FRIEDRICH-ALEXANDER-UNIVERSITÄT ERLANGEN-NÜRNBERG
INSTITUT FÜR ANGEWANDTE MATHEMATIK

Vector Optimization with a Variable Ordering Structure

by

G. Eichfelder

No. 328 2009

Institut für Angewandte Mathematik
Martensstraße 3
D-91058 Erlangen

Lehrstuhl AM I:	Lehrstuhl AM II:	Lehrstuhl AM III:
Tel.: ++49/9131/85-27015	Tel.: ++49/9131/85-27510	Tel.: ++49/9131/85-25226
Fax: ++49/9131/85-27670	Fax: ++49/9131/85-28126	Fax: ++49/9131/85-25228

internet: `http://www.am.uni-erlangen.de`

ISSN 1435-5833, 2009, No. 328

Vector Optimization with a Variable Ordering Structure

Gabriele Eichfelder*

July 28, 2009

Abstract

In this work vector optimization problems are examined which have a variable ordering structure defined by a set-valued map which associates to each element in the objective space a cone of preferred or dominated directions. These considerations are motivated by a recent application in medical image registration where the preferences vary dependently on the actually considered element in the objective space.

Several optimality concepts and characterizations of optimal solutions are discussed. Existence and scalarization results as well as optimality conditions and duality results are presented.

Key Words: Vector optimization, variable ordering structure, variable domination structure, cone-valued map, linear scalarization, duality.

Mathematics subject classifications (MSC 2000): 90C29, 90C30,90C46, 90C48.

1 Introduction

We consider in this work vector optimization problems where optimality is defined based on a variable ordering structure. This structure is defined in the objective space by a set-valued map which associates to each element a cone of preferred or dominated directions.

Our main motivation for examining vector optimization problems with a variable ordering structure is a recent application problem in image registration in medical engineering [20]: There it is the aim to merge several medical images gained by different methods as for instance computer tomography (CT), magnetic resonance tomography (MRT), positron emission tomography (PET) or ultrasound (US). Thus one searches for a best transformation map, also called registration. (For a short introduction to image registration see for instance [11].) To measure the quality of such a transformation, a multitude of similarity measures is known, which all possess different properties and advantages. The values of the different measures can be interpreted as objectives which have to be minimized all at the same time.

*Department Mathematik, Universität Erlangen-Nürnberg, Martensstr. 3, D-91058 Erlangen, Germany, email: Gabriele.Eichfelder@am.uni-erlangen.de

However, dependent on the objective values, it is advantageous to put a higher weight on some of these objectives than on others. To each element $\bar{y}$ in the objective space one can thus relate a weight vector $w(\bar{y})$ and consider a related cone $\mathcal{D}(w(\bar{y}))$ which depends on the weight vector and hence on the considered point $\bar{y}$. Then one searches for a minimal solution of a vector optimization problem with a variable ordering structure. We discuss this application in more detail in Section 2.2.

Already in [16] it is recognized that the importance of criteria may change during the decision making process and that it may depend on the current criteria values. Wiecek gives in [26] an example for that. In [1] Baatar and Wiecek examine the concept of equitability, which has applications in portfolio optimization and location problems. There it is assumed that the criteria are not only comparable but also anonymous, which makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria. They show that this minimality notion is related to a finite number of ordering cones instead of a unique ordering cone. The objective space is departed into sections and the related ordering is variable and depends on the section in which an element lies.

Engau examines in [10] the role of variable domination structures in preference modeling. He gives examples showing the limitations of preference modeling using only one ordering cone. Variable structures defined by convex cones containing the positive orthant and being some kind of symmetric are studied. In [28] examples are given for variable domination cones to illustrate the importance of variable ordering structures to model the preferences of decision makers adequately, see also [7].

Because variable ordering structures become more important in applications we present here a general theory which can form a basis for further developments. Theoretical examinations of vector optimization problems with a variable ordering structure in the context of vector complementarity problems are already done by Huang et al. in [13].

In Section 2 we present various optimality concepts, we discuss the application in image registration mentioned above in more detail, and we give first characterizations of minimal and nondominated elements of a set. This is continued in Section 3 where we examine linear scalarizations of the vector optimization problem. We derive an existence result for minimal and nondominated elements respectively. Necessary and sufficient optimality conditions for a vector optimization problem with a variable ordering structure are presented in Section 4 and in Section 5 we conclude with some duality results.

2 Optimal elements and their characterization

First we have to define optimality for a variable ordering (or domination) structure. Thereby two concepts are of interest. The importance of variable ordering structures is illustrated with an application in image registration. We also present characterizations of optimal elements.

2.1 Variable ordering structure and optimality notions

In general, in vector optimization with a non-variable ordering structure, one assumes a partial ordering $\geq_K$ in a real linear space Y to be given by a nontrivial convex cone $K \subset Y$. Then we write $x \leq_K y$ for $y - x \in K$. A minimal element $\bar{y} \in A$ of a nonempty subset A of Y w.r.t. the cone $K \subset Y$ is defined by $(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K$ ([15]). For pointed cones this definition reduces to

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}. \tag{1}$$

Recall that a set K is called a cone if $\lambda x \in K$ for all $\lambda \geq 0$ and $x \in K$. And a cone is convex if $K + K \subset K$. A cone satisfying $K \cap (-K) = \{0_Y\}$ is called pointed. For getting a pointed cone $\tilde{K}$ from a non-pointed cone K, Borwein [3] replaces K in (1) by $\tilde{K} := (K \setminus (K \cap (-K))) \cup \{0_Y\}$.

The definition in (1) is also used for an arbitrary set K with $0_Y \in K$, see [2, 25], or even $0_Y \notin K$, see [23]. In this work we also base out considerations on the condition in (1), which is equivalent to that there is no $y \in A$ with

$$\bar{y} \in \{y\} + K \setminus \{0_Y\}. \tag{2}$$

We always assume the set K to be a cone. In (2) the cone K can be interpreted as the set of dominated directions of the element y while in (1) the cone $-K$ represents the set of preferred directions of the element $\bar{y}$. Searching for minimal elements equals thus the search for nondominated elements or for the most preferred elements in the set A respectively. For a literature survey about the usage of these concepts for K a set, a cone or the positive orthant, see [10].

The concepts (1) and (2) are equivalent, but for a variable ordering structure we have to differentiate between these two ideas of preference and domination. We need the following assumptions.

Assumption 2.1. *Let A be a nonempty subset of a real linear spaces Y. Let $\mathcal{D}\colon Y \to 2^Y$ be a set-valued map with $\mathcal{D}(y)$ a nontrivial cone with nonempty algebraic interior $cor(\mathcal{D}(y))$ for all $y \in Y$.*

Using the cone-valued map $\mathcal{D}$ we can define two different relations by

$$y \leq_1 \bar{y} \quad \text{if} \quad \bar{y} \in \{y\} + \mathcal{D}(\bar{y}) \tag{3}$$

or

$$y \leq_2 \bar{y} \quad \text{if} \quad \bar{y} \in \{y\} + \mathcal{D}(y). \tag{4}$$

We speak here of a variable ordering (structure), also variable order relation in [13], given by the ordering map $\mathcal{D}$, despite these binary relations given above are in general not transitive or even compatible with positive scalar multiplication, to express that the partial ordering given by a cone is replaced by $\mathcal{D}$. In [10] $\mathcal{D}$ is also called a variable domination structure and in [26] $\mathcal{D}$ is denoted as structure of domination. Vector optimization problems with $\mathcal{D}(y) \equiv K$ a closed cone which is not necessarily convex are also discussed in [18, 14, 28].

These two relations lead to the following two minimality notions for an ordering map $\mathcal{D}$. The first, based on (4), is due to Yu [27, 28], see also [19, 13]:

Definition 2.2. *Let the Assumption 2.1 be satisfied. An element $\bar{y} \in A$ is called a* nondominated element *of the set A w.r.t. the ordering map $\mathcal{D}$, if there is no $y \in A$ such that*

$$\bar{y} \in \{y\} + \mathcal{D}(y) \setminus \{0_Y\}. \tag{5}$$

This notion is also used in [21, 26]. Note, that this definition by Yu is incorrectly cited in [5, p.98],[6, Definition 1.11]. There, (5) is replaced by

$$A \cap (\{\bar{y}\} - \mathcal{D}(y)) = \{\bar{y}\} \;\text{ for all }\; y \in A.$$

In the definition of nondominated elements the cone $\mathcal{D}(y) = \{d \in Y \mid y + d \text{ is dominated by } y\} \cup \{0_Y\}$, also called domination cone, is the set of dominated directions or domination factors for each element $y \in Y$, see also [2, 26, 28]. An equivalent formulation of Definition 2.2 is that an element $\bar{y} \in A$ is a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$ if

$$\bar{y} \notin \bigcup_{y \in A} \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}).$$

For $\mathcal{D}(y) = \{d \in Y \mid y - d \text{ is preferred to } y\} \cup \{0_Y\}$ interpreted as the set of preferred directions we get the following minimality notion. This notion, based on (3), is called nondominated-like minimal solution of A by Chen in [4, 5, 6] or nondominated w.r.t. $\mathcal{D}$ by Engau in [10] and is also used in [13, 17].

Definition 2.3. *Let the Assumption 2.1 be satisfied. An element $\bar{y} \in A$ is called a* minimal element *of the set A w.r.t. the ordering map $\mathcal{D}$, if*

$$(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}. \tag{6}$$

Thus, if there is no y in A, which is preferred to $\bar{y}$, i.e. such that $y \in \bar{y} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\})$, then $\bar{y}$ is called a minimal element of the set A w.r.t. the ordering map $\mathcal{D}$.

For $\mathcal{D}(y) \equiv K$ both Definitions 2.2 and 2.3 are equivalent, i.e. the minimal elements are exactly the nondominated elements. In that case we speak of minimal/nondominated elements of a set A w.r.t. the cone K. In Section 5 about duality we get an important relation between minimal elements of a set and nondominated elements of another so called dual set. For that reason it is useful to study both optimality concepts despite we believe that the concept of minimal elements is more suitable for preference modeling in applications (see [10]). To illustrate that the concept of nondominated elements is a more restrictive concept compared to the concept of minimal elements, we give the following example.

Example 2.4. *Let $Y = \mathbb{R}^2$, $A := \{y \in \mathbb{R}^2 \mid \|y\|_2 \le 1\}$, and let $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by*

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for } y \in \mathbb{R}^2 \setminus \{(0,-1)\} \\ \{y \in \mathbb{R}^2 \mid y_2 \ge 0\} & \text{for } y = (0,-1) \end{cases}$$

Only $(0,-1)$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ while all elements of the set

$$\{y \in \mathbb{R}^2 \mid \|y\|_2 = 1,\; y_1 \le 0,\; y_2 \le 0\}$$

are minimal elements w.r.t. $\mathcal{D}$.

For further examples comparing these two definitions see [5].

Like it is known from vector optimization without a variable ordering structure one can also define weakly nondominated or weakly minimal elements by replacing $\mathcal{D}(y)$ and $\mathcal{D}(\bar{y})$ in (5) and (6) by the algebraic interior $\mathrm{cor}(\mathcal{D}(y))$ and $\mathrm{cor}(\mathcal{D}(\bar{y}))$ respectively (assuming it is nonempty). Weakly elements are useful as they are completely characterized by linear scalarizations, but in applications one is in general not interested in only weakly optimal elements.

Definition 2.5. *Let the Assumption 2.1 be satisfied An element $\bar{y} \in A$ is called a* weakly nondominated element *of the set A w.r.t. the ordering map $\mathcal{D}$, if there is no $y \in A$ such that*

$$\bar{y} \in \{y\} + cor(\mathcal{D}(y)). \tag{7}$$

An element $\bar{y} \in A$ is called a weakly minimal element *of the set A w.r.t. the ordering map $\mathcal{D}$, if*

$$(\{\bar{y}\} - cor(\mathcal{D}(\bar{y}))) \cap A = \emptyset. \tag{8}$$

Thus any nondominated element of A is also a weakly nondominated element and any minimal element of A is also a weakly minimal element of A w.r.t. $\mathcal{D}$.

There also exists the definition of strongly minimal elements.

Definition 2.6. *Let the Assumption 2.1 be satisfied. An element $\bar{y} \in A$ is called a* strongly minimal element *of the set A w.r.t. the ordering map $\mathcal{D}$), if*

$$A \subset \{\bar{y}\} + \mathcal{D}(\bar{y}).$$

Then one also says that A is minorized [6]. Note that strongly minimal elements do not need to be nondominated or minimal elements of A w.r.t. $\mathcal{D}$.

