0, \quad i = 1, \ldots, m, \\ t \in \boldsymbol{R},\; x \in \Omega. \end{gathered}$$

This is because of $w_i > 0,\ i = 1, \ldots, m$, equivalent to

$$\begin{array}{c}\min t\\ \text{under the constraints}\\ w_i(f_i(x) - a_i) \leq t, \quad i = 1, \ldots, m,\\ t \in \boldsymbol{R},\ x \in \Omega\end{array}$$

but this is according to [18], p.305, or [21], p.14, just a reformulation of (4.8) with an additional variable introduced.

Thus a variation of the weights in the norm corresponds to a variation of the direction r, and a variation of the reference point (or goal in the goal attainment method by Gembicki and Haimes, [10]) is also a variation of the parameter a in the Pascoletti-Serafini method.

A variation of the goal a for generating equidistant approximation points can be done with the Algorithm 1 for a constant weight w and hence a constant parameter r. For a variation of the parameter r the results from Theorem 4.2 in [8] can be used where the sensitivity of the minimal value of the Pascoletti-Serafini problem on the parameter r is examined.

As a consequence of Theorem 3.1 any solution of (4.8) is at least weakly EP-minimal and any weakly EP-minimal point can be found as solution of (4.8). In [30], Example 7.7.1, it is shown, that $f(\Omega) \subset a + \boldsymbol{R}^m_+$ is necessary for the last statement.

For $a = 0_m$ problem (4.8) reduces to the weighted minimax method as discussed in [20].

4.5 Problem according to Gourion and Luc

This problem is described in [12] for finding EP-maximal points of a multi-objective optimization problem with $f(\Omega) \subset \boldsymbol{R}^m_+$. This corresponds to the multi-objective optimization problem (2.1) w. r. t. the ordering cone $K = \boldsymbol{R}^m_-$. The parameter dependent scalar optimization problems according to Gourion and Luc read as follows

$$\begin{array}{c}\max s\\ \text{under the constraints}\\ f(x) \geq s\,\alpha,\\ s \in \boldsymbol{R},\ x \in \Omega\end{array} \tag{4.9}$$

introducing the new variable $s \in \boldsymbol{R}$ and with the parameter $\alpha \in \boldsymbol{R}^m_+$. We will see that the scalarization of Gourion and Luc can be seen as a special case of the Pascoletti-Serafini method with a variation of the parameter $r = -\alpha$ only and with the parameter $a = 0_m$ constant.

Theorem 4.7 *A point* $(\bar{s}, \bar{x})$ *is a maximal solution of* (4.9) *with parameter* $\alpha \in \boldsymbol{R}^m_+$ *if and only if* $(-\bar{s}, \bar{x})$ *is a minimal solution of* (3.1) *with* $a = 0_m$, $r = -\alpha \in \boldsymbol{R}^m_-$ *and* $K = \boldsymbol{R}^m_-$.

Proof By defining $r = -\alpha \in \boldsymbol{R}^m_-$ and $t = -s$ problem (4.9) can be written as

$$\begin{array}{c}\max(-t)\\ \text{under the constraints}\\ f(x) \geq (-r) \cdot (-t),\\ t \in \boldsymbol{R},\ x \in \Omega\end{array}$$

being equivalent to

$$\begin{array}{c}-\min t\\ \text{under the constraints}\\ t\,r - f(x) \in K,\\ t \in \boldsymbol{R},\ x \in \Omega.\end{array}$$

with $K = \boldsymbol{R}^m_-$, i. e. to the Pascoletti-Serafini scalarization (3.1) with $a = 0_m$ and $K = \boldsymbol{R}^m_-$.

It follows as a direct consequence of this theorem together with Theorem 3.1b) that if $(\bar{s}, \bar{x})$ is a maximal solution of (4.9) then $\bar{x}$ is weakly $\boldsymbol{R}^m_-$-minimal, i. e. $\bar{x}$ is weakly EP-maximal for the corresponding multi-objective optimization problem.

Then again, for the dependence of the optimal value of the scalar problem (4.9) on a variation of $\alpha = -r$ the sensitivity results from Theorem 4.2 in [8] can be applied and based on this the choice of the parameter $\alpha = -r$ can be controlled adaptively for generating an approximation of the efficient set with a good quality.

4.6 Generalized weighted sum method

Before we come to the usual weighted sum method we consider a more general formulation having not only a weighted sum as objective function but also similar constraints to the already discussed ε-constraint method (compare [30], p.136)

$$\begin{array}{c} \min \sum_{i=1}^{m} w_i f_i(x) = w^\top f(x) \\ \text{under the constraints} \\ f_i(x) \leq \varepsilon_i, \quad \forall i \in P, \\ x \in \Omega \end{array} \tag{4.10}$$

with $P \subsetneqq \{1, \ldots, m\}$, $\varepsilon_i \in \boldsymbol{R}$ for all $i \in P$, and weights $w \in \boldsymbol{R}^m \setminus \{0_m\}$. We start with the connection to the Pascoletti-Serafini problem and from that we conclude some properties of the problem (4.10).

Theorem 4.8 *A point $\bar{x}$ is a minimal solution of* (4.10) *for the parameter $w \in \boldsymbol{R}^m$ with $\sum_{i \notin P} w_i > 0$ if and only if there is a $\bar{t}$ such that $(\bar{t}, \bar{x})$ is a minimal solution of* (3.1) *with $a_i = \varepsilon_i$ for $i \in P$, a_i arbitrary for $i \in \{1, \ldots, m\} \setminus P$, $r_i = 0$ for $i \in P$, $r_i = 1$ for $i \in \{1, \ldots, m\} \setminus P$ and $K = K_w := \{y \in \boldsymbol{R}^m \mid y_i \geq 0, \quad \forall i \in P, \ w^\top y \geq 0\}$, i. e. of*

$$\begin{array}{c} \min t \\ \textit{under the constraints} \\ a + t\, r - f(x) \in K_w, \\ t \in \boldsymbol{R}, \ x \in \Omega. \end{array} \tag{4.11}$$

Proof The optimization problem (4.11) is equivalent to

$$\begin{array}{c} \min t \\ \text{under the constraints} \\ w^\top (a + t\, r - f(x)) \geq 0, \\ a_i + t\, r_i - f_i(x) \geq 0, \quad \forall i \in P, \\ t \in \boldsymbol{R}, \ x \in \Omega. \end{array} \tag{4.12}$$

As $a_i = \varepsilon_i$ and $r_i = 0$ for $i \in P$ and because of $w^\top r = \sum_{i \notin P} w_i > 0$ a point $(\bar{t}, \bar{x})$ is a minimal solution of (4.12) if and only if $\bar{x}$ is a minimal solution of

$$\begin{array}{c} \min \frac{w^\top f(x) - w^\top a}{w^\top r} \\ \text{under the constraints} \\ f_i(x) \leq \varepsilon_i, \quad \forall i \in P, \\ x \in \Omega. \end{array} \tag{4.13}$$

Because we can ignore the constant term $-\frac{w^\top a}{w^\top r}$ in the objective function of (4.13) and because of $w^\top r > 0$ a point $\bar{x}$ is a minimal solution of (4.13) if and only if it is a minimal solution of (4.10).

The set K_w is a closed convex cone and for $|P| = m-1$ the cone K_w is even pointed ([30], Lemma 7.11.1).

Corollary 4.9 *Let* $\bar{x}$ *be a minimal solution of* (4.10) *and let* $K \subset \{y \in \boldsymbol{R}^m \mid y_i \geq 0, \quad \forall i \in P\}$, $w \in K^* \setminus \{0_m\}$ *and* $\sum_{i \notin P} w_i > 0$ *then* $\bar{x} \in M_w(f(\Omega), K)$.

Proof Applying Theorem 4.8 and Theorem 3.1b) it follows $\bar{x} \in \mathcal{M}_w(f(\Omega), K_w)$ with $K_w = \{y \in \boldsymbol{R}^m \mid y_i \geq 0, \quad \forall i \in P, \; w^\top y \geq 0\}$. As for all $y \in K$ we have $w^\top y \geq 0$ it follows $K \subset K_w$ and hence (see [27], Prop. 3.2.1) $M_w(f(\Omega), K_w) \subset M_w(f(\Omega), K)$.

Thus e. g. for $K = \boldsymbol{R}^m_+$ and $w_i > 0$ for all $i \in P$ all minimal solutions of (4.10) are at least weakly EP-minimal. For $w_i > 0, \; i = 1, \ldots, m$, they are even EP-minimal (see [30], Theorem 7.11.1c)).

Of course we can find all EP-minimal points by solving (4.10) if we set $P = \{1, \ldots, m-1\}$, $w_i = 0$ for $i \in P$, and $w_m = 1$. Then $K_w = \boldsymbol{R}^m_+$ and (4.10) equals the ε-constraint problem (4.1) for $k = m$. Then by choosing $\varepsilon_i = f_i(\bar{x})$ for all $i \in P$ it is known that $\bar{x}$ is a minimal solution of the ε-constraint problem and hence of the problem (4.10).

The case $P = \{1, \ldots, m\}$ of problem (4.10) is introduced and discussed in Charnes and Cooper ([2]) and later in Wendell and Lee ([31]). Weidner has shown (see [30], p.130) that there is no equivalent formulation between (4.10) with $P = \{1, \ldots, m\}$ and (3.1).

4.7 Weighted sum method

Now we come to the usual weighted sum method ([32])

$$\begin{array}{c} \min w^\top f(x) \\ \text{under the constraint} \\ x \in \Omega \end{array} \tag{4.14}$$

for weights $w \in K^* \setminus \{0_m\}$ which is just a special case of (4.10) for $P = \emptyset$. Because it is such an important method we adapt Theorem 4.8 for this special case (see also Fig. 4.5):

Theorem 4.10 *A point* $\bar{x}$ *is a minimal solution of* (4.14) *for the parameter* $w \in K^* \setminus \{0_m\}$ *if and only if there is a* $\bar{t}$ *such that* $(\bar{t}, \bar{x})$ *is a minimal solution of* (3.1) *with* $a \in \boldsymbol{R}^m$ *arbitrarily chosen, cone* $K_w := \{y \in \boldsymbol{R}^m \mid w^\top y \geq 0\}$ *and* $r \in int(K_w)$.

Proof Problem (3.1) with cone K_w reads as follows:

$$\begin{array}{c} \min t \\ \text{under the constraints} \\ w^\top (a + t\, r - f(x)) \geq 0, \\ t \in \boldsymbol{R}, \; x \in \Omega. \end{array} \tag{4.15}$$

Because of $r \in \operatorname{int}(K_w)$ we have $w^\top r > 0$ and hence (4.15) is equivalent to

$$\begin{array}{c} \min \frac{w^\top f(x) - w^\top a}{w^\top r} \\ \text{under the constraint} \\ x \in \Omega. \end{array}$$

With the same arguments as used in the proof to Theorem 4.8 this is equivalent to (4.14).

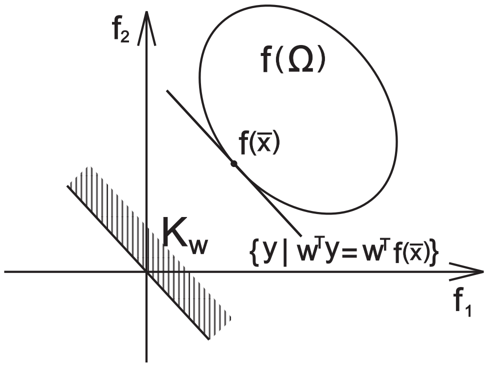

Fig. 4.5 Connection between the weighted sum and the Pascoletti-Serafini problem.

Hence a variation of the weights $w \in K^* \setminus \{0_m\}$ corresponds to a variation of the ordering cone K_w and we get a new interpretation for the weighted sum method. The cone K_w is a closed convex polyhedral cone, but K_w is not pointed and that is the reason why the results from Theorem 3.1a) cannot be applied to the weighted sum method and why it is in general (in the non-convex case) not possible to find all K-minimal points of (2.1) by using the weighted sum method. But we can conclude from Theorem 3.1b) the following well known result:

Corollary 4.11 *Let $\bar{x}$ be a minimal solution of (4.14) with parameter $w \in K^* \setminus \{0_m\}$, then $\bar{x}$ is weakly K-minimal for (MOP).*

Proof According to Theorem 4.10 there is a $\bar{t}$ such that $(\bar{t}, \bar{x})$ is a minimal solution of (3.1) with cone K_w and hence $\bar{x}$ is according to Theorem 3.1b) a weakly K_w-minimal point. Because $w \in K^* \setminus \{0_m\}$ we have $w^\top y \geq 0$ for all $y \in K$ and hence $K \subset K_w$. Thus it is $\mathcal{M}_w(f(\Omega), K_w) \subset \mathcal{M}_w(f(\Omega), K)$ and therefore $\bar{x}$ is a weakly K-minimal point, too.

The weighted sum method has the same drawback against the Pascoletti-Serafini method as the ε-constraint method has as it was shown in Example 4.2. In [1], Ex. 7.3, Brosowski gives a simple example where the weighted sum problem delivers only for one choice of weights a minimal solution and where it is not solvable for all other weights despite the set $f(\Omega) + K$ is closed and convex in opposition to Theorem 3.2.

We further want to mention the weighted p-power method (see [20]) where the scalarization is given by

$$\min_{x \in \Omega} \sum_{i=1}^{m} w_i f_i^p(x) \tag{4.16}$$

for $p \geq 1$, $w \in \boldsymbol{R}^m_+ \setminus \{0_m\}$ (or $w \in K^* \setminus \{0_m\}$). For $p = 1$ (4.16) is equal to the weighted sum method. For arbitrary p the problem (4.16) can be seen as an application of the weighted sum method to the multi-objective optimization problem

$$\min_{x \in \Omega} \begin{pmatrix} f_1^p(x) \\ \vdots \\ f_m^p(x) \end{pmatrix}.$$

Another generalization of the weighted sum method is discussed in [30], p.111f. There k ($k \in \boldsymbol{N},\ k \leq m$) linearly independent weights $w^1, \ldots, w^k \in \boldsymbol{R}^m_+$ are allowed representing

e. g. the preferences of k decision makers, and a reference point $v \in \boldsymbol{R}^k$ is given. Then the problem

$$\min_{x\in\Omega} \max_{i\in\{1,\ldots,k\}} (w^i)^\top f(x) - v_i \tag{4.17}$$

is solved. The connection to the parameters of the Pascoletti-Serafini problem is given by the equations

$$\begin{array}{ll} (w^i)^\top a = v_i, & i = 1,\ldots,k, \\ (w^i)^\top r = 1, & i = 1.\ldots,k, \end{array}$$

and it is $K = \{y \in \boldsymbol{R}^m \mid (w^i)^\top y \geq 0,\ i = 1,\ldots,k\}$. Then K is a closed and convex cone and K is pointed if and only if $k = m$. A minimal solution of (4.17) is not only weakly K-minimal but because of $\boldsymbol{R}^m_+ \subset K$ also weakly EP-minimal (see also [30] 7.2.1a)). For $k = 1$ and $v = 0$ (4.17) is equivalent to (4.14). In the following section we will discuss a special case of problem (4.17) for the case $k = m$.

4.8 Problem according to Kaliszewski

In [19] Kaliszewski discusses among others the problem called (P^∞):

$$\min_{x\in\Omega} \max_{i\in\{1,\ldots,m\}} \lambda_i \left((f_i(x) - y^*_i) + \rho \sum_{j=1}^m (f_j(x) - y^*_j) \right) \tag{4.18}$$

for a closed set $f(\Omega) \subset y^* + \mathrm{int}(\boldsymbol{R}^m_+)$, $y^* \in \boldsymbol{R}^m$, and $\rho > 0$, $\lambda_i > 0$, $i = 1,\ldots,m$, for determining properly efficient solutions (see Theorem 4.2 in [19]). Proper efficiency (here a definition according to Geoffrion, see [11], [27]) is a stronger minimality notion than EP-minimality.

Definition 4.12 Let $K = \boldsymbol{R}^m_+$. A point $\bar{x}$ is a properly efficient solution of (2.1) if it is EP-minimal and if there is some real $M > 0$ such that for each i and each $x \in \Omega$ satisfying $f_i(x) < f_i(\bar{x})$, there exists at least one j such that $f_j(\bar{x}) < f_j(x)$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

The connection to the Pascoletti-Serafini problem is given by the following theorem (compare [30], p.118f).

Theorem 4.13 *A point $\bar{x}$ is a minimal solution of (P^∞) for $f(\Omega) \subset y^* + int(\boldsymbol{R}^m_+)$, $y^* \in \boldsymbol{R}^m$, $\rho > 0$, and $\lambda_i > 0$, $i = 1,\ldots,m$, if and only if the point $(\bar{t}, \bar{x})$ with*

$$\bar{t} = \max_{i\in\{1,\ldots,m\}} \lambda_i \left((f_i(\bar{x}) - y^*_i) + \rho \sum_{j=1}^m (f_j(\bar{x}) - y^*_j) \right) \tag{4.19}$$

is a minimal solution of (3.1) with $a = y^*$, $r \in \boldsymbol{R}^m$ with

$$r_i + \rho \sum_{j=1}^m r_j = \frac{1}{\lambda_i}, \qquad \forall i = 1,\ldots,m, \tag{4.20}$$

and $K = \{y \in \boldsymbol{R}^m \mid y_i + \rho \sum_{i=1}^m y_j \geq 0,\ i = 1,\ldots,m\}$.

Proof For a and K as in the Theorem the problem (3.1) reads as follows

$$\begin{gathered}\min t\\ \text{under the constraints}\\ y_i^* + t\,r_i - f_i(x) + \rho\sum_{j=1}^m (y_j^* + t\,r_j - f_j(x)) \geq 0, \qquad \forall i = 1,\dots,m,\\ t \in \boldsymbol{R},\ x \in \Omega,\end{gathered}$$

which is because of $r_i + \rho\sum_{j=1}^m r_j = \frac{1}{\lambda_i} > 0$ equivalent to

$$\begin{gathered}\min t\\ \text{under the constraints}\\ t \geq \frac{f_i(x) - y_i^* + \rho\sum\limits_{j=1}^m (f_j(x) - y_j^*)}{r_i + \rho\sum\limits_{j=1}^m r_j}, \qquad \forall i = 1,\dots,m,\\ t \in \boldsymbol{R},\ x \in \Omega.\end{gathered}$$

Using (4.20) a point $(\bar{t},\bar{x})$ is a minimal solution of this problem if and only if $\bar{x}$ is a solution of (4.18) with $\bar{t}$ as in (4.19).

The set K is a closed convex pointed cone and we have $r \in K$. For $m = 2$ the cone K is given by the set

$$K = \{y \in \boldsymbol{R}^2 \mid \begin{pmatrix} 1+\rho & \rho \\ \rho & 1+\rho \end{pmatrix} y \geq 0_2\}.$$

Hence the parameter ρ controls the cone K and a variation of the parameters λ and ρ lead to a variation of the parameter r while the parameter a is chosen constant as y^*. Because of $\boldsymbol{R}_+^m \subset K$ and $r \in K$ for $\lambda_i > 0$, $i = 1\dots,m$, we have $\mathcal{E}_w(f(\Omega),K) \subset \mathcal{E}_w(f(\Omega),\boldsymbol{R}_+^m)$ and thus a minimal solution of (P^∞) is an at least weakly EP-minimal point of (2.1).

5 Conclusion

We have shown that many scalarization problems can be seen as a special case of the Pascoletti-Serafini method and hence that the results for the general problem can be applied to these special cases, too. The enumeration of special cases is not complete. For example in [30] a problem called hyperbola efficiency going back to [9] is discussed. But for a connection to the Pascoletti-Serafini problem K has to be defined as a convex set which is not a cone. Also a generalization of the weighted Chebyshev norm is mentioned there which can be connected to problem (3.1) then using a closed pointed convex cone K.

Based on the demonstrated connections it was possible to apply the adaptive parameter control developed for the Pascoletti-Serafini problem to other scalarization approaches. Because in this algorithm only the parameter a is controlled and no sensitivity results w. r. t. the parameter r are used, it would be of interest, too, to examine the dependence on the parameter r further.

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