FRIEDRICH-ALEXANDER-UNIVERSITÄT ERLANGEN-NÜRNBERG
INSTITUT FÜR ANGEWANDTE MATHEMATIK

Optimality conditions for vector optimization problems with variable ordering structures

by

G. Eichfelder & T.X.D. Ha

No. 338 2010

Institut für Angewandte Mathematik
Martensstraße 3
D-91058 Erlangen

Lehrstuhl AM I:	Lehrstuhl AM II:	Lehrstuhl AM III:
Tel.: ++49/9131/85-27015	Tel.: ++49/9131/85-27510	Tel.: ++49/9131/85-25226
Fax: ++49/9131/85-27670	Fax: ++49/9131/85-28126	Fax: ++49/9131/85-25228

internet: http://www.am.uni-erlangen.de

ISSN 1435-5833, 2010, No. 338

Optimality conditions for vector optimization problems with variable ordering structures

Gabriele Eichfelder* and Truong Xuan Duc Ha**

March 31, 2010

Abstract

Our main concern in this paper are concepts of nondominatedness w.r.t. a variable ordering structure introduced by P.L.Yu in 1974. Our studies are motivated by some recent applications e.g. in medical image registration. Restricting ourselves to the case when the values of a cone-valued map defining the ordering structure are Bishop-Phelps cones, we obtain for the first time scalarizing functionals for nondominated elements, Fermat rule, Lagrange multiplier rule and duality results for a single- or set-valued vector optimization problem with a variable ordering structure.

Key Words: Vector optimization, set optimization, variable ordering structure, optimality conditions, Fermat rule, Lagrange multiplier rule.

Mathematics subject classifications (MSC 2000): 90C29, 90C30, 90C46, 90C48.

1 Introduction

In the pioneering book in 1896 Pareto presented the concept of optimal elements of a set in a vector space and it just began a new branch in optimization – vector optimization – which has been extensively studied in the last decades. Here one assumes in general that a vector space is partially ordered by a convex cone and an element is *Pareto efficient* if it is not dominated by any other reference element w.r.t. *the ordering of the space*.

But already in the first publications in the 1970s related to the definition of optimal elements in vector optimization [32, 3] also the idea of variable ordering structures was

*Department Mathematik, Universität Erlangen-Nürnberg, Martensstr. 3, D-91058 Erlangen, Germany, email: Gabriele.Eichfelder@am.uni-erlangen.de

**Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Viet Nam, email: txdha@math.ac.vn

given in the sense that there is a set-valued map with cone values which associates to each element an ordering, and a candidate element is called *a nondominated element* if it is not dominated by other reference elements w.r.t. *the corresponding ordering of these other ones*. Besides the notion of nondominated elements, it has also been considered another notion of optimal elements in [5, 6, 7], namely a candidate element is called *a minimal element* (also called nondominated-like) if it is not dominated by any other reference element w.r.t. *the ordering of the candidate element*. Note that since the works [32, 3] not much progress has been made in the study of vector optimization problems with variable ordering structures (see [5, 6, 7, 8, 16, 22, 33]).

Only in the last years based on some interesting applications in image registration, portfolio optimization and location theory [2] there is an increasing interest in such problems. In medical image registration for instance several distance measures are discussed to evaluate the similarity of two images. In [29, 11] it is proposed to consider all these measures at the same time as a vector-valued objective function instead of summing up some of these measures or of picking one special measure. In the optimization process however some of these measures turn out to be not appropriate for the actual problem and are thus considered to be less important or are even neglected. This is updated in each step which results in a variable ordering structure represented by cones of preferred directions associated to each point in the image space. Also in [13, 31] the importance of variable ordering structures for modeling preferences of decision makers are pointed out. Recently, some first nonlinear scalarization results for minimal elements are presented in [6, 7] and linear scalarizations and several characterizations of optimal elements as well as optimality conditions and duality results are considered in [11]. Observe that minimal elements w.r.t. a variable ordering structure can be transferred to Pareto efficient points w.r.t. a non-variable ordering structure and in some cases can be handled with the use of techniques of classical vector optimization. In opposition to minimal elements it is more difficult to deal with nondominated elements; for instance, scalarization results in [11] were obtained only for weakly nondominated elements (for the definition, see Section 2) under some convexity assumptions and, to our knowledge, no scalarizing functions for nondominated elements are known.

Our main concern in this paper are nondominated elements and nondominated solutions w.r.t. a variable ordering structure of single- or set-valued vector optimization problems (VOP). Restricting ourselves to the case when the values of a cone-valued map defining the ordering structure are Bishop-Phelps cones (BP cones for short), we obtain for the first time scalarizing functionals for these nondominated elements. With these functionals in hand we then convert the VOP into an optimization problem with a scalar objective function, and obtain optimality conditions for the VOP in the forms of the Fermat rule or Lagrange multiplier rule. Besides, we establish duality results for the VOP. Note that it is very natural to restrict the examinations to BP cones because many cones such as the Lorentz cone and its extensions using various l_p norms in any finite dimensional space and nonnegative orthants in the classical Banach spaces $L^1_{[0,1]}$, l^1 are BP cones. Moreover, any nontrivial convex cone with a closed and bounded base in a real normed space, for instance, any closed pointed convex cone in $\mathbb{R}^n$, is representable as a BP cone [19, Remark 2.16], [27, 21]. Also in the recent work devoted to variable ordering structures [13, Remark 8] the proposed variable cones for modeling preferences of decision makers are in fact special BP cones.