Example 2.7. *Consider Y and A as in Example 2.4 but replace the map $\mathcal{D}$ by*

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for } y \in \mathbb{R}^2 \setminus \{(0,0),(0,-1)\}, \\ \{y \in \mathbb{R}^2 \mid y_2 \leq 0\} & \text{for } y = (0,0), \\ \{y \in \mathbb{R}^2 \mid y_1 \geq 0 \ \vee \ y_2 \geq 0\} & \text{for } y = (0,-1). \end{cases}$$

Then the point $(0,-1)$ is strongly minimal but not a minimal element of A w.r.t. $\mathcal{D}$, and because of the point $(0,0)$ it is also not a nondominated element of A w.r.t. $\mathcal{D}$.

Lemma 2.8. *Let the Assumption 2.1 be satisfied and let $\bar{y} \in A$ be a strongly minimal element of A w.r.t. the ordering map $\mathcal{D}$ with the cone $\mathcal{D}(\bar{y})$ pointed. Then $\bar{y}$ is also a minimal element of A w.r.t. $\mathcal{D}$.*

Proof. Assume $\bar{y} \in A$ is not a minimal element. Then there exists $y \in A$, $y \neq \bar{y}$ with $y \in \{\bar{y}\} - \mathcal{D}(\bar{y})$. Because $y \in A$ and $A \subset \{\bar{y}\} + \mathcal{D}(\bar{y})$ we also get $y \in \{\bar{y}\} + \mathcal{D}(\bar{y})$ and thus $y - \bar{y} \in \mathcal{D}(\bar{y}) \cap (-\mathcal{D}(\bar{y}))$ in contradiction to $\mathcal{D}(\bar{y})$ pointed and $y \neq \bar{y}$. $\square$

In Section 5 we also need the definition of (weakly) maximal and (weakly) max-nondominated elements w.r.t. the ordering map $\mathcal{D}$ for associating a dual problem to the primal problem of finding minimal/nondominated elements of a set. We start with the notions related to nondominated elements.

Definition 2.9. *Let the Assumption 2.1 be satisfied. An element $\bar{y} \in A$ is called a* max-nondominated element *of the set A w.r.t. the ordering map $\mathcal{D}$, if there is no $y \in A$ such that*

$$\bar{y} \in \{y\} - \mathcal{D}(y) \setminus \{0_Y\}. \tag{9}$$

An element $\bar{y} \in A$ is called a weakly max-nondominated element *of the set A w.r.t. the ordering map $\mathcal{D}$, if there is no $y \in A$ such that*

$$\bar{y} \in \{y\} - cor(\mathcal{D}(y)). \tag{10}$$

Analogously, related to the concept of minimal elements, we define:

Definition 2.10. *Let the Assumption 2.1 be satisfied. An element $\bar{y} \in A$ is called a* maximal element *of the set A (w.r.t. the ordering map $\mathcal{D}$), if*

$$(\{\bar{y}\} + \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}. \tag{11}$$

An element $\bar{y} \in A$ is called a weakly maximal element *of the set A (w.r.t. the ordering map $\mathcal{D}$), if*

$$(\{\bar{y}\} + cor(\mathcal{D}(\bar{y}))) \cap A = \emptyset. \tag{12}$$

For $\mathcal{D}(y) \equiv K$ a cone the conditions (7) and (8) as well as (9) and (11) and (10) and (12) are equivalent.

2.2 Variable ordering structure in image registration

As already described in the introduction the aim in (medical) image registration is to match two data sets A and B, gained for instance by CT or MRT. Hence one searches for a transformation map T which has to be exact, robust and which should, for some applications, be found automatically without a human decision maker. The quality of the transformation, i.e. the similarity of the transformed data set to the target set, can be measured by several distance measures $g_i \colon (T, A, B) \to \mathbb{R}$. Then one searches for the transformation with the smallest distance measure.

However there exists a plenitude of measures which evaluate distinct characteristics like the sum of square differences, mutual information or cross-correlation. Different measures can lead to different best transformations. Some measures fail on special data sets and can lead to mathematical correct but useless results. Thus it is important to combine several measures. Possible approaches are a weighted sum of different measures. But difficulties appear as badly scaled functions or non-convex functions.

The new approach discussed by Wacker in [20] proposes instead a multiobjective view of this problem by combining the several distance measures in an objective vector $f := (g_1, \ldots, g_m)^\top$ and to solve for given sets A and B a vector optimization problem

$$\min_T f(T, A, B).$$

For defining minimality a variable ordering structure is introduced. Therefore a weighting vector $w = w(y) \in \mathbb{R}^m_+$ is generated to each point $y \in \mathbb{R}^m$ in the image space.

This weight depends on gradient information, conformity and continuity aspects and reflects the preference of a totally rational decision maker who puts a higher weight on promising measures dependent on the value $f(T, A, B)$ of the actual transformation. This can also be interpreted as some kind of voting between the several measures. Also a weight component equal to zero is allowed which corresponds to the negligence of one measure, which for instance seems to fail on the data set.

To such a weight at a point $\bar{y}$ a cone of more or equally preferred directions is defined by

$$\mathcal{D}^w := \mathcal{D}(w(\bar{y})) := \{d \in \mathbb{R}^m \mid \sum_{i=1}^{m} \operatorname{sgn}(d_i) w_i(\bar{y}) \geq 0\}$$

where

$$\operatorname{sgn}(d_i) := \begin{cases} 1 & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0, \\ -1 & \text{if } d_i < 0. \end{cases}$$

The first results presented in [20] are promising for this approach in the sense of a robust, fast and correct algorithm.

To illustrate the varying nature of such a variable ordering structure we determine $\mathcal{D}^w \subset \mathbb{R}^2$ for all possible weights $w \in \mathbb{R}^2_+ \setminus \{0_2\}$:

$$\mathcal{D}^w := \begin{cases} \{d \in \mathbb{R}^2 \mid d_1 \geq 0,\ d_2 \in \mathbb{R}\} & \text{for } w_1 > 0,\ w_2 = 0, \\ \{d \in \mathbb{R}^2 \mid d_1 \in \mathbb{R},\ d_2 \geq 0\} & \text{for } w_1 = 0,\ w_2 > 0, \\ \{d \in \mathbb{R}^2 \mid (d_1 \geq 0 \wedge d_2 \geq 0) \vee (d_1 < 0 \wedge d_2 > 0)\} & \text{for } w_2 \geq w_1 > 0, \\ \{d \in \mathbb{R}^2 \mid (d_1 \geq 0 \wedge d_2 \geq 0) \vee (d_1 > 0 \wedge d_2 < 0)\} & \text{for } w_1 \geq w_2 > 0. \end{cases}$$

Of course in higher dimensions ($m \geq 3$) also the values of the components of w_i are important. Note that for nonnegative weights $w \in \mathbb{R}^m_+$ the positive orthant $\mathbb{R}^m_+$ is always included in $\mathcal{D}^w$. Besides $\mathcal{D}^w$ is generally a non-pointed cone. The cone $\mathcal{D}^w$ can also be nonconvex. For instance consider the cone $\mathcal{D}^w \subset \mathbb{R}^3$ for $w = (4, 3, 2)$ with $d^1 = (1, -3, 1) \in \mathcal{D}^w$ and $d^2 = (-3, 1, 1) \in \mathcal{D}^w$ but $d^1 + d^2 = (-2, -2, 2) \notin \mathcal{D}^w$.

2.3 Characterization of solutions

Under very weak assumptions we get that all weakly nondominated and also all weakly minimal elements lie on the algebraic boundary ∂A of the set A

Lemma 2.11. *Let the Assumption 2.1 be satisfied.*

- ***(a)*** *If $\bigcap_{y \in A} \operatorname{cor}(\mathcal{D}(y)) \neq \emptyset$ and $\bar{y} \in A$ is a weakly nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$, then $\bar{y} \in \partial A$.*

- ***(b)*** *If $\bar{y} \in A$ is a weakly minimal element of the set A w.r.t. the ordering map $\mathcal{D}$, then $\bar{y} \in \partial A$.*

Proof. (a) We assume that $\bar{y} \in \operatorname{cor}(A)$. Then for any $d \in \bigcap_{y \in A} \operatorname{cor}(\mathcal{D}(y))$ there exists some $\lambda > 0$ with $\bar{y} - \lambda d \in A$. As $-\lambda d \in -\bigcap_{y \in A} \operatorname{cor}(\mathcal{D}(y)) \subset -\operatorname{cor}(\mathcal{D}(\bar{y} - \lambda d))$ we have

$$\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - \operatorname{cor}(\mathcal{D}(\bar{y} - \lambda d)))$$

or $\bar{y} \in \{\bar{y} - \lambda d\} + \operatorname{cor}(\mathcal{D}(\bar{y} - \lambda d))$, being a contradiction to $\bar{y}$ weakly nondominated.

(b) Choosing any $d \in \operatorname{cor}(\mathcal{D}(\bar{y}))$ the proof is analogously.

□

The following example demonstrates that we need an assumption like

$$\bigcap_{y \in A} \operatorname{cor}(\mathcal{D}(y)) \neq \emptyset$$

for the statement in (a).

Example 2.12. *For the set $A = [1,3] \times [1,3] \subset \mathbb{R}^2$ and the ordering map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$,*

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for all } y \in \mathbb{R}^2,\ y_1 \geq 2, \\ \{z \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{else,} \end{cases}$$

the point $\bar{y} = (2,2)$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$ but $\bar{y} \notin \partial A$.

In [5, Example 2.1] an example is given in which for $\bigcap_{y \in A} \operatorname{cor}(\mathcal{D}(y)) = \emptyset$ no nondominated element and no minimal element of A w.r.t. $\mathcal{D}$ exists at all.

It is easy to see that $A_1 \subset A_2$ implies that an element $\bar{y} \in A_1$, which is a minimal (or nondominated) element of A_2, is also a minimal (respectively nondominated) element of A_1. In the following lemma we examine the dependence on the variable domination structure. For the case of $\mathcal{D}$ a constant cone-valued map the correspondent result is for instance given in [8, Lemma 1.7].

Lemma 2.13. *Let A be a nonempty subset of a real linear spaces Y. Let $\mathcal{D}_1, \mathcal{D}_2\colon Y \to 2^Y$ be cone-valued maps with $\mathcal{D}_1(y)$ and $\mathcal{D}_2(y)$ a nontrivial cone with nonempty algebraic interior for all $y \in Y$.*

- ***(a)*** *If $\mathcal{D}_1(y) \subset \mathcal{D}_2(y)$ for all $y \in A$, then each (weakly) nondominated element of the set A w.r.t. the ordering map $\mathcal{D}_2$ is also a (weakly) nondominated element of the set A w.r.t. the ordering map $\mathcal{D}_1$.*
- ***(b)*** *If $\bar{y}$ is a (weakly) minimal element of the set A w.r.t. the ordering map $\mathcal{D}_2$ and $\mathcal{D}_1(\bar{y}) \subset \mathcal{D}_2(\bar{y})$, then $\bar{y}$ is also a (weakly) minimal element of the set A w.r.t. the ordering map $\mathcal{D}_1$.*

Proof. The proofs follow directly from the Definitions 2.2 and 2.3. □

Corollary 2.14. *Let the Assumption 2.1 be satisfied.*

- ***(a)*** *Let $\bar{y} \in A$ be a (weakly) nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$. Then for any cone K with $K \subset \mathcal{D}(y)$ for all $y \in A$, $\bar{y}$ is also a (weakly) minimal element of the set A w.r.t. the cone K, i.e. (1) and (2) are satisfied.*
- ***(b)*** *Let $\bar{y} \in A$ be a (weakly) minimal element of the set A w.r.t. the ordering map $\mathcal{D}$. Then for any cone K with $K \subset \mathcal{D}(\bar{y})$, $\bar{y}$ is also a (weakly) minimal element of the set A w.r.t. the cone K, i.e. (1) and (2) are satisfied.*

The converse statement is given in the following corollary.

Corollary 2.15. *Let the Assumption 2.1 be satisfied. Let $\bar{y} \in A$ be a (weakly) minimal (and thus a nondominated) element of the set A w.r.t. the cone $K \subset Y$, i.e. (1) and (2) are satisfied.*

(a) *If $\mathcal{D}(y) \subset K$ for all $y \in A$, then $\bar{y}$ is also a (weakly) nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$.*

(b) *If $\mathcal{D}(\bar{y}) \subset K$, then $\bar{y}$ is also a (weakly) minimal element of the set A w.r.t. the ordering map $\mathcal{D}$.*

Thus, we have the following characterization, which can directly be proofed using the Definitions 2.3 and 2.5:

Lemma 2.16. *Let the Assumption 2.1 be satisfied. Let $K := \mathcal{D}(\bar{y})$. $\bar{y} \in A$ is a (weakly) minimal element of A w.r.t. the ordering map $\mathcal{D}$ if and only if it is a (weakly) minimal element w.r.t. the cone K, i.e. (1) and (2) are satisfied.*

Lemma 2.17. *Let A, B be nonempty subsets of a real linear space Y. Let $\mathcal{D}\colon Y \to 2^Y$ be set-valued with $\mathcal{D}(y)$ a nontrivial cone for all $y \in Y$ and let $\mathcal{D}$ satisfy*

$$\mathcal{D}(y^A) + \mathcal{D}(y^B) \subset \mathcal{D}(y^A + y^B) \qquad \forall\, y^A \in A,\ y^B \in B. \tag{13}$$

(a) *If $\bar{y} = \bar{y}^A + \bar{y}^B \in A + B$ with $\bar{y}^A \in A$, $\bar{y}^B \in B$ is a nondominated element of $A + B$ w.r.t. the ordering map $\mathcal{D}$, then $\bar{y}^A$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and $\bar{y}^B$ is a nondominated element of B w.r.t. the ordering map $\mathcal{D}$.*

(b) *If $\bar{y} = \bar{y}^A + \bar{y}^B \in A + B$ with $\bar{y}^A \in A$, $\bar{y}^B \in B$ is a minimal element of $A+B$ w.r.t. the ordering map $\mathcal{D}$, then $\bar{y}^A$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and $\bar{y}^B$ is a minimal element of B w.r.t. the ordering map $\mathcal{D}$.*

Proof. (a) We assume $\bar{y}^A$ is not a nondominated element of A w.r.t. the ordering map $\mathcal{D}$. Then there exists $y \in A$ with $\bar{y}^A \in \{y\} + (\mathcal{D}(y) \setminus \{0_Y\})$ and thus

$$\bar{y} = \bar{y}^A + \bar{y}^B \in \{y + \bar{y}^B\} + (\mathcal{D}(y) \setminus \{0_Y\}).$$

With $0_Y \in \mathcal{D}(\bar{y}^B)$ and because $\mathcal{D}$ satisfies (13) we get

$$\mathcal{D}(y) \subset \mathcal{D}(y) + \mathcal{D}(\bar{y}^B) \subset \mathcal{D}(y + \bar{y}^B).$$

Thus

$$\bar{y} \in \{\underbrace{y + \bar{y}^B}_{\in A+B}\} + (\mathcal{D}(y + \bar{y}^B) \setminus \{0_Y\})$$

in contradiction to $\bar{y}$ a nondominated element of $A + B$ w.r.t. the ordering map $\mathcal{D}$. The same for $\bar{y}^B$.

(b) We assume $\bar{y}^A$ is not a minimal element for A w.r.t. the ordering map $\mathcal{D}$. Then there exists $y \in A$ with $\bar{y}^A \in \{y\} + (\mathcal{D}(\bar{y}^A) \setminus \{0_Y\})$ and thus

$$\bar{y} = \bar{y}^A + \bar{y}^B \in \{y + \bar{y}_B\} + (\mathcal{D}(\bar{y}^A) \setminus \{0_Y\}).$$

With the same arguments as before we conclude

$$\bar{y} \in \{\underbrace{y + \bar{y}^B}_{\in A+B}\} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\})$$

in contradiction to $\bar{y}$ a minimal element of $A + B$ w.r.t. the ordering map $\mathcal{D}$. The same for $\bar{y}^B$.

□

For a variable ordering structure satisfying (13) see Example 2.23. If the set-valued map $\mathcal{D}$ is subadditive w.r.t. the cone $\{0_Y\}$ on Y (for a definition of subadditivity see [15, Definition 15.20]), then condition (13) is satisfied. The converse statement of Lemma 2.17 does even not hold in general in the case of a non-variable ordering structure, see [19, Remark 3.1.3].

Lemma 2.18. *Let the Assumption 2.1 be satisfied and $\alpha \in \mathbb{R}$, $\alpha > 0$, be given. Let $\mathcal{D}(y') = \mathcal{D}(\lambda y')$ for all $\lambda > 0$ and all $y' \in Y$.*

(a) *Then y is a nondominated element of the set $\alpha\, A$ w.r.t. the ordering map $\mathcal{D}$ if and only if $\frac{1}{\alpha} y$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.*

(b) *Then y is a minimal element of the set $\alpha\, A$ w.r.t. the ordering map $\mathcal{D}$ if and only if $\frac{1}{\alpha} y$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.*

Proof. (a) y is a nondominated element of the set $\alpha\, A$ w.r.t. the ordering map $\mathcal{D}$ if and only if

$$\begin{aligned} & y \notin \{y'\} + (\mathcal{D}(y') \setminus \{0_Y\}) \qquad \forall y' \in \alpha A \\ \Leftrightarrow \quad & \tfrac{1}{\alpha} y \notin \{\tfrac{1}{\alpha} y'\} + (\mathcal{D}(y') \setminus \{0_Y\}) \qquad \forall \tfrac{1}{\alpha} y' \in A, \\ \Leftrightarrow \quad & \tfrac{1}{\alpha} y \notin \{z\} + (\mathcal{D}(z) \setminus \{0_Y\}) \qquad \forall z \in A \end{aligned}$$

because $\mathcal{D}(y')$ is a cone and $\mathcal{D}(y') = \mathcal{D}(\frac{1}{\alpha} y')$. Thus if and only if $\frac{1}{\alpha} y$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

(b) y is a minimal element of the set $\alpha\, A$ w.r.t. the ordering map $\mathcal{D}$ if and only if

$$(\{y\} - \mathcal{D}(y)) \cap (\alpha\, A) = \{y\} \;\Leftrightarrow\; (\{\frac{1}{\alpha} y\} - \mathcal{D}(y)) \cap A = \{\frac{1}{\alpha} y\}.$$

Because $\mathcal{D}(y) = \mathcal{D}(\frac{1}{\alpha} y)$ we obtain

$$(\{\frac{1}{\alpha} y\} - \mathcal{D}(\frac{1}{\alpha} y)) \cap A = \{\frac{1}{\alpha} y\}.$$

Thus $\frac{1}{\alpha} y$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.

□

In the following we give an example for a domination structure $\mathcal{D}$ satisfying $\mathcal{D}(y) = \mathcal{D}(\lambda y)$ for all $\lambda > 0$.

Example 2.19. *Consider the set-valued map* $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ *with*

$$\mathcal{D}(y) := \{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \mid r \geq 0,\ \varphi \in [\bar{\varphi}_y - \pi/4, \bar{\varphi}_y + \pi/4] \cap [0, \pi/2]\}$$

for $y \neq 0_2$ *and* $\mathcal{D}(0_2) := \mathbb{R}^2_+$*, with* $\bar{\varphi}_y \in [0, \pi/2)$ *defined by*

$$y = (r_y \cos(l\bar{\varphi}_y), r_y \sin(l\bar{\varphi}_y)) \text{ for some } l \in \mathbb{N} \text{ and some } r_y \in \mathbb{R},\ r_y > 0.$$

Then $\mathcal{D}(y) = \mathcal{D}(y/\|y\|_2)$ *for all* $y \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$ *and the condition* $\mathcal{D}(y) = \mathcal{D}(\lambda y)$ *for all* $\lambda > 0$ *and all* $y \in Y$ *is satisfied.*

For $\mathcal{D}(y) \equiv K$ with K a pointed convex cone, it is known, see for instance [15, Lemma 4.7],[27, Lemma 4.1], that an element is a minimal element of A if and only if it is a minimal element of the set $A + K$. This can be generalized only under strong assumptions.

Lemma 2.20. *Let the Assumption 2.1 be satisfied and define*

$$M := \bigcup_{y \in A} \{y\} + \mathcal{D}(y).$$

(a) *(i) If* $\bar{y} \in M$ *is a nondominated element of* M *w.r.t.* $\mathcal{D}$*, then* $\bar{y} \in A$ *and* $\bar{y}$ *is also a nondominated element of* A *w.r.t.* $\mathcal{D}$*.*

(ii) Assume that $\mathcal{D}(y)$ *is a pointed convex cone for all* $y \in Y$*. If* $\bar{y} \in A$ *is a nondominated element of* A *w.r.t.* $\mathcal{D}$*, and if* $\mathcal{D}(y + d) \subset \mathcal{D}(y)$ *for all* $y \in A$ *and for all* $d \in \mathcal{D}(y)$*, then* $\bar{y}$ *is a nondominated element of* M *w.r.t.* $\mathcal{D}$*.*

(b) *(i) If* $\bar{y} \in A$ *is a minimal element of* M *w.r.t.* $\mathcal{D}$*, then it is also a minimal element of* A *w.r.t.* $\mathcal{D}$*.*

(ii) If $\bar{y} \in A$ *is a minimal element of the set* A *w.r.t.* $\mathcal{D}$*, if* $\mathcal{D}(\bar{y})$ *is a pointed convex cone and if* $\mathcal{D}(a) \subset \mathcal{D}(\bar{y})$ *for all* $a \in A$*, then* $\bar{y}$ *is also a minimal element of the set* M *w.r.t* $\mathcal{D}$*.*

Proof. (a) If $\bar{y} \in M \setminus A$ then $\bar{y} \in \{y\} + (\mathcal{D}(y) \setminus \{0_Y\})$ for some $y \in A \subset M$ in contradiction to $\bar{y}$ a nondominated element of M w.r.t. $\mathcal{D}$. Thus $\bar{y} \in A$. Due to $A \subset M$, $\bar{y}$ is then also a nondominated element of A w.r.t. $\mathcal{D}$. Thus (i) is shown. Now we assume for (ii) that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ but not of M, i.e. there exists $a \in A$ and $d_a \in \mathcal{D}(a) \setminus \{0_Y\}$ with $\bar{y} \in \{a + d_a\} + (\mathcal{D}(a + d_a) \setminus \{0_Y\})$. As $\mathcal{D}(a)$ is a pointed convex cone this implies

$$\begin{array}{rcl} \bar{y} & \in & \{a\} + (\mathcal{D}(a) \setminus \{0_Y\}) + (\mathcal{D}(a + d_a) \setminus \{0_Y\}) \\ & \subset & \{a\} + (\mathcal{D}(a) \setminus \{0_Y\}) + (\mathcal{D}(a) \setminus \{0_Y\}) \\ & \subset & \{a\} + (\mathcal{D}(a) \setminus \{0_Y\}). \end{array}$$

This is a contradiction to $\bar{y}$ a nondominated element of A.

(b) The first implication (i) follows again from $A \subset M$. Now we assume for (ii) that $\bar{y}$ is minimal for A but not for M, i.e. there exists $a \in A$ and $d_a \in \mathcal{D}(a) \setminus \{0_Y\}$

with $a + d_a \in \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\})$. Again, as $\mathcal{D}(\bar{y})$ is a pointed convex cone, this implies

$$\begin{array}{rcl} a & \in & \{\bar{y}\} - (\mathcal{D}(a) \setminus \{0_Y\}) - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) \\ & \subset & \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) \\ & \subset & \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}). \end{array}$$

This is a contradiction to $\bar{y}$ a minimal element of A.

□

A variable domination structure satisfying the condition in Lemma 2.20,(a)(ii) is given in the following example.

Example 2.21. *Define the set-valued map* $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ *by*

$$\mathcal{D}(y_1, y_2) := \begin{cases} \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, \pi/8]\} & \text{for all } y_1 \geq \pi/2, \\ \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, \frac{\pi}{2} + \frac{\pi}{8} - y_1]\} & \text{for all } y_1 \in (\pi/8, \pi/2), \\ \mathbb{R}^2_+ & \text{for all } y_1 \leq \pi/8. \end{cases}$$

Then $\mathcal{D}$ *depends only on* y_1 *and for* $y_1 \geq \bar{y}_1$ *we conclude* $\mathcal{D}(y) \subset \mathcal{D}(\bar{y})$. *As for any* $y \in \mathbb{R}^2$ *and any* $d \in \mathcal{D}(y)$ *we have* $d_1 \geq 0$ *and thus* $y_1 + d_1 \geq y_1$ *we conclude*

$$\mathcal{D}(y + d) \subset \mathcal{D}(y) \text{ for any } y \in \mathbb{R}^2 \text{ and any } d \in \mathcal{D}(y). \tag{14}$$

For $\mathcal{D}(y)$ a convex cone for all $y \in Y$ the condition (14) can also be written as $\mathcal{D}(y+d) + \mathcal{D}(y) \subset \mathcal{D}(y)$ for all $y \in Y$ and all $d \in \mathcal{D}(y)$. This condition corresponds to the property of transitivity of a binary relation (see [7]): If y^1 is dominated by y^2 (in the sense of (4)), i.e. $y^1 \in \{y^2\} + \mathcal{D}(y^2)$, and if y^2 is dominated by y^3, i.e. $y^2 \in \{y^3\} + \mathcal{D}(y^3)$, then $y^1 \in \{y^2\} + \mathcal{D}(y^3) \subset \{y^3\} + \mathcal{D}(y^3)$, i.e. y^1 is dominated by y^3.

Using the contraire assumption then in Lemma 2.20,(a)(ii) we can extend the results of Lemma 2.20,(b)(i):

Lemma 2.22. *Let the Assumption 2.1 be satisfied and define* M *as in Lemma 2.20. If* $\bar{y} \in M$ *is a minimal element of* M *w.r.t. the ordering map* $\mathcal{D}$ *and if* $\mathcal{D}(y + d) \supset \mathcal{D}(y)$ *for all* $y \in A$ *and for all* $d \in \mathcal{D}(y)$*, then* $\bar{y}$ *is also a minimal element of* A *w.r.t.* $\mathcal{D}$.

Proof. With Lemma 2.20 it remains to show $\bar{y} \in A$. For that assume $\bar{y} \in M \setminus A$, i.e. $\bar{y} = \bar{a} + \bar{d}_a$ for some $\bar{a} \in A$ and some $\bar{d}_a \in \mathcal{D}(\bar{a}) \setminus \{0_Y\}$. Then for $y := \bar{a} + \frac{1}{2}\bar{d}_a \in M \setminus \{\bar{y}\}$ we get

$$y = \bar{y} - \frac{1}{2}\bar{d}_a \in \{\bar{y}\} - \frac{1}{2}(\mathcal{D}(\bar{a}) \setminus \{0_Y\}) \subset \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}).$$

This is a contradiction to the minimality of $\bar{y}$ for the set M w.r.t. $\mathcal{D}$. □

The condition $\mathcal{D}(y) \subset \mathcal{D}(y + d)$ for all $y \in A$ and all $d \in \mathcal{D}(y)$ of Lemma 2.22 is similar to the f-inclusive condition in [13]. We adapt the example presented there to give an example for a variable ordering structure satisfying this condition. Based on this variable domination structure we can also give an Example for (13).

Example 2.23. *Define the set-valued map* $\mathcal{D} : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ *by*

$$\mathcal{D}(y_1, y_2) := \begin{cases} \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, \pi/8]\} & \text{for all } y_1 \leq \pi/8, \\ \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, y_1]\} & \text{for all } y_1 \in (\pi/8, \pi/2), \\ \mathbb{R}^2_+ & \text{for all } y_1 \geq \pi/2. \end{cases}$$

Here, $\mathcal{D}$ *depends only on* y_1. *Then it is easy to see that the cones* $\mathcal{D}(y)$ *are closed pointed and convex and satisfy* $\mathcal{D}(y) \subset \mathcal{D}(y+d)$ *for all* $y \in \mathbb{R}^2$, $d \in \mathcal{D}(y)$.

Besides, for $A, B \subset \mathbb{R}^2_+$ *we get for any* $y^A \in A$, $y^B \in B$ *that* $y_1^A + y_1^B \geq y_1^A$ *and also* $y_1^A + y_1^B \geq y_1^B$. *Because* $\mathcal{D}(y') \supset \mathcal{D}(y)$ *for any* $y, y' \in \mathbb{R}^2_+$, $y_1' \geq y_1$, *we conclude, using the convexity of* $\mathcal{D}(y)$ *for any* $y \in \mathbb{R}^2$*:*

$$\mathcal{D}(y^A) + \mathcal{D}(y^B) \subset \mathcal{D}(y^A + y^B) + \mathcal{D}(y^A + y^B) = \mathcal{D}(y^A + y^B).$$

The following characterization of minimal elements uses linear scalarizations which will also be discussed in the following section.

Lemma 2.24. *Additionally to the Assumption 2.1 let* Y *be a real locally convex topological linear space.*

(a) *Let a convex closed cone* $K \subset Y$ *with* $K \subset \mathcal{D}(y)$ *for all* $y \in A$ *be given. If* $\bar{y} \in A$ *is a nondominated element of* A *w.r.t. the ordering map* $\mathcal{D}$ *then for every* $y \in A \setminus \{\bar{y}\}$ *there is a continuous linear functional* $l \in K^* \setminus \{0_{Y^*}\}$ *with*

$$l(\bar{y}) < l(y).$$

(b) *Let a cone* $K \subset Y$ *with* $K \supset \mathcal{D}(y)$ *for all* $y \in A$ *be given and let* $\bar{y} \in A$. *If for every* $y \in A \setminus \{\bar{y}\}$ *there is a continuous linear functional* $l \in K^* \setminus \{0_{Y^*}\}$ *with*

$$l(\bar{y}) < l(y), \tag{15}$$

then $\bar{y}$ *is a nondominated element of the set* A *w.r.t. the ordering map* $\mathcal{D}$.

(c) *Let the cone* $\mathcal{D}(\bar{y})$ *be convex and closed for some* $\bar{y} \in A$. *Then* $\bar{y}$ *is a minimal element of the set* A *w.r.t. the ordering map* $\mathcal{D}$ *if and only if for every* $y \in A \setminus \{\bar{y}\}$ *there is a continuous linear functional* $l \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ *with*

$$l(\bar{y}) < l(y).$$

Here, $\mathcal{D}(\bar{y})^* := \{l \in Y^* \mid l(y) \geq 0\ \forall y \in \mathcal{D}(\bar{y})\}$ *denotes the topological dual cone of the cone* $\mathcal{D}(\bar{y})$.

Proof. (a) Because $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$, $\bar{y}$ is according to Corollary 2.14,(a) also a nondominated element of A w.r.t. the cone K, i.e.

$$y \notin \{\bar{y}\} - K \quad \text{for all} \quad y \in A \setminus \{\bar{y}\}. \tag{16}$$

Since K is closed and convex, the set $\{\bar{y}\} - K$ is closed and convex, and using a separation theorem ([15, Theorem 3.18]) (16) is equivalent to: for every $y \in A \setminus \{\bar{y}\}$ there is some $l \in Y^* \setminus \{0_{Y^*}\}$ and some $\alpha \in \mathbb{R}$ with

$$l(y) > \alpha \geq l(\bar{y} - k) \quad \text{for all} \quad k \in K.$$

Because $l(k) < 0$ for some $k \in K$ leads to a contradiction, as K is a cone, we conclude $l \in K^* \setminus \{0_{Y^*}\}$.

(b) The condition (15) is equivalent to $l(\bar{y} - y) < 0$ and because of $l \in K^*$ to $\bar{y} \notin \{y\} + (K \setminus \{0_Y\})$. Thus for all $y \in A \setminus \{\bar{y}\}$ we have $\bar{y} \notin \{y\} + (K \setminus \{0_Y\})$ and hence
$$\bar{y} \notin \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}).$$

(c) Using Lemma 2.16 this result can be proofed like a similar result for a non-variable ordering structure given in [15, Theorem 5.5] using Eidelheit's separation theorem.

□

For the assumption in (b) we need $\mathcal{D}(A) \subset K$ with $\mathcal{D}(A)$ denoting the image set of $\mathcal{D}$:
$$\mathcal{D}(A) := \bigcup_{y \in A} \mathcal{D}(y.) \tag{17}$$
Note that the set $K^* \setminus \{0_Y\}$ might be empty.

A similar statement as in the previous lemma can be given for strongly minimal elements. Note that no convexity of the set A is assumed.

Lemma 2.25. *Let A be a nonempty subset of a real locally convex topological linear space. Let $\mathcal{D}\colon Y \to 2^Y$ be a set-valued map with $\mathcal{D}(y)$ a nontrivial closed convex cone for all $y \in Y$. An element $\bar{y} \in A$ is a strongly minimal element of the set A if and only if for every $l \in \mathcal{D}(\bar{y})^*$*
$$l(\bar{y}) \le l(y) \text{ for all } y \in A,$$
i. e. $\bar{y}$ is a minimal solution of the scalarized problem
$$\min_{y \in A} l(y).$$

Proof. Since $\mathcal{D}(\bar{y})$ is closed and convex we have according to [15, Lemma 3.21,(a)]
$$\mathcal{D}(\bar{y}) = \{y \in Y \mid l(y) \ge 0 \text{ for all } l \in \mathcal{D}(\bar{y})^*\}.$$
$\bar{y} \in A$ is a strongly minimal element of A if and only if $A - \{\bar{y}\} \subset \mathcal{D}(\bar{y})$, i.e. if and only if
$$A - \{\bar{y}\} \subset \{y \in Y \mid l(y) \ge 0 \text{ for all } l \in \mathcal{D}(\bar{y})^*\},$$
or if and only if for all $y \in A$
$$l(y - \bar{y}) \ge 0 \text{ for all } l \in D(\bar{y})^*.$$

□

3 Linear scalarization and existence results

In this section we continue the consideration of the scalarization
$$\min_{y \in A} l(y)$$
for linear functionals $l \in Y'$ to present necessary as well as sufficient conditions for minimal and nondominated elements and to conclude existence results.

We start with a necessary condition for a minimal element:

Lemma 3.1. *Let the Assumption 2.1 be satisfied and additionally let A be convex. Then for any weakly minimal element $\bar{y} \in A$ of the set A w.r.t. the ordering map $\mathcal{D}$, with $D(\bar{y})$ convex, there exists a linear functional $l \in \mathcal{D}(\bar{y})' \setminus \{0_{Y'}\}$ with*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A.$$

Here, $\mathcal{D}(\bar{y})'$ denotes the dual cone of $\mathcal{D}(\bar{y})$ in the algebraic dual space.

Proof. Since $\bar{y}$ is a weakly minimal element of A the intersection of the sets $\{\bar{y}\} - \operatorname{cor}(\mathcal{D}(\bar{y}))$ and A is empty. The set $\operatorname{cor}(\mathcal{D}(\bar{y}))$ is convex as $\mathcal{D}(\bar{y})$ is convex, see [15, Lemma 1.9]. Applying a separation theorem ([15, Theorem 3.14]) there exists a linear function $l \in Y' \setminus \{0_{Y'}\}$ and a real number α with

$$l(\bar{y} - d) \leq \alpha \leq l(y) \text{ for all } d \in \mathcal{D}(\bar{y}) \text{ and for all } y \in A.$$

As $\mathcal{D}(\bar{y})$ is a cone, we conclude $l(d) \geq 0$ for all $d \in \mathcal{D}(\bar{y})$ and thus $l \in \mathcal{D}(\bar{y})' \setminus \{0_{Y'}\}$, and due to $0_Y \in D(\bar{y})$ we obtain $l(\bar{y}) \leq l(y)$ for all $y \in A$. □

As all minimal elements are also weakly minimal elements, this result holds also for minimal elements.

We proceed with two sufficient conditions for minimal elements. We need the image set of $\mathcal{D}$,

$$\bar{D} := \mathcal{D}(A) = \bigcup_{y \in A} \mathcal{D}(y), \tag{18}$$

see also (17), and the algebraic dual of this cone $\bar{D}' := \{l \in Y' \mid l(y) \geq 0 \ \forall y \in \bar{D}\} = \bigcap_{y \in A} \mathcal{D}(y)'$. Note that $\bar{D}$ is not necessarily a convex cone and that in many cases $\bar{D}' = \{0_{Y'}\}$.

For $l \in \bar{D}'$, l is a monotonically increasing function, i.e., $\bar{y} - y \in \bar{D}$ implies $l(\bar{y}) \geq l(y)$ for all $y, \bar{y} \in Y$. For $l \in \bar{D}' \setminus \{0_{Y'}\}$ and $\bar{D}$ a convex cone with $\operatorname{cor}(\bar{D}) \neq \emptyset$, l is a strictly monotonically increasing function, i.e., $\bar{y} - y \in \operatorname{cor}(\bar{D})$ implies $l(\bar{y}) > l(y)$ for all $y, \bar{y} \in Y$, see Lemma 3.21(b) in [15]. Let $\bar{D}^{\#}_{Y'} := \{l \in Y' \mid l(y) > 0 \ \forall y \in \bar{D} \setminus \{0_Y\}\}$ denote the quasi-interior of the dual cone. Then any $l \in \bar{D}^{\#}_{Y'}$ is a strongly monotonically increasing function, i.e. $\bar{y} - y \in \bar{D} \setminus \{0_Y\}$ implies $l(\bar{y}) > l(y)$ for all $y, \bar{y} \in Y$ (see also [15, Example 5.2]). For the definition of monotonically increasing, strictly monotonically and strongly monotonically increasing functions see for instance [15, Def. 5.1], and for scalarization results w.r.t. non-variable dominance sets D and monotone functions see [22].

Lemma 3.2. *Let the Assumption 2.1 be satisfied.*

(a) *If for some $l \in \bar{D}'$ and some $\bar{y} \in A$ we have*

$$l(\bar{y}) < l(y) \text{ for all } y \in A \setminus \{\bar{y}\},$$

then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.

(b) *Let $\bar{D}$ be convex. If for some $l \in \bar{D}' \setminus \{0_{Y'}\}$ and some $\bar{y} \in A$ we have*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A,$$

then $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathcal{D}$.

Proof. (a) Let $l \in \bar{D}'$ be arbitrarily chosen. Then l is a monotonically increasing function, and thus $\bar{y} - y \in \mathcal{D}(\bar{y}) \subset \bar{D}$ implies $l(\bar{y}) \geq l(y)$ for all $\bar{y}, y \in A$. Assume that $\bar{y}$ is not a minimal element. Then there exists some $y \in A$ with $\bar{y} - y \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$ and thus $l(\bar{y}) \geq l(y)$, which is a contradiction.

(b) Let $l \in \bar{D}' \setminus \{0_{Y'}\}$ be arbitrarily chosen. Then l is a strictly monotonically increasing function, and thus $\bar{y} - y \in \operatorname{cor}(\mathcal{D}(\bar{y}))$ implies $l(\bar{y}) > l(y)$ for all $\bar{y}, y \in A$. Now assume that $\bar{y}$ is not a weakly minimal element. Then there exists some $y \in A$ with $\bar{y} - y \in \operatorname{cor}(\mathcal{D}(\bar{y}))$ and thus $l(\bar{y}) > l(y)$, which is a contradiction.

□

Remark 3.3. *Note that for the statement in Lemma 3.2,(a) it suffices that $l \in \mathcal{D}(\bar{y})'$ and for the statement in (b) it suffices that $\mathcal{D}(\bar{y})$ is convex and $l \in \mathcal{D}(\bar{y})' \setminus \{0_{Y'}\}$.*

Lemma 3.4. *Let the Assumption 2.1 be satisfied. If for some $l \in \bar{D}^{\#}_{Y'}$ and some $\bar{y} \in A$ we have*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A,$$

then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.

Proof. Any $l \in \bar{D}^{\#}_{Y'}$ is a strongly monotonically increasing function, and thus $\bar{y} - y \in \mathcal{D}(\bar{y})$, $y \neq \bar{y}$, implies $l(\bar{y}) > l(y)$ for all $\bar{y}, y \in A$. Now assume that $\bar{y}$ is not a minimal element. Then there exists some $y \in A$ with $\bar{y} - y \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$ and thus $l(\bar{y}) > l(y)$, which is a contradiction. □

Note, that in Lemma 3.4 it suffices that $l \in \mathcal{D}(\bar{y})^{\#}_{Y'}$.

A necessary condition for $\bar{D}^{\#}_{Y'}$ to be nonempty, if $\bar{D}$ is a convex cone, is the pointedness of $\bar{D}$ ([15, Lemma 1.27]). According to the Krein-Rutmann-Theorem (see for instance [15, Theorem 3.38]), in a real separable normed space Y with $\bar{D}$ closed convex and pointed, the quasi interior of the topological dual cone $\bar{D}^{\#}_{Y^*} := \{l \in Y^* \mid l(y) > 0 \ \forall y \in \bar{D} \setminus \{0_Y\}\}$ is nonempty.

For $\bar{D}^{\#}_{Y^*}$ nonempty we get the following existence result for minimal elements.

Theorem 3.5. *Let the Assumption 2.1 be satisfied and let Y be a real topological linear space. Let $\bar{D}^{\#}_{Y^*}$ be nonempty and let the set A be compact. Then there exists a minimal element of the set A w.r.t. the ordering map $\mathcal{D}$.*

Proof. Let $l \in \bar{D}^{\#}_{Y^*}$ be arbitrarily chosen. Then according to the Weierstraß theorem there exists a minimal solution of the optimization problem

$$\min_{y \in A} l(y).$$

According to Lemma 3.4 this minimal solution is then a minimal element of the set A w.r.t. $\mathcal{D}$. □

In the following we study linear scalarizations for nondominated elements w.r.t. the ordering map $\mathcal{D}$. We start with a necessary condition.

Lemma 3.6. *Let the Assumption 2.1 be satisfied and additionally let A be convex, let*

$$\hat{D} := \bigcap_{y \in A} \mathcal{D}(y)$$

be convex and let $cor(\hat{D})$ be nonempty. Then for any weakly nondominated element $\bar{y} \in A$ of A w.r.t. the ordering map $\mathcal{D}$ there exists a linear functional $l \in \hat{D}' \setminus \{0_{Y'}\}$ with

$$l(\bar{y}) \leq l(y) \quad \textit{for all} \quad y \in A.$$

Proof. Since $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. the ordering map $\mathcal{D}$ we have

$$\bar{y} \notin \{y\} + \mathrm{cor}(\mathcal{D}(y)) \quad \text{for all} \quad y \in A$$

and thus $\bar{y} \notin \{y\} + \mathrm{cor}(\hat{D})$ for all $y \in A$. Then $(\{\bar{y}\} - \mathrm{cor}(\hat{D})) \cap A = \emptyset$ and with a separation theorem (see Proof of Lemma 3.1) this results in

$$l(\bar{y}) \leq l(y) \quad \text{for all} \quad y \in A$$

for some $l \in \hat{D}' \setminus \{0_{Y'}\}$. $\square$

If the cone $\hat{D}$ is convex and if $\mathrm{cor}(\hat{D}) \neq \emptyset$ then according to [15, Lemma 1.27] the dual cone $\hat{D}'$ is pointed.

Lemma 3.7. *Let the Assumption 2.1 be satisfied and let $\bar{D}$ be defined as in (18).*

(a) *If for some $l \in \bar{D}'$ and some $\bar{y} \in A$ we have*

$$l(\bar{y}) < l(y) \textit{ for all } y \in A \setminus \{\bar{y}\},$$

then $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

(b) *Let $\bar{D}$ be convex. If for some $l \in \bar{D}' \setminus \{0_{Y'}\}$ and some $\bar{y} \in A$ we have*

$$l(\bar{y}) \leq l(y) \textit{ for all } y \in A,$$

then $\bar{y}$ is a weakly nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

(c) *If for some $l \in \bar{D}^{\#}_{Y'}$ and some $\bar{y} \in A$ we have*

$$l(\bar{y}) \leq l(y) \textit{ for all } y \in A,$$

then $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

Proof. (a) Let $l \in \bar{D}'$ be arbitrarily chosen. Then l is a monotonically increasing function on A, i.e. $\bar{y} - y \in \bar{D}$ implies $l(\bar{y}) \geq l(y)$ for all $\bar{y}, y \in A$. Assume that $\bar{y}$ is not a nondominated element. Then there exists some $y \in A$ with $\bar{y} - y \in \mathcal{D}(y) \subset \bar{D}$ and thus $l(\bar{y}) \geq l(y)$, which is a contradiction.
(b) and (c) are proofed analogously. $\square$

We can also conclude an existence result for nondominated elements of a set.

Theorem 3.8. *Let the Assumption 2.1 be satisfied and let Y be a real topological linear space. Let $\bar{D}^{\#}_{Y^*}$ be nonempty and let the set A be compact. Then there exists a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$.*

Proof. Let $l \in \bar{D}^{\#}_{Y^*}$ be arbitrarily chosen. Then according to the Weierstraß theorem there exists a minimal solution of the optimization problem

$$\min_{y \in A} l(y).$$

According to Lemma 3.7,(c) the minimal solution is then a nondominated element of the set A. □

Comparing Lemma 3.2 and Lemma 3.4 with Lemma 3.7 it turns out that the same sufficient conditions guarantee (weakly) minimal and nondominated elements w.r.t. $\mathcal{D}$ at the same time. These sufficient conditions are very strong, as they demand $\bar{D}' \neq \{0_{Y'}\}$. Note, that for the sufficient conditions for (weakly) minimal elements (Lemma 3.2 and Lemma 3.4) the assumptions can be weakened to the consideration of $\mathcal{D}(\bar{y})'$ instead of $\bar{D}'$, see Remark 3.3.

4 Optimality conditions

In this section we consider vector optimization problems with the image space equipped with a variable ordering structure introduced by $\mathcal{D}$. We have the following assumptions.

Assumption 4.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and $(Y, \|\cdot\|_Y)$ a normed space. Let Y be equipped with a variable ordering structure defined by a cone-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a convex cone with nonempty topological interior int$(\mathcal{D}(y))$ for all $y \in Y$. Let $f\colon X \to Y$ be a given map and $S \subset X$ a nonempty set.*

Under this assumption we consider the vector optimization problem

$$\min_{x \in S} f(x). \tag{19}$$

Then an element $\bar{x} \in S$ is called a *minimal* or a *nondominated solution* of the problem (19), if $f(\bar{x})$ is a minimal or a nondominated element of the image set $f(S)$ respectively. The same for the notions of weakly minimal/nondominated solutions and (weakly) maximal/max-nondominated solutions.

In [13] minimality of (19) is defined based on a family of closed, pointed convex cones $\{\mathcal{P}(x) \mid x \in X\}$ in Y. Then $\bar{x} \in S$ is called a minimal solution of (19) if there is no $x \in S$ such that $f(\bar{x}) \in f(x) + \mathcal{P}(\bar{x}) \setminus \{0_Y\}$. This fits in the theory presented here by setting $\mathcal{P}(x) := \mathcal{D}(f(x))$ for all $x \in X$. For the case of f a bijective map we can also define $\mathcal{D}(y) := \mathcal{P}(f^{-1}(y))$ for all $y \in Y$, see also [19, Remark 2.3.3].

In addition to the necessary and sufficient conditions in the preceding section using linear scalarizations we present a generalized Lagrange multiplier rule for the optimization problem (19) in this section. We restrict us thereby to the notion of (weakly) minimal solutions.

4.1 Necessary condition

Theorem 4.2. *Let the Assumption 4.1 be satisfied. Let $(Z_2, \|\cdot\|_{Z_2})$ be a real Banach space and let $(Z_1, \|\cdot\|_{Z_1})$ be a partially ordered normed space with the ordering cone C_{Z_1} with nonempty topological interior. Let $\hat{S}$ be a nonempty convex subset of X which has a nonempty interior. Let $g\colon X \to Z_1$ and $h\colon X \to Z_2$ be given maps defining the constraint set*

$$S := \{x \in \hat{S} \mid g(x) \in -C_{Z_1},\ h(x) = 0_{Z_2}\}$$

which is assumed to be nonempty. Let $\bar{x}$ be a weakly minimal solution of the vector optimization problem (19) w.r.t. the ordering map $\mathcal{D}$. Let f and g be Fréchet differentiable at $\bar{x}$ and let h be continuously Fréchet differentiable at $\bar{x}$. Let $h'(\bar{x})(X)$ be closed.

Then there are continuous linear functions $t \in \mathcal{D}(f(\bar{x}))^*$, $u \in C_{Z_1}^*$, $v \in Z_2^*$ with $(t, u, v) \neq 0_{Y^* \times Z_1^* \times Z_2^*}$ so that

$$(t \circ f'(\bar{x}) + u \circ g'(\bar{x}) + v \circ h'(\bar{x}))(x - \bar{x}) \geq 0 \textit{ for all } x \in \hat{S}$$

and

$$(u \circ g)(\bar{x}) = 0.$$

If additionally there is some $\hat{x} \in int(\hat{S})$ with $g(\bar{x}) + g'(\bar{x})(\hat{x} - \bar{x}) \in -int(C_{Z_1})$ and $h'(\bar{x})(\hat{x} - \bar{x}) = 0_{Z_2}$ and if the map $h'(\bar{x})$ is surjective, then $t \neq 0_{Y^*}$.

Proof. Using Lemma 2.16 any weakly minimal solution of the problem (19) w.r.t. the ordering map $\mathcal{D}$ is also a weakly minimal solution of (19) w.r.t. the convex cone $K := \mathcal{D}(f(\bar{x}))$ with nonempty topological interior $\mathrm{int}(K)$. The standard necessary optimality condition for vector optimization problems with a non-variable ordering structure (see [15, Theorem 7.4]) delivers then the given multiplier rule. □

As all minimal elements are also weakly minimal elements, this is also a necessary condition for minimal solutions of the problem (19).

4.2 Sufficient condition

For a sufficient condition we need convexity assumptions. Thereby some kind of quasiconvexity is sufficient. We first recall the definition of differentiably K-quasiconvex maps [15].

Definition 4.3. *Let S be a nonempty subset of a real linear space X and let K be a nonempty subset of a real linear space Y. Let $\bar{x} \in S$ and let a map $f\colon S \to Y$ have a directional variation at $\bar{x}$ w.r.t. some superset of K. Then f is called* differentiably K-quasiconvex *at $\bar{x}$ if the following holds: whenever there is some $x \in S \setminus \{\bar{x}\}$ with $f(x) - f(\bar{x}) \in K$, then there exists some $\tilde{x} \in S \setminus \{\bar{x}\}$ with $\lambda\tilde{x} + (1 - \lambda)\bar{x} \in S$ for all $\lambda \in (0, 1]$ and $f'(\bar{x})(\tilde{x} - \bar{x}) \in K$.*

Note, that a Fréchet differentiable map $f\colon X \to Y$ (with $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ normed spaces) has a directional variation w.r.t. any open set in Y.

Definition 4.4. *Let S be a nonempty convex subset of a real linear space X and let Y be a linear space with a convex cone $K \subset Y$. A map $f\colon S \to Y$ is called K-convex, if for all $x, y \in S$ and $\lambda \in [0,1]$*

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \in K.$$

K-convex maps ca be characterized using the Fréchet derivative, see [15, Theorem 2.20]:

Lemma 4.5. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let S be a nonempty open convex subset of X, let $K \subset Y$ be a closed convex cone and let $f\colon S \to Y$ be Fréchet differentiable at every $x \in S$. Then f is K-convex if and only if*

$$f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x}) \in K \textit{ for all } x, \bar{x} \in S.$$

We obtain the following sufficient optimality condition.

Theorem 4.6. *Let the Assumption 4.1 be satisfied. Let $\hat{S}$ be a nonempty subset of X, let $(Z_1, \|\cdot\|_{Z_1})$ and $(Z_2, \|\cdot\|_{Z_2})$ be partially ordered real normed spaces with ordering cones C_{Z_1} and C_{Z_2} respectively with the convex cone C_{Z_2} pointed. Let $g\colon \hat{S} \to Z_1,\ h\colon \hat{S} \to Z_2$ be given maps defining the constraint set*

$$S := \{x \in \hat{S} \mid g(x) \in -C_{Z_1},\ h(x) = 0_{Z_2}\}$$

which is assumed to be nonempty. Let f, g and h be Fréchet differentiable on X.

Let (g, h) be differentiably C-quasiconvex at some $\bar{x} \in S$ with

$$C := (-C_{Z_1} + \mathit{cone}(\{g(\bar{x})\}) - \mathit{cone}(\{g(\bar{x})\})) \times \{0_{Z_2}\}$$

and let f be differentiably $-\mathit{cor}(\mathcal{D}(f(\bar{x})))$-quasiconvex at $\bar{x}$. Here $\mathit{cone}(\{g(\bar{x})\})$ denotes the cone generated by the set $\{g(\bar{x})\}$.

Assume that for some $\bar{x} \in S$ there exists some

$$t \in \mathcal{D}(f(\bar{x}))' \setminus \{0_{Y'}\},\ u \in C'_{Z_1} \textit{ and } v \in Z'_2$$

with

$$(t \circ f'(\bar{x}) + u \circ g'(\bar{x}) + v \circ h'(\bar{x}))(x - \bar{x}) \geq 0 \textit{ for all } x \in \hat{S}$$

and

$$(u \circ g)(\bar{x}) = 0.$$

Then $\bar{x}$ is a weakly minimal solution of (19) w.r.t. the ordering map $\mathcal{D}$.

Proof. According to the assumptions the composite map (f, g, h) is differentiably $\tilde{C}$-quasiconvex in $\bar{x} \in S$ for $\tilde{C} := (-\mathrm{cor}(K)) \times C$ with the convex cone $K := \mathcal{D}(f(\bar{x}))$. Then, according to [15, Corollar 7.21] $\bar{x}$ is a weakly minimal solution of the problem (19) w.r.t. the cone $K = \mathcal{D}(f(\bar{x}))$. Because $\mathcal{D}(f(\bar{x}))$ is a convex cone with nonempty topological interior $\mathrm{int}(\mathcal{D}(f(\bar{x})))$ we get $\mathrm{cor}(K) = \mathrm{int}(\mathcal{D}(f(\bar{x}))) \neq \emptyset$ ([15, Lemma 1.32]). According to Corollary 2.15,(b) $\bar{x}$ is also a weakly minimal solution w.r.t. the ordering map $\mathcal{D}$. $\square$

The assumptions of Theorem 4.6 can be weakened by only assuming X, Y, Z_1 and Z_2 to be linear spaces and f, g and h having directional variations w.r.t. special sets instead of being Fréchet differentiable (see [15, Theorem 7.20]).

Note that for $C_{Z_1} = \mathbb{R}^k_+$ and in $\bar{x}$ partial differentiable functions $g\colon \mathbb{R}^n \to \mathbb{R}^k$ and $h\colon \mathbb{R}^n \to \mathbb{R}^p$ with $h_1, \ldots, h_p, -h_1, \ldots, h_p$, g_i, $i \in I(\bar{x}) := \{i \in \{1, \ldots, k\} \mid g_i(\bar{x}) = 0\}$ quasiconvex, the composite function (g, h) is differentiably C-quasiconvex. We also examine the property of f being differentiably $-\mathrm{cor}(\mathcal{D}(f(\bar{x})))$-quasiconvex in more detail.

Lemma 4.7. *Let the Assumption 4.1 together with the assumptions of Lemma 4.5 be satisfied with $K := \mathcal{D}(f(\bar{x}))$ for some $\bar{x} \in S$. Let f be $\mathcal{D}(f(\bar{x}))$-convex on X. Then f is also differentiably $-cor(\mathcal{D}(f(\bar{x})))$-quasiconvex at $\bar{x}$.*

Proof. According to Lemma 4.5 it holds for any $x \in S$

$$f'(\bar{x})(x - \bar{x}) \in f(x) - f(\bar{x}) - \mathcal{D}(f(\bar{x})).$$

Thus for any $x \in S \setminus \{\bar{x}\}$ with $f(x) - f(\bar{x}) \in -\mathrm{cor}(\mathcal{D}(f(\bar{x})))$ we conclude with [15, Lemma 1.12,(b)])

$$f'(\bar{x})(x - \bar{x}) \in -\mathrm{cor}(\mathcal{D}(f(\bar{x}))) - \mathcal{D}(f(\bar{x})) = -\mathrm{cor}(\mathcal{D}(f(\bar{x})))$$

and hence f is differentiably $-\mathrm{cor}(\mathcal{D}(f(\bar{x})))$-quasiconvex. $\square$

Remark 4.8. *If f is K-convex for*

$$K \subset \bigcap_{x \in S} \mathcal{D}(f(x)),$$

then $f\colon S \to Y$ is $\mathcal{D}(f(x))$-convex for any $x \in S$.

In [10] Engau considers only variable preferences in $\mathbb{R}^m$ with $\mathbb{R}^m_+ \subset \mathcal{D}(y)$ for all $y \in \mathbb{R}^m$. Then $\mathbb{R}^m_+$-convexity of $f = (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ implies that f is $\mathcal{D}(f(x))$-convex for any $x \in S$. Thereby, it is equivalent to f being $\mathbb{R}^m_+$-convex that

$$\lambda f_i(x) + (1-\lambda) f_i(y) \geq f_i(\lambda x + (1-\lambda) y) \text{ for all } x, y \in \mathbb{R}^n,\ \lambda \in [0,1],\ i = 1, \ldots, m,$$

i.e., that the functions $f_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are convex.

5 Duality

It is already known that, under appropriate assumptions, a maximization problem can be associated to a minimization problem also in vector optimization [15]. We start by considering duality concepts in a very general way for sets, and in the subsequent section we consider duality results for vector optimization problems.

5.1 General duality

Let again $\bar{D} := \mathcal{D}(A)$ and define the set $P := A + \bar{D}$. In the following we associate the dual set

$$T := Y \setminus \tilde{P} \;\text{ with }\; \tilde{P} := A + (\bar{D} \setminus \{0_Y\}) \subset P$$

to the primal set A. In [12] the following basic condition for a dual set $H \subset Y$, called *weak duality*, is defined for an ordering cone $K \subset Y$:

$$A \cap (H - (K \setminus \{0_Y\})) = \emptyset.$$

This condition, which is here equivalent to $(A + K \setminus \{0_Y\}) \cap H = \emptyset$, is called *dominance requirement*. For the case of a variable ordering structure this condition can be formulated as

$$A \cap (\{y\} - (\mathcal{D}(y) \setminus \{0_Y\})) = \emptyset \qquad \forall y \in H \tag{20}$$

or

$$(\{y\} + (\mathcal{D}(y) \setminus \{0_Y\})) \cap H = \emptyset \qquad \forall y \in A. \tag{21}$$

For the dual set T both weak duality conditions are satisfied.

Lemma 5.1. *Let the Assumption 2.1 be satisfied. For the dual set $H := T$ the weak duality (20) and (21) are satisfied.*

Proof. First we assume that there exists $\bar{y} \in A$ with $\bar{y} \in \{y\} - (\mathcal{D}(y) \setminus \{0_Y\})$ for some $y \in T$. Then

$$y \in \{\bar{y}\} + (\mathcal{D}(y) \setminus \{0_Y\}) \subset A + (\bar{D} \setminus \{0_y\}) = \tilde{P}$$

in contradiction to $y \in T = Y \setminus \tilde{P}$. Thus (20) is satisfied.

Now assume that there exists $\bar{y} \in A$ with $(\{\bar{y}\} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\})) \cap T \neq \emptyset$. But $(\{\bar{y}\} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\})) \subset A + \bar{D} \setminus \{0_Y\} = \tilde{P}$ which is a contradiction. Thus also (21) is satisfied. $\square$

In the following we discuss under which assumptions a minimal element of A is also a maximal element of the dual set T w.r.t. the ordering map $\mathcal{D}$ and vice versa.

Lemma 5.2. *Let the Assumption 2.1 be satisfied. If $\bar{y} \in A \cap T$, then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and $\bar{y}$ is a maximal element of T w.r.t. the ordering map $\mathcal{D}$.*

Proof. As $\bar{y} \notin \tilde{P}$ we conclude

$$\begin{array}{rrcl} & (\{\bar{y}\} - (\bar{D} \setminus \{0_Y\})) \cap A & = & \emptyset \\ \Rightarrow & (\{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\})) \cap A & = & \emptyset \\ \Rightarrow & (\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A & = & \{\bar{y}\} \end{array}$$

and thus $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$. Since $\emptyset = T \cap \tilde{P} = T \cap (A + \bar{D} \setminus \{0_Y\})$ we conclude $T \cap (\{\bar{y}\} + \bar{D} \setminus \{0_Y\}) = \emptyset$ and hence $T \cap (\{\bar{y}\} + \mathcal{D}(\bar{y})) = \{\bar{y}\}$. Thus $\bar{y}$ is a maximal element of the set T w.r.t. the ordering map $\mathcal{D}$. $\square$

Lemma 5.3. *Under the assumptions of Lemma 5.2, if $\bar{y} \in A \cap T$, then $\bar{y}$ is also a nondominated element of A w.r.t. $\mathcal{D}$. If additionally $\mathcal{D}(y) \subset \bar{D}$ for all $y \in Y$, then $\bar{y}$ is also a max-nondominated element of T w.r.t. the ordering map $\mathcal{D}$.*

Proof. As $\bar{y} \in T$ we conclude $\bar{y} \notin \tilde{P}$, i.e. $\{\bar{y}\} \notin \{y\} + (\bar{D} \setminus \{0_Y\})$ for all $y \in A$, hence $\{\bar{y}\} \notin \{y\} + (\mathcal{D}(y) \setminus \{0_Y\})$ for all $y \in A$ and thus $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

Now assume $\bar{y} \in A$ is not a max-nondominated element. Then there exists $y \in T = Y \setminus \tilde{P}$ with $\bar{y} \in \{y\} - (\mathcal{D}(y) \setminus \{0_Y\})$. Because $\mathcal{D}(y) \subset \bar{D}$ for all $y \in Y$ we get

$$y \in \{\bar{y}\} + (\mathcal{D}(y) \setminus \{0_Y\}) \subset A + (\bar{D} \setminus \{0_Y\}) = \tilde{P}$$

in contradiction to $y \in Y \setminus \tilde{P}$. □

We get the following converse duality theorem:

Theorem 5.4. *Let the Assumption 2.1 be satisfied. If $Y \setminus P$ is algebraically open then for every maximal element $\bar{y}$ of the set T w.r.t. the ordering map $\mathcal{D}$ it is $\bar{y} \in A \cap T$ and thus $\bar{y}$ is also a minimal element of the set A w.r.t. the ordering map $\mathcal{D}$.*

Proof. Let $\bar{y}$ be an arbitrary maximal element of T w.r.t. $\mathcal{D}$, i.e.

$$(\{\bar{y}\} + \mathcal{D}(\bar{y})) \cap T = \{\bar{y}\}.$$

We assume $\bar{y} \notin P$. As $Y \setminus P$ is algebraically open, there exists for every $d \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$ a scalar $\lambda > 0$ with $\bar{y} + \lambda\, d \in Y \setminus P \subset T$. As $\bar{y} + \lambda\, d \in \{\bar{y}\} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\})$ this is a contradiction to the maximality of $\bar{y}$. Thus $\bar{y} \in P = A + \bar{D}$ and due to $\bar{y} \notin \tilde{P} = A + (\bar{D} \setminus \{0_Y\})$ we conclude $\bar{y} \in A$. Together with Lemma 5.2 we get that $\bar{y}$ is also a minimal element of A. □

Note, that under the assumptions of Theorem 5.4 together with the assumptions of Lemma 5.3 $\bar{y}$ is also a nondominated element of A w.r.t. $\mathcal{D}$ and a max-nondominated element of T w.r.t. the ordering map $\mathcal{D}$.

As stated in [15, p. 191] the set $Y \setminus P$ is for instance algebraically open, if the set P is convex and algebraically closed. But in general we need no convexity assumptions in Theorem 5.4.

However, a standard non-converse duality theorem can only be stated under very strong assumptions, as generally a minimal element of the set A need not to be included in the set T and hence cannot be a maximal element of the set T:

Example 5.5. *We consider the set $A = [1,3] \times [1,3]$ and the ordering map*

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for all } y \in \mathbb{R}^2 \text{ with } y_1 > 1, \\ \{z \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0,\ z_1 - z_2 \geq 0\} & \text{else.} \end{cases}$$

The set of minimal elements of A w.r.t. this ordering map is $\{y \in A \mid y_1 = 1\}$ while the dual set is given by

$$T = \{y \in \mathbb{R}^2 \mid y_1 < 1\} \cup \{y \in \mathbb{R}^2 \mid y_1 \geq 1,\ y_1 + y_2 < 2\} \cup \{(1,1)\}.$$

The point $(1,1)$ is maximal for T and minimal for A w.r.t. $\mathcal{D}$, but all the other minimal elements of A are not included in T.

Theorem 5.6. *Let the Assumption 2.1 be satisfied and let $\bar{y} \in A$ be a minimal element of A w.r.t. the ordering map $\mathcal{D}$. Then $\bar{y}$ is also a maximal element of T w.r.t. the ordering map $\mathcal{D}$ if and only if it is also a minimal element of the set A w.r.t. the cone $K = \bar{D}$.*

Proof. Let $\bar{y} \in A$ be a minimal element of the set A w.r.t. the cone $K = \bar{D}$. If $\bar{y} \notin T$ then $\bar{y} \in A + \bar{D} \setminus \{0_Y\}$, i.e. $(\bar{y} - \bar{D}) \cap A \neq \{\bar{y}\}$, which contradicts the minimality of A w.r.t. the cone $K = \bar{D}$. Thus $\bar{y} \in A \cap T$ and the assertion follows with Lemma 5.2.

Now assume $\bar{y} \in A$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ but not w.r.t. the cone $K = \bar{D}$. Then there exists some $y \in A$ with $\bar{y} \in \{y\} + (\bar{D} \setminus \{0_Y\})$ and thus $\bar{y} \in \tilde{P}$ and $\bar{y} \notin T$. This is a contradiction to $\bar{y}$ a maximal element of T w.r.t. $\mathcal{D}$. □

Remark 5.7. *Note, that according to Corollary 2.15, if $\bar{y}$ is a minimal element of A w.r.t. the cone $K = \bar{D}$, then it is also a nondominated element of A w.r.t. the ordering map $\mathcal{D}$. Thus it is a necessary condition for a minimal element $\bar{y}$ of A w.r.t. $\mathcal{D}$ to be a maximal element of T w.r.t. $\mathcal{D}$, that $\bar{y}$ is also a nondominated element of A w.r.t. $\mathcal{D}$.*

Hence, while considering the maximal elements of the dual set, only those minimal elements $\bar{y}$ of the set A w.r.t. $\mathcal{D}$ can be found which are also $\bar{D}$ minimal.

However, we can associate another dual set to the primal set A such that we get duality results as well as converse duality results for the nondominated elements of the primal set and the maximal elements of the dual set. Therefore we define the set

$$M := \bigcup_{y \in A} \{y\} + \mathcal{D}(y)$$

(see also Lemma 2.20) as well as

$$\tilde{M} := \bigcup_{y \in A} \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}) \quad \text{and} \quad Q := Y \setminus \tilde{M}.$$

We start with a result on weak duality.

Lemma 5.8. *Let the Assumption 2.1 be satisfied. For the dual set $H := Q$ the weak duality (21) is satisfied.*

Proof. For $\bar{y} \in A$ we conclude

$$\bar{y} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) \subset \tilde{M}.$$

As $Q = Y \setminus \tilde{M}$ we get $(\{y\} + (\mathcal{D}(y) \setminus \{0_Y\}) \cap Q = \emptyset$ for all $y \in A$. □

Lemma 5.9. *Let the Assumption 2.1 be satisfied. If $\bar{y} \in A \cap Q$, then $\bar{y}$ is a nondominated element of A and $\bar{y}$ is a maximal element of Q w.r.t. the ordering map $\mathcal{D}$.*

Proof. As $\bar{y} \in Q = Y \setminus \tilde{M}$ and hence $\bar{y} \notin \tilde{M}$ we immediately see that $\bar{y}$ is nondominated w.r.t. the ordering map $\mathcal{D}$ for the set A. Since $Q = Y \setminus \tilde{M}$ we have $Q \cap \tilde{M} = \emptyset$ and hence $Q \cap (\{\bar{y}\} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\})) = \emptyset$. Thus $\bar{y}$ is a maximal element of the set Q w.r.t. the ordering map $\mathcal{D}$. □

We can formulate the following duality theorem:

Theorem 5.10. *Let the Assumption 2.1 be satisfied. If $\bar{y} \in A$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$, then $\bar{y}$ is also a maximal element of Q w.r.t. the ordering map $\mathcal{D}$.*

Proof. Let $\bar{y} \in A$ be a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$ and assume $\bar{y} \notin Q$. Then $\bar{y} \in \tilde{M}$ which is a contradiction to $\bar{y}$ nondominated. Thus $\bar{y} \in A \cap Q$ and the assertion follows with Lemma 5.9. $\square$

Together with Lemma 2.20,(a)(i) we conclude:

Corollary 5.11. *Let the Assumption 2.1 be satisfied. If $\bar{y} \in M$ is a nondominated element of M w.r.t. the ordering map $\mathcal{D}$, then $\bar{y}$ is also a maximal element of Q w.r.t. the ordering map $\mathcal{D}$.*

The next theorem is the associated converse duality theorem.

Theorem 5.12. *Let the Assumption 2.1 be satisfied. If $Y \setminus M$ is algebraically open then every maximal element of the set Q w.r.t. the ordering map $\mathcal{D}$ is also a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$.*

Proof. Let $\bar{y}$ be an arbitrary maximal element of Q w.r.t. $\mathcal{D}$. With the same arguments as in the proof of Theorem 5.4 we derive $\bar{y} \in M$. Due to $\bar{y} \notin \tilde{M}$ we get $\bar{y} \in A$. Together with Lemma 5.9 we conclude that $\bar{y}$ is a nondominated element of A. $\square$

Example 5.13. *We consider again the set A and the ordering map $\mathcal{D}$ of Example 5.5. The above sets are in this example then*

$$\tilde{M} = \{y \in \mathbb{R}^2 \mid y_1 > 1,\ y_2 \geq 2 - y_1\}$$

and

$$M = \tilde{M} \cup \{y \in \mathbb{R}^2 \mid y_1 = 1,\ y_2 \in [1, 3]\}.$$

Thus

$$Q = \{y \in \mathbb{R}^2 \mid y_1 \leq 1\ \vee\ (y_1 > 1\ \wedge\ y_1 + y_2 < 2)\}.$$

The set of nondominated elements of A w.r.t. $\mathcal{D}$ is $\{y \in \mathbb{R}^2 \mid y_1 = 1,\ y_2 \in [1, 3]\}$ and the set of maximal elements of Q w.r.t. $\mathcal{D}$ is $\{y \in \mathbb{R}^2 \mid y_1 = 1,\ y_2 \geq 1\}$. Thus $Q \cap A$ equals the set of nondominated elements of A which is a strict subset of the set of maximal elements of Q. Thus not all maximal elements of Q w.r.t. $\mathcal{D}$ refer to a nondominated element of A w.r.t. $\mathcal{D}$. The set $Y \setminus M$ is not algebraically open.

5.2 Duality for vector optimization problems

In this section we consider vector optimization problems

$$\min_{x \in S} f(x)$$

under the following assumptions:

Assumption 5.14. *Let $\hat{S} \neq \emptyset$ be a convex subset of a real linear space X. Let the real topological linear space Y be equipped with a variable ordering structure defined by a cone-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a nontrivial convex cone with nonempty topological interior for all $y \in Y$. Let the real topological linear space Z be ordered by a convex cone C_Z. Let $g\colon \hat{S} \to Z$ be a C_Z-convex map, let $f\colon \hat{S} \to Y$ be a map, and let the constraint set*

$$S := \{x \in \hat{S} \mid g(x) \in -C_Z\}$$

be nonempty.

Under these assumptions the set S is convex. We associate a dual problem and we present a weak and a strong duality result. For the case of a non-variable ordering structure these examinations can be found in [15, Chapter 8]. We set

$$D_1 := \{y \in Y \quad | \quad \begin{array}{l} \exists t \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\} \text{ and } \exists u \in C_Z^* \text{ with} \\ (t \circ f + u \circ g)(x) \geq t(y) \text{ for all } x \in \hat{S}\}. \end{array} \tag{22}$$

We consider weakly max-nondominated elements of the set D_1 w.r.t. $\mathcal{D}$, i.e. elements $\bar{y} \in D_1$ such that there exists no $y \in D_1$ with

$$y \in \{\bar{y}\} + \operatorname{int}(\mathcal{D}(y)).$$

Note that for $\mathcal{D}(y)$ convex and $\operatorname{int}(\mathcal{D}(y)) \neq \emptyset$ we have $\operatorname{int}(\mathcal{D}(y)) = \operatorname{cor}(\mathcal{D}(y))$.

First we present a weak duality theorem based on scalarization.

Theorem 5.15. *Let the Assumption 5.14 be satisfied and consider the set D_1 in (22). Then for every $\bar{y} \in D_1$ there is some $t \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ with*

$$t(\bar{y}) \leq t(y) \textit{ for all } y \in f(S). \tag{23}$$

Proof. We choose $\bar{y} \in D_1$ arbitrarily. Then there exists $t \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ and $u \in C_Z^*$ with

$$(t \circ f + u \circ g)(x) \geq t(\bar{y}) \text{ for all } x \in \hat{S}$$

which implies due to $g(x) \in -C_Z$ for all $x \in S$

$$(t \circ f)(x) \geq t(\bar{y}) \text{ for all } x \in S.$$

□

If $\bar{y} \in f(S)$ and (23) is satisfied, then according to Remark 3.3 $\bar{y}$ is a weakly minimal element of $f(S)$ w.r.t. the ordering map $\mathcal{D}$.

If we define

$$\tilde{D}_1 := \{y \in Y \quad | \quad \begin{array}{l} \exists t \in \bar{D}^* \setminus \{0_{Y^*}\} \text{ and } \exists u \in C_Z^* \text{ with} \\ (t \circ f + u \circ g)(x) \geq t(y) \ \forall x \in \hat{S}\} \end{array} \tag{24}$$

we even get under the conditions of Theorem 5.15 that $t(y) \geq t(\bar{y})$ for all $y \in f(S) + \bar{D}$ and thus also especially for all $y \in \bigcup_{y \in A} f(y) + \mathcal{D}(y)$. However note, that the set $\bar{D}^* \setminus \{0_{Y^*}\}$ might be empty and thus also the set $\tilde{D}_1$.

Theorem 5.16. *Let the Assumption 5.14 be satisfied and consider the set $\tilde{D}_1$ in (24). Then for every $\bar{y} \in \tilde{D}_1$ there is some $t \in \bar{D}^* \setminus \{0_{Y^*}\}$ with*

$$t(\bar{y}) \leq t(y) \text{ for all } y \in f(S) + \bar{D}.$$

Proof. Following the proof of Theorem 5.15 we conclude that there exists some $t \in \bar{D}^* \setminus \{0_{Y^*}\}$ and some $u \in C_Z^*$ with

$$(t \circ f + u \circ g)(x) \geq t(\bar{y}) \text{ for all } x \in \hat{S}.$$

With $g(x) \in -C_Z$ for all $x \in S$ and with $t(d) \geq 0$ for all $d \in \bar{D}$ we get

$$t(d + f(x)) = t(d) + (t \circ f)(x) \geq t(\bar{y}) \text{ for all } x \in S, \ d \in \bar{D}.$$

□

According to Lemma 3.2,(b), if $\bar{y} \in f(S)$ and if $\bar{D}$ is convex, then $\bar{y}$ is a weakly minimal element of $f(S)$ w.r.t. the ordering map $\mathcal{D}$.

Theorem 5.17. *Let the Assumption 5.14 be satisfied. Let $\bar{y}$ be a weakly minimal element of $f(S)$, let f be $\mathcal{D}(\bar{y})$-convex and let $t \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ with*

$$t(\bar{y}) \leq t(y) \text{ for all } y \in f(S)$$

be given. Additionally let the scalar optimization problem

$$\inf_{x \in S} (t \circ f)(x) \tag{25}$$

be stable, i.e.

$$\inf_{x \in S} (t \circ f)(x) = \sup_{u \in C_Z^*} \inf_{x \in \hat{S}} (t \circ f + u \circ g)(x)$$

and the problem on the right hand side has at least one solution. Then $\bar{y}$ is also a weakly max-nondominated element of D_1.

Proof. Because $\bar{y} \in f(S)$ and $t(\bar{y}) \leq t(y)$ for all $y \in f(S)$ we have for $f(\bar{x}) := \bar{y}$:

$$t(f(\bar{x})) \leq t(f(x)) \text{ for all } x \in S.$$

As f is $\mathcal{D}(\bar{y})$-convex we have for arbitrary $x, y \in S$ and $\lambda \in [0, 1]$:

$$\lambda f(x) + (1 - \lambda) f(y) \in \{f(\lambda x + (1 - \lambda) y)\} + \mathcal{D}(\bar{y}).$$

This implies with $t \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$

$$(t \circ f)(\lambda x + (1 - \lambda) y) \leq \lambda (t \circ f)(x) + (1 - \lambda)(t \circ f)(y).$$

Thus the composite map $t \circ f$ is convex and $\bar{x}$ is a minimal solution of the convex optimization problem (25) which is assumed to be stable.

As (25) is stable there exists some $\bar{u} \in C_Z^*$ with

$$\inf_{x \in S} (t \circ f)(x) = \inf_{x \in \hat{S}} (t \circ f + \bar{u} \circ g)(x)$$

and thus

$$(t \circ f + \bar{u} \circ g)(\hat{x}) \geq \inf_{x \in S} (t \circ f)(x) = (t \circ f)(\bar{x})$$

for all $\hat{x} \in \hat{S}$, i.e. $\bar{y} \in D_1$ and hence $\bar{y} \in f(S) \cap D_1$.

We now show that there is no $d \in D_1$ so that $\bar{y} \in \{d\} - \mathrm{int}(\mathcal{D}(d)\}$, i.e. that $\bar{y}$ is a weakly max-nondominated element of D_1 w.r.t. $\mathcal{D}$. For that we first show that for any $a \in f(S)$ and any $d \in D_1$ we have $d - a \notin \mathrm{int}(\mathcal{D}(d))$. Otherwise, for $d - a \in \mathrm{int}(\mathcal{D}(d))$, we conclude $t(d-a) > 0$ for all $t \in \mathcal{D}(d)^* \setminus \{0_{Y^*}\}$ according to [15, Lemma 3.21(c)] and thus $t(d) > t(a)$ for all $t \in \mathcal{D}(d)^* \setminus \{0_{Y^*}\}$ in contradiction to Theorem 5.15. Thus, due to $\bar{y} \in f(S)$ we obtain $d - \bar{y} \notin \mathrm{int}(\mathcal{D}(d))$ for all $d \in D_1$ and hence there is no $d \in D_1$ so that $\bar{y} \in \{d\} - \mathrm{int}(\mathcal{D}(d))$. $\square$

Such a $t \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ with

$$t(\bar{y}) \leq t(y) \quad \text{for all } y \in f(S)$$

exists according to Lemma 3.1 for instance if $f(S)$ is convex.

6 Outlook

The applicability of variable ordering structures in practise and a more intensive study of the known applications is of special importance. Thereby also the study of domination sets instead of domination cones [2, 7, 24, 25] can be of interest. Subject to further research is also the examination of scalarizations (see for instance [6, 8, 9]) and the development of numerical methods for solving vector optimization problems with variable ordering structures.

References

[1] D. Baatar and M.M. Wiecek, "Advancing equitability in multiobjective programming", *Comput. Math. Appl.* 52 (2006) 225–234.

[2] K Bergstresser, A. Charnes and P.L. Yu, "Generalization of domination structures and nondominated solutions in multicriteria decision making", *J. Optim. Theory Appl.* 18 (1976) 3–13.

[3] J.M. Borwein, "The geometry of Pareto efficiency over cones", *Optimization* 11 (1980) 235–248.

[4] G.Y. Chen, "Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia Theorem", *J. Optim. Theory Appl.* 74 (1992) 445–456.

[5] G.Y. Chen and X.Q. Yang, "Characterizations of variable domination structures via nonlinear scalarization", *J. Optim. Theory Appl.* 112(1) (2002) 97–110.

[6] G.-Y. Chen, X. Huang and X. Yang, *Vector Optimization, Set-Valued and Variational Analysis* (Springer, Berlin, 2005).

[7] K.L. Chew, "Domination structures in abstract spaces", *Southeast Asian Bulletin of Mathematics*, Proc. first franco-southeast Asian Math. Conference (1979) 190–204.

[8] G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization* (Springer, Berlin, 2008).

[9] G. Eichfelder, "An adaptive scalarization method in multiobjective optimization", *SIAM Journal on Optimization* 19 (2009) 1694-1718.

[10] A. Engau, "Variable preference modeling with ideal-symmetric convex cones", *J. Global Optim.* 42 (2008) 295-311.

[11] B. Fischer, E. Haber and J. Modersitzki, "Mathematics meets medicine - an optimal alignment", *SIAG/OPT Views and News* 19(2) (2008) 1-7.

[12] C. Gerstewitz (Tammer), "Nichtkonvexe Dualität in der Vektoroptimierung", *Wissenschaftl. Zeitschrift TH Leuna-Merseburg* 25 (1983) 357–364.

[13] N.J. Huang, X.Q. Yang and W.K. Chan, "Vector complementarity problems with a variable ordering relation", *European Journal of Operational Research* 176 (2007) 15–26.

[14] N.J. Huang, A.M. Rubinov, X.Q. Yang, "Vector optimization problems with nonconvex preferences", *J. Global Optim.* 40 (2008) 765–777.

[15] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions* (Springer, Berlin, 2004).

[16] E.K. Karaskal and W. Michalowski, "Incorporating wealth information into a multiple criteria decision making model", *European Journal of Operational Research* 150 (2003) 204–219.

[17] S.J. Li and M.H. Li, "Levitin-Polyak well-posedness of vector equilibrium problems", *Math. Meth. Oper. Res.* 69 (2009) 125-140.

[18] A.M. Rubinov and R.N. Gasimov, "Scalarization and nonlinear scalar duality for vector optimization with preferences that are not necessarily a pre-order relation", *J. Global Optim.* 29 (2004) 455–477.

[19] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization* (Academic Press, London, 1985).

[20] M. Wacker, *Multikriterielle Optimierung bei Registrierung medizinischer Daten* (Diplomarbeit, Univ. Erlangen-Nürnberg, 2008).

[21] P. Weidner, "Dominanzmengen und Optimalitätsbegriffe in der Vektoroptimierung", *Wissenschatl. Zeitschrift Ilmenau* 31 (1985) 133-146.

[22] P. Weidner, "On the characterization of efficient points by means of monotone functionals", *Optimization* 19 (1988) 53-69.

[23] P. Weidner, "Problems in models and methods of vector optimization", *Wissenschatl. Schriftenreihe der TU Karl-Marx-Stadt* 5 (1989) 47-57.

[24] P. Weidner, *Ein Trennungskonzept und seine Anwendung auf Vektoroptimierungsverfahren* (Habilitation, Martin-Luther-Universität Halle-Wittenberg, Halle, 1990).

[25] P. Weidner, "Problems in scalarizing multicriteria approaches", in: Köksalen, M., Zionts, S. (eds.), *Multiple Criteria Decision Making in the New Millenium*, (Springer, 2001) 199-209.

[26] M.M. Wiecek, "Advances in cone-based preference modeling for decision making with multiple criteria", *Decision Making in Manufacturing and Services* 1 (2007) 153–173.

[27] P.L. Yu, "Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives", *J. Optim. Theory Appl.* 14 (1974) 319–377.

[28] P.L. Yu, *Multiple-criteria Decision Making: Concepts, Techniques and Extensions* (Plenum Press, New York, 1985).

PREPRINTS

DES INSTITUTS FÜR ANGEWANDTE MATHEMATIK
DER UNIVERSITÄT ERLANGEN-NÜRNBERG

ZULETZT ERSCHIENENE BEITRÄGE:

304 Michael Kočvara , Michael Stingl: *On the solution of large-scale SDP problems by the modified barrier method using iterative solvers*

305 Kathrin Klamroth ,Jørgen Tind: *Constrained Optimization Using Multiple Objective Programming*

306 S. Kräutle , P. Knabner: *A Reduction Scheme for Coupled Multicomponent Transport-Reaction Problems in Porous Media: Generalization to problems with heterogeneous equilibrium reactions*

307 G. Leugering , K. Klamroth , O. V. Museyko , M. Stiglmayr: *Image Registration: Several Approaches Involving Segmentation*

308 G. Eichfelder: *An Adaptive Scalarization Method in Multi-Objective Optimization*

309 W. Alt , N. Bräutigam: *Discretization of Elliptic Control Problems with Time Dependent Parameters*

310 J. Gorski , K. Klamroth , S. Ruzika: *Connectedness of Efficient Solutions in Multiple Objective Combinatorial Optimization*

311 G. Eichfelder: *Scalarizations For Adaptively Solving Multi-Objective Optimization Problems*

312 G. Eichfelder: *Multiobjective Bilevel Optimization*

313 M. Bischoff, K. Klamroth: *Two Branch & Bound Methods for a Generalized Class of Location-Allocation Problems*

314 M. Bischoff , K. Dächert: *Allocation Search Methods for a Generalized Class of Location-Allocation Problems*

315 P. Knabner: *Open Questions and Research Directions in Parameter Identification for Multicomponent Reactive Transport in Porous Media*

316 G. Eichfelder , J. Jahn: *Set-Semidefinite Optimization*

317 M. Stingl , M. Kočvara , G. Leugering: *A Sequential Convex Semidefinite Programming Algorithm for Multiple-Load Free Material Optimization*

318 J. Jahn , E. Schaller: *A Global Solver for Multiobjective Nonlinear Bilevel Optimization Problems*

319 M. Stingl , M. Kočvara , G. Leugering: *Free Material Optimization with Control of the Fundamental Eigenfrequency*

320 G. Eichfelder: *Solving Nonlinear Multiobjective Bilevel Optimization Problems with Coupled Upper Level Constraints*

321 O. Museyko , G. Leugering , M. Stiglmayr , K. Klamroth: *On the application of the Monge-Kantorovich problem to image registration*

322 C. Hopfgartner , I. Scholz , M. Gugat , G. Leugering , J. Hornegger: *Intensity based Three-Dimensional Reconstruction with Nonlinear Optimization*

323 P. Hastreiter , G. Leugering , O. Museyko: *A free-discontinuity problem for the registration of images with incomplete information*

324 N. Suciu , C. Vamoş , H. Vereecken , K. Sabelfeld , P. Knabner: *Dependence on Initial Conditions, Memory Effects, and Ergodicity of Transport in Heterogeneous Media*

325 J. Jahn: *Bishop-Phelps Cones in Optimization*

326 J. Hoffmann: *Results of the GdR MoMaS Reactive Transport Benchmark with RICHY2D*

327 A. Khludnev , G. Leugering: *On Elastic Bodies with Thin Rigid Inclusions and Cracks*

328 G. Eichfelder: *Vector Optimization with a Variable Ordering Structure*