We proceed as follows. Section 2 is devoted to auxiliary concepts as the notion of BP cones and the concepts of subdifferential and coderivative. Various concepts of non-dominatedness w.r.t. a variable ordering structure are recalled in Section 3 together with scalar characterization of nondominated elements of a set by new scalarizing functionals and properties of these functionals and a existence result. In Section 4 optimality conditions for nondominated solutions of vector optimization problems, i.e. Fermat rule and Lagrange multiplier rule, are given. Duality results are formulated in Section 5. Some concluding remarks as well as a comment on scalarizing functionals for minimal elements are presented in Section 6.

2 Auxiliary: Bishop Phelps cones and differentiability concepts

2.1 Bishop-Phelps cones

In 1962, Bishop and Phelps [4] introduced a class of ordering cones which have a rich mathematical structure and are very useful in functional analysis and vector optimization. This subsection is devoted to these cones.

In this subsection we consider a real normed space Y with the dual Y^*. The notations $\|.\|$ and $\|.\|_*$ denote the norms in Y and Y^*, respectively. A Bishop-Phelps cone (BP cone for short) is defined by an element ϕ from the dual space Y^* as follows.

Definition 2.1. *For an arbitrary continuous linear functional ϕ on the normed space Y the cone*

$$C(\phi) := \{y \in Y \mid \|y\| \leq \phi(y)\} \tag{1}$$

is called Bishop-Phelps cone*.*

Note that the definition of BP cone introduced in [4] is slightly different from the above one; namely, Bishop and Phelps required that $\|\phi\|_* = 1$ and $t\|y\| \leq \phi(y)$ for some scalar $t \in (0, 1)$. Nowadays, several authors do not use the constant t and the assumption $\|\phi\|_* = 1$ and Definition 2.1 follows this line. We present examples of BP cones in the following.

Example 2.2. *(i) Let $Y = \mathbb{R}^n$ and $C_p := \{y \in \mathbb{R}^n \mid \|(y_1, ..., y_{n-1})\|_p \leq y_n\}$ for an l_p norm $\|.\|_p$ with $p \in [1, \infty]$. It has been established that C_p is a BP cone [19]. Note that C_2 is the Lorentz cone (also called second-order cone or ice cream cone) which is a well-known concept in second-order cone programming.*

(ii) The natural ordering cones in the classical Banach spaces $L^1_{[0,1]}$ and l^1 are BP cones.

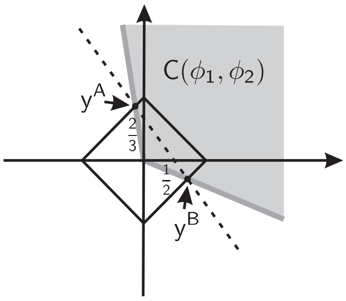

Figure 1: BP cone $C(\phi_1, \phi_2)$ of Example 2.2(iii) for $\phi_1 = 2$ and $\phi_2 = 3/2$, as well as the unit ball w.r.t. the l_1-norm and (in dashed line) the set $\{(y_1, y_2) \in \mathbb{R}^2 \mid (\phi_1, \phi_2)^\top (y_1, y_2) = 1\}$.

(iii) Let $Y = \mathbb{R}^2$ and assume that the space is equipped with the l_1-norm. Then for instance for $(\phi_1, \phi_2) = (1, 1)$ we have $C(\phi_1, \phi_2) = \mathbb{R}^2_+$. Assume $\phi_1, \phi_2 \geq 1$, then $C(\phi_1, \phi_2) \supset \mathbb{R}^2_+$, $(0, 1/\phi_2) \in C(\phi_1, \phi_2)$, $(1/\phi_1, 0) \in C(\phi_1, \phi_2)$ and

$$C(\phi_1, \phi_2) = \text{cone conv}\{y^A, y^B\}$$

with

$$y^A := \left(\frac{1 - \phi_2}{\phi_1 + \phi_2}, \frac{1 + \phi_1}{\phi_1 + \phi_2} \right)^\top \quad \text{and} \quad y^B := \left(\frac{1 + \phi_2}{\phi_1 + \phi_2}, \frac{1 - \phi_1}{\phi_1 + \phi_2} \right)^\top,$$

see Fig. 1.

Below we collect some properties of a BP cone from [19].

Proposition 2.3. *Let $\phi \in Y^*$ be given.*

(i) $C(\phi)$ is closed, pointed and convex.

(ii) If $\|\phi\|_* > 1$ then $C(\phi)$ is nontrivial; if $\|\phi\|_* < 1$ then $C(\phi) = \{0\}$.

(iii) If $\|\phi\|_* > 1$ then the interior of $C(\phi)$ is nonempty and is the set

$$\{y \in Y \mid \|y\| < \phi(y)\}.$$

(iv) $\phi \in C(\phi)^\# := \{y^* \in Y^* \mid y^*(y) > 0 \ \forall y \in C(\phi) \setminus \{0\}\}$.

(v) The set $\{y \in C(\phi) \mid \phi(y) = 1\}$ is a closed and bounded base for the cone $C(\phi)$.

In [27] Petschke considered cones which are representable as BP cone in the sense that they become BP cones when the spaces are equipped with some equivalent norms. It has been established that every nontrivial convex cone in $\mathbb{R}^n$ is representable as a BP cone if and only if it is closed and pointed and in general real normed spaces every nontrivial convex cone with a closed and bounded base is representable as a BP cone [27, 19].

2.2 Subdifferentials and coderivatives

In this subsection, we recall the concepts of subdifferential of a function and coderivative of a set-valued map that will be used to formulate optimality conditions. Throughout the section, X and Y are normed spaces. We will use the same notation $\|.\|$ for the norms in X, Y and $\|.\|_*$ for the norms in X^*, Y^*. Assume that $g : X \to \mathbb{R} \cup \{\infty\}$ is a function and $F : X \to 2^Y$ is a set-valued map (for the sake of convenience we assume that $F(x)$ is nonempty for all $x \in X$). The domain, epigraph of g and the graph of F are the sets $\mathrm{dom} g = \{x \in X \mid g \text{ is finite at } x\}$, $\mathrm{epi} g = \{(x,t) \in X \times \mathbb{R} \mid g(x) \leq t\}$ and $\mathrm{gr} F = \{(x,y) \in X \times Y \mid y \in F(x)\}$, respectively. For a nonempty set $A \subset X$, $d(x,A)$ is the distance from x to A and $\chi_A(x)$ is the indicator function associated to A, i.e., $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$, otherwise.

To define the concept of subdifferential of g, suppose first that g is locally Lipschitz near $x \in \mathrm{dom} g$. The *Ioffe approximate subdifferential* of g at x [17] is the set

$$\partial_A g(x) = \bigcap_{L \in \mathcal{F}} \limsup_{(\varepsilon,y)\to(0^+,x)} \partial_\varepsilon^- g_{y+L}(y),$$

where $\mathcal{F}$ is the collection of all finite dimensional subspaces of X, $g_{y+L}(u) = g(u)$ if $u \in y + L$ and $g_{y+L}(u) = +\infty$ otherwise, for $\varepsilon \geq 0$

$$\begin{aligned}\partial_\varepsilon^- g_{y+L}(y) = \{x^* \in X^* \mid \quad & x^*(v) \leq \varepsilon\|v\| + \\ & + \liminf\nolimits_{t\to 0^+} t^{-1}[g_{y+L}(y+tv) - g_{y+L}(y)], \forall v \in X\}.\end{aligned}$$

The *Clarke generalized subdifferential* of g at x [9] is the set

$$\partial_C g(x) = \{x^* \in X^* \mid x^*(v) \leq g^0(x;v),\ \forall v \in X\},$$

where $g^0(x;v)$ is the generalized directional derivative of g at x in the direction v

$$g^0(x;v) = \limsup_{y\to x,\ t\to 0^+} \frac{g(y+tv) - g(y)}{t}.$$

Let Ω be a nonempty subset of X different from X and $x \in \mathrm{cl}\Omega$ ("cl" stands for "close hull"). The *Ioffe approximate normal cone* to Ω at $x \in \Omega$ [17] is given by

$$N_A(x;\Omega) = \bigcup_{\lambda>0} \lambda \partial_A d(x;\Omega)$$

and the *Clarke normal cone* to Ω at $x \in \Omega$ [9] is given by

$$N_C(x;\Omega) = \mathrm{cl}\left(\bigcup_{\lambda>0} \lambda \partial_C d(x;\Omega)\right).$$

The *Mordukhovich normal cone* to Ω at x [23, 24, 25, 26] is defined by

$$N_M(x;\Omega) = \limsup_{x' \xrightarrow{\Omega} x, \varepsilon \downarrow 0^+} \hat{N}_\varepsilon(x';\Omega),$$

where the limit in the right-hand side means the sequential Kuratowski-Painlevé upper limit with respect to the norm topology in X and the weak-star ω^* topology in X^*, $x' \xrightarrow{\Omega} x$ refers to all sequences converging to x which remain in Ω and $\hat{N}_\varepsilon(x';\Omega)$ is the set of Fréchet ε-normals to Ω at x' given by

$$\hat{N}_\varepsilon(x';\Omega) = \left\{ x^* \in X^* \;\middle|\; \limsup_{x'' \xrightarrow{\Omega} x'} \frac{x^*(x''-x')}{\|x''-x'\|} \le \varepsilon \right\}.$$

Now assume that the function g is lower semicontinuous and the set-valued map F is closed (i.e., its graph is a closed set). The subdifferentials for g and coderivatives for F in the sense of Ioffe, Clarke or Mordukhovich are defined through the corresponding normal cone as follows [17, 9, 23, 24, 25, 26] (for the sake of convenience, we make the convention that the same notations N, ∂g and D^*F are used for the normal cone, subdifferentials and coderivatives in the above senses and that the spaces under consideration are Asplund, i.e., each of its separable subspace has a separable dual, whenever the subdifferential and the coderivative are understood in the sense of Mordukhovich)

$$\partial g(x) = \{x^* \in X^* \mid (x^*, -1) \in N((x, g(x)); \mathrm{epi} g)\}$$

and

$$D^*F(x,y)(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N((x,y); \mathrm{gr}F)\}.$$

Next, we recall some properties of the above normal cones, subdifferentials and coderivatives that will be used in the sequel. Denote by $\mathcal{L}(X,Y)$ the space of continuous linear maps from X to Y. Let be given a map $h : X \to Y$. Recall that h is said to admit a strict derivative at x, an element of $\mathcal{L}(X,Y)$ denoted $h'(x)$, provided that for each v, the following holds:

$$\lim_{x' \to x,\ t \downarrow 0} \frac{h(x'+tv) - h(x')}{t} = h'(x)(v),$$

and provided the convergence is uniform for v in compact sets (this condition automatically holds if h is Lipschitz near x). We need the following characterization (see [9, Prop. 2.2.1]).

Proposition 2.4. *Let h map a neighborhood of x to Y, and let ζ be an element of $\mathcal{L}(X,Y)$. The following are equivalent:*

(i) h is strictly differentiable at x and $h'(x) = \zeta$.

(ii) h is Lipschitz near x, and for each v in X one has

$$\lim_{x' \to x,\ t \downarrow 0} \frac{h(x'+tv) - h(x')}{t} = \zeta(v).$$

Proposition 2.5. *[9, 17, 23, 24, 25, 26] Assume that $g : X \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous and Ω is a nonempty closed subset of X.*

(i) If the function g is strictly differentiable near x then $\partial g(x) = \{g'(x)\}$.

(ii) If g is convex and Lipschitz near x, the above subdifferentials reduce to the subdifferential of convex analysis, i.e.,

$$\partial g(x) = \{x^* \in X^* \mid x^*(x') - x^*(x) \leq g(x') - g(x) \text{ for all } x' \in \mathrm{dom} g\}.$$

(iii) If $g(x') \geq g(x)$ for all x' in a neighborhood of $x \in domg$, then $0 \in \partial g(x)$.

(iv) (sum rule) Assume that $h : X \to R \cup \{+\infty\}$ is Lipschitz near $x \in domg\cap$ domh, then $\partial(g+h)(x) \subset \partial g(x) + \partial h(x)$ and the equality holds if at least one function is strictly differentiable near x.

(v) $\partial \chi_\Omega(x) = N(x;\Omega)$ and if Ω is convex then the above normal cones reduce to the normal cone of convex analysis, i.e. to the set

$$\{x^* \in X^* \mid x^*(x' - x) \leq 0 \textit{ for all } x' \in \Omega\}.$$

(vi) For the norm $\|.\|$ in X one has: $\partial\|.\|(0) = B_{X^*}$, $\partial(-\|.\|)(0) = B_{X^*}$ if the subdifferential is understood in the sense of Clarke and $\partial(-\|.\|)(0) = S_{X^*}$ if the subdifferential is understood in the senses of Ioffe or Mordukhovich. Here B_{X^*} and S_{X^*} are the closed unit ball and the unit sphere in X^*, respectively.

Proposition 2.6. *Assume that a map $g\colon X \to Y$ is strictly differentiable near $\bar{x}$ then $D^*g(\bar{x}, g(\bar{x})) = \{[g'(\bar{x})]^*\}$. Here, $[g'(\bar{x})]^*\colon Y^* \to X^*$ is the adjoint map to $g'(\bar{x})$ defined by $[g'(\bar{x})]^*(y^*)(x) = y^*(g'(\bar{x})(x))$ for any $(x, y^*) \in X \times Y^*$.*

3 Nondominated elements and their scalarizing functionals

3.1 Definitions of nondominated elements

In this subsection we recall, for the reader's convenience, some concepts of nondominated elements and minimal elements w.r.t. a variable ordering structure of a set.

Let Y be a topological space. For a nonempty set $A \subset Y$, cl$A$, int$A$, cone$A$ and conv$A$ denote the close hull, the interior, the conic hull and the convex hull of A, respectively. Let $\mathcal{D} : Y \to 2^Y$ be a cone-valued map with $\mathcal{D}(y)$ a nontrivial cone for all $y \in Y$. Based on this map one can define two different relations: for $y, \bar{y} \in Y$

$$y \leq_1 \bar{y} \quad \text{if} \quad \bar{y} \in \{y\} + \mathcal{D}(y) \tag{2}$$

and

$$y \leq_2 \bar{y} \quad \text{if} \quad \bar{y} \in \{y\} + \mathcal{D}(\bar{y}). \tag{3}$$

The first relation leads to the concept of nondominated elements, which was defined by Yu [32, 33] and considered in [16, 28, 30, 31]. The second relation leads to various concepts of minimal elements, which were used in [13, 16, 22] and were called nondominated-like solution by Chen in [5, 6, 7].

Definition 3.1. *Let A be a nonempty subset of Y and $\bar{a} \in A$. We say that*

(i) $\bar{a}$ *is a* nondominated element *of A w.r.t. the ordering map $\mathcal{D}$ if there is no $a \in A \setminus \{\bar{a}\}$ such that $\bar{a} \in \{a\} + \mathcal{D}(a)$, i.e., $a \not\leq_1 \bar{a}$.*

(ii) $\bar{a}$ *is a* strongly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ if $\bar{a} \in \{a\} - \mathcal{D}(a)$ for all $a \in A$.*

(iii) Supposing that int $\mathcal{D}(a) \neq \emptyset$ for all $a \in A$, $\bar{a}$ is a weakly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ if there is no $a \in A$ such that $\bar{a} \in \{a\} + \mathrm{int}\mathcal{D}(a)$.*

(iv) $\bar{a}$ *is a* minimal element *of A w.r.t. the ordering map $\mathcal{D}$ if there is no $a \in A \setminus \{\bar{a}\}$ such that $\bar{a} \in \{a\} + \mathcal{D}(\bar{a})$, i.e., $a \not\leq_2 \bar{a}$.*

(v) $\bar{a}$ *is a* strongly minimal *element of A w.r.t. the ordering map $\mathcal{D}$ if $A \subset \{\bar{a}\} + \mathcal{D}(\bar{a})$.*

Note that when $\mathcal{D}(y) \equiv \mathcal{K}$, where $\mathcal{K}$ is a pointed convex cone and the space Y is partially ordered by $\mathcal{K}$, the concepts of nondominated, strongly nondominated and weakly nondominated elements w.r.t. the ordering map $\mathcal{D}$ reduce to the classical concepts of Pareto efficient, strongly efficient and weakly efficient elements w.r.t. the cone $\mathcal{K}$ (see, for instance, [18]). Observe further that if $\mathcal{D}(\bar{a})$ is a pointed convex cone, then $\bar{a}$ is a minimal element w.r.t. $\mathcal{D}$ if and only if it is a Pareto efficient element of the set A w.r.t. a non-variable ordering structure in Y given by the cone $\mathcal{K} := \mathcal{D}(\bar{a})$, i.e. $(\{\bar{a}\} - \mathcal{K}) \cap A = \{\bar{a}\}$.

Replacing $\mathcal{D}$ by $\tilde{\mathcal{D}}$ with $\tilde{\mathcal{D}}(y) = -\mathcal{D}(y)$ for all $y \in Y$ in Definition 2.1, we obtain corresponding concepts of *max-nondominated* and *maximal elements* of a set A w.r.t. the ordering map $\mathcal{D}$. Replacing A by $\mathcal{N}(\bar{a}) \cap A$ for some neighborhood $\mathcal{N}(\bar{a})$ of $\bar{a}$ in all definitions, we can define corresponding local concepts. Note that minimal and nondominated elements are connected via duality results, see [11] and Section 5.

From the definitions one can easily derive the following.

Proposition 3.2. *Let A be a nonempty subset of Y.*

(i) If $\mathcal{D}(a)$ is pointed for all $a \in A$, then any strongly nondominated element of A w.r.t. $\mathcal{D}$ is also a nondominated element of A w.r.t. $\mathcal{D}$.

(ii) If int $\mathcal{D}(a) \neq \emptyset$ for all $a \in A$, then any nondominated element of A w.r.t. $\mathcal{D}$ is also a weakly nondominated element of A w.r.t. $\mathcal{D}$.

(iii) If $\bar{a}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$, then the set of minimal elements of A w.r.t. $\mathcal{D}$ is empty or equals $\{\bar{a}\}$. If $\bar{D} := \bigcup_{a \in A} \mathcal{D}(a)$ is pointed, then $\bar{a}$ is the unique minimal element of A w.r.t. $\mathcal{D}$.

(iv) If $\bar{a} \in A$ is a strongly minimal element of A and if $\mathcal{D}(\bar{a}) \subset \mathcal{D}(a)$ for all $a \in A$, then $\bar{a}$ is also a strongly nondominated element of A.

We illustrate Definition 3.1 by the following examples.

Example 3.3. *Let $Y = \mathbb{R}^2$, the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by*

$$\mathcal{D}(y_1, y_2) := \begin{cases} \text{cone conv}\{(y_1, y_2), (1, 0)\} & \text{if } (y_1, y_2) \in \mathbb{R}^2_+,\ y_2 \neq 0 \\ \mathbb{R}^2_+ & \text{otherwise} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq 0,\ y_2 \geq 1 - y_1\}.$$

One can check that $\{(y_1, y_2) \in A \mid y_1 + y_2 = 1\}$ is the set of all nondominated elements of A and $\{(y_1, y_2) \in A \mid y_1 + y_2 = 1 \vee y_1 = 0 \vee y_2 = 0\}$ is the set of all weakly nondominated elements of A.

Example 3.4. *Let $Y = \mathbb{R}^2$, the cone-valued map $\mathcal{D}_1\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by*

$$\mathcal{D}_1(y_1, y_2) := \begin{cases} \text{cone conv}\{(-1, 1), (1, 0)\} & \text{if } y_2 \geq 0 \\ \mathbb{R}^2_+ & \text{otherwise} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\}.$$

Then $(-1, 0)$ is a nondominated but not a minimal element of A w.r.t. $\mathcal{D}_1$.

Considering instead the cone-valued map $\mathcal{D}_2\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$\mathcal{D}_2(y_1, y_2) := \begin{cases} \text{cone conv}\{(1, -1), (0, 1)\} & \text{if } y_2 \geq 0 \\ \mathbb{R}^2_+ & \text{otherwise} \end{cases},$$

then $(0, -1)$ is minimal but not a nondominated element of A w.r.t. $\mathcal{D}_2$.

Example 3.5. *Let $Y = \mathbb{R}^2$, the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by*

$$\mathcal{D}(y_1, y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y_2 = 0 \\ \text{cone conv}\{(y_1, |y_2|), (1, 0)\} & \text{otherwise} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq y_2 \leq 2y_1\}.$$

One can check that $(0, 0) \in A$ is a strongly minimal and also a strongly nondominated element of A.

3.2 Variable ordering structures defined by BP cones

In the remaining of Section 3 we assume that $(Y, \|.\|)$ is a normed space. The variable ordering structures on Y are defined by set-valued maps $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a special BP cone for all $y \in Y$ (or for all elements y of a subset A of Y) as follows. To any $y \in Y$ we associate a norm $\|.\|_y$ equivalent to but eventually different from the norm of the space and for a given map ℓ from Y to Y^* we define

$$\mathcal{D}(y) = C(\ell(y)) := \{u \in Y \mid \|u\|_y \leq \ell(y)(u)\} \text{ for all } y \in Y.$$

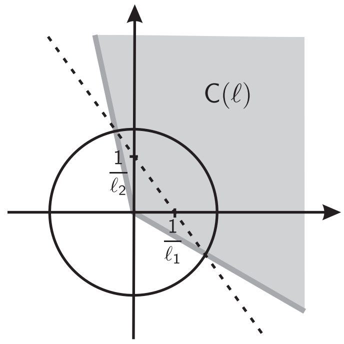

Figure 2: BP cone $C(\ell(y))$ of Example 3.6 for $\ell_1 = \ell_1(y)$ and $\ell_2 = \ell_2(y)$, as well as the unit ball w.r.t. the Euclidean norm and (in dashed line) the line connecting the points $(1/\ell_1, 0)$ and $(0, 1/\ell_2)$.

Note that in $\mathbb{R}^n$ one might need different equivalent norms to represent different nontrivial convex closed pointed cones as BP cones and this motivates us to consider the above BP cones. In $\mathbb{R}^2$ it is fine to choose just one norm but already in $\mathbb{R}^3$ one has to use different norms to model for instance a polyhedral cone and a Lorentz cone. In an application we might have a variable structure with different cones $\mathcal{D}(y)$ but presumably they will all be of the same type, for instance all polyhedral, and can all be modeled with the same norm, compare [13, Remark 8]. In particular, when the norm $\|.\|_y$ in the definition of the special BP cones $\mathcal{D}(y)$ is assumed to equal the norm $\|.\|$ of the space Y and is thus equal for all $y \in Y$, these cones reduce to the BP cones

$$\mathcal{D}(y) = C(\ell(y)) = \{u \in Y \mid \|u\| \leq \ell(y)(u)\}.$$

Below is an example of the variable ordering structure given by such cones. As the reader can see from the example, even in such a case the images of $\mathcal{D}$ still cover a wide range of different cones.

Example 3.6. *Let Y be the Euclidean space $\mathbb{R}^2$, $\|.\|_y := \|.\|_2$ for all $y \in \mathbb{R}^2$ and define $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$\ell(y_1, y_2) := (\ (3 + \sin y_1)/2\, ,\ (3 + \cos y_2)/2\ ) \in [1, 2] \times [1, 2]. \tag{4}$$

Then $\mathbb{R}^2_+ \subset C(\ell(y))$ for all $y \in \mathbb{R}^2$. The cones $C(\ell(y))$ can be visualized as follows: The two extreme rays of the convex pointed cone $C(\ell(y))$ are given by two half rays starting in the origin being defined by the two intersection points of the unit circle and the line connecting the points $(1/\ell_1, 0)$ and $(0, 1/\ell_2)$, see Fig. 2. For instance $C(\ell(3\pi/2, \pi)) = \mathbb{R}^2_+$.

3.3 Scalar characterization of nondominated elements

In this subsection, we obtain for the first time scalarization results for (strongly, weakly) nondominated elements w.r.t. variable ordering structures defined by BP cones. This delivers the base for the Fermat rule and Lagrange multiplier rule for vector optimization problems with a variable ordering structure which will be presented in Section 4.

To any fixed $\bar{y} \in Y$ we associate the following functionals:

$$\begin{array}{rl} \theta(y) & := \ell(y)(y) \\ \eta_{\bar{y}}(y) & := \ell(y)(y-\bar{y}) \\ \gamma_{\bar{y}}(y) & := \ell(y)(y-\bar{y}) - \|y-\bar{y}\|_y \\ \xi_{\bar{y}}(y) & := \ell(y)(y-\bar{y}) + \|y-\bar{y}\|_y \end{array} \tag{5}$$

for each $y \in Y$. Using the functionals (5) we can characterize a (weakly, strongly) non-dominated element of a set A w.r.t. $\mathcal{D}$. Note that neither additional assumptions on $\mathcal{D}$ – besides that the images of $\mathcal{D}$ are BP cones – nor convexity assumptions on the set A are presumed.

Theorem 3.7. *Let A be a nonempty subset of Y and $\bar{a} \in A$.*

(i) $\bar{a} \in A$ is a strongly nondominated element of A w.r.t. the ordering map $\mathcal{D}$ if and only if the functional $\gamma_{\bar{a}}$ attains its minimum over A at $\bar{a}$, which means that

$$\gamma_{\bar{a}}(a) \geq \gamma_{\bar{a}}(\bar{a}) = 0, \ \forall a \in A.$$

(ii) $\bar{a} \in A$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ if and only if the functional $\xi_{\bar{a}}$ attains its strict minimum over A at $\bar{a}$, which means that

$$\xi_{\bar{a}}(a) > \xi_{\bar{a}}(\bar{a}) = 0, \ \forall a \in A \setminus \{\bar{a}\}.$$

(iii) Supposing that $\|\ell(a)\|_* > 1$ (and hence, int $\mathcal{D}(a) \neq \emptyset$) for all $a \in A$, $\bar{a} \in A$ is a weakly nondominated element of A w.r.t. the ordering map $\mathcal{D}$ if and only if the functional $\xi_{\bar{a}}$ attains its minimum over A at $\bar{a}$, which means that

$$\xi_{\bar{a}}(a) \geq \xi_{\bar{a}}(\bar{a}) = 0, \ \forall a \in A.$$

Proof. (i) $\bar{a}$ is strongly nondominated w.r.t. the ordering map $\mathcal{D}$ if and only if

$$\begin{array}{rl} & a-\bar{a} \in \mathcal{D}(a) = \{y \in Y \mid \|y\|_a \leq \ell(a)(y)\}, \ \forall a \in A \\ \Leftrightarrow & \ell(a)(a-\bar{a}) - \|a-\bar{a}\|_a \geq 0, \ \forall a \in A \\ \Leftrightarrow & \gamma_{\bar{a}}(a) \geq \gamma_{\bar{a}}(\bar{a}) = 0, \ \forall\ a \in A. \end{array}$$

(ii) $\bar{a}$ is a nondominated element w.r.t. the ordering map $\mathcal{D}$ if and only if

$$\begin{array}{rl} & \bar{a}-a \notin \mathcal{D}(a), \ \forall a \in A \setminus \{\bar{a}\} \\ \Leftrightarrow & \|a-\bar{a}\|_a > \ell(a)(\bar{a}-a), \ \forall a \in A \setminus \{\bar{a}\} \\ \Leftrightarrow & \ell(a)(a-\bar{a}) + \|a-\bar{a}\|_a > 0, \ \forall a \in A \setminus \{\bar{a}\} \\ \Leftrightarrow & \xi_{\bar{a}}(a) > \xi_{\bar{a}}(\bar{a}) = 0, \ \forall\ a \in A \setminus \{\bar{a}\}. \end{array}$$

(iii) $\bar{a}$ is weakly nondominated w.r.t. the ordering map $\mathcal{D}$ if and only if $\bar{a}-a \notin \mathrm{int}\mathcal{D}(a)$ for all $a \in A$. According to Prop. 2.3 we have $\mathrm{int}\mathcal{D}(a) = \{y \in Y \mid \|y\|_a < \ell(a)(y)\} \neq \emptyset$ and hence

$$\begin{aligned} & \bar{a}-a \notin \mathrm{int}\mathcal{D}(a),\ \forall a \in A \\ \Leftrightarrow\ & \ell(a)(a-\bar{a}) + \|a-\bar{a}\|_a \geq 0,\ \forall a \in A \\ \Leftrightarrow\ & \xi_{\bar{a}}(a) \geq \xi_{\bar{a}}(\bar{a}) = 0,\ \forall\ a \in A. \end{aligned}$$

□

Remark 3.8. *It is easy to see that any BP cone is pointed. Hence, according to Proposition 3.2(i), any strongly nondominated element is nondominated and the "only if" part of the assertion (ii) in the above theorem also is necessary for $\bar{a}$ to be a strongly nondominated element of A w.r.t. $\mathcal{D}$. Additionally, the "if" part of the assertion (i) in the above theorem is also sufficient for $\bar{a}$ to be a nondominated element of A w.r.t. $\mathcal{D}$.*

Example 3.9. *Let Y be the Euclidean space $\mathbb{R}^2$ and the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $\mathcal{D}(y) := C(\ell(y))$ with $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ as in (4) and with $\|.\|_y := \|.\|_2$ for all $y \in \mathbb{R}^2$ and let*

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq 0,\ y_2 \geq \pi - y_1\}.$$

Then $\mathbb{R}^2_+ \subset \mathcal{D}(y) \subset \mathrm{cone\ conv}\{y^A, y^B\}$ with $y^A := 0.25(1+\sqrt{7}, 1-\sqrt{7})$, $y^B = 0.25(1-\sqrt{7}, 1+\sqrt{7})$ for all $y \in \mathbb{R}^2$. By Theorem 3.7, $\bar{y} = (0, \pi)$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ because it holds

$$\xi_{\bar{y}}(y) = \frac{3+\sin y_1}{2}(y_1 - 0) + \frac{3+\cos y_2}{2}(y_2 - \pi) + \|y - (0,\pi)^\top\|_2 \geq 0 = \xi_{\bar{y}}(\bar{y})$$

for all $y \in A$ but it is not a strongly nondominated element of A w.r.t. $\mathcal{D}$ because it holds

$$\gamma_{\bar{y}}(3\pi/2, 0) = (1,2)(3\pi/2, -\pi)^\top - \|(3\pi/2, -\pi)^\top\|_2 < 0 = \gamma_{\bar{y}}(\bar{y}).$$

Note that a special scalarizing functional for minimal elements w.r.t. a variable ordering structure was examined for the first time in [6]: for some $k \in \mathrm{int}\left(\bigcap_{y\in Y} \mathcal{D}(y)\right)$ the functional $\tilde{\xi}\colon Y \times Y \to \mathbb{R}$ defined by

$$\tilde{\xi}(y,z) := \inf\{t \in \mathbb{R} \mid t\,k - z \in \mathcal{D}(y)\} \ \text{ for all } y, z \in Y$$

was discussed. Under some assumptions including the linearity of the cone-valued map $\mathcal{D}$

$$\mathcal{D}(\alpha\, y_1 + \beta\, y_2) = \alpha\, \mathcal{D}(y_1) + \beta\, \mathcal{D}(y_2) \text{ for all } y_1, y_2 \in Y,\ \alpha, \beta \in \mathbb{R}, \tag{6}$$

it was established that $\tilde{\xi}$ is convex. However, according to the following remark this linearity condition implies that $\mathcal{D}$ is a trivial cone-valued map.

Remark 3.10. *Any cone-valued map $\mathcal{D}\colon Y \to 2^Y$ which satisfies (6) is constant with $\mathcal{D}(y) = \{0\}$ for all $y \in Y$: due to $\mathcal{D}(\alpha\, y_1) = \alpha\, \mathcal{D}(y_1)$ for all $\alpha \in \mathbb{R}$ we conclude $\mathcal{D}(0) = \{0\}$ and thus for arbitrary $y \in Y$*

$$\{0\} = \mathcal{D}(0) = \mathcal{D}(y) - \mathcal{D}(y)$$

which implies, as the sets $\mathcal{D}(y)$ are cones, $D(y) = \{0\}$.

From Theorem 3.7 one can easily deduce the following scalar characterization for Pareto efficient, strongly efficient and weakly efficient elements.

Theorem 3.11. *Suppose that the normed space $(Y, \|.\|)$ is partially ordered by a BP cone $\mathcal{K}$ given by*

$$\mathcal{K} = \{y \in Y \mid \|y\| \leq \phi(y)\}, \tag{7}$$

where ϕ is an arbitrary continuous linear functional from the dual space Y^*. Let A be a nonempty subset of Y and $\bar{a} \in A$. Define functionals $\tilde{\xi}_{\bar{a}}$ and $\tilde{\gamma}_{\bar{a}}$ as follows: for any $y \in Y$

$$\tilde{\gamma}_{\bar{a}}(y) = \phi(y - \bar{a}) - \|y - \bar{a}\|$$

and

$$\tilde{\xi}_{\bar{a}}(y) = \phi(y - \bar{a}) + \|y - \bar{a}\|.$$

(i) $\bar{a} \in A$ is a strongly efficient element of A w.r.t. the cone $\mathcal{K}$ if and only if the functional $\tilde{\gamma}_{\bar{a}}$ attains its minimum over A at $\bar{a}$.

(ii) $\bar{a} \in A$ is a Pareto efficient element of A w.r.t. the cone $\mathcal{K}$ if and only if the functional $\tilde{\xi}_{\bar{a}}$ attains its strict minimum over A at $\bar{a}$.

(iii) Supposing that $\|\phi\|_* > 1$ (and hence, int $\mathcal{K} \neq \emptyset$), $\bar{a} \in A$ is a weakly efficient element of A w.r.t. the cone $\mathcal{K}$ if and only if the functional $\tilde{\xi}_{\bar{a}}$ attains its minimum over A at $\bar{a}$.

Remark 3.12. *Recently, the functional $\tilde{\xi}_{\bar{a}}$ has been used in [20, Theorem 5.8] to characterize an element $\bar{a}$ which is a properly efficient element of A in the senses of Henig or Benson.*

In the remaining of this work we assume $\|.\|_y := \|.\|$ for all $y \in Y$. As noted above, the values of the map $\mathcal{D}$ reduce to the BP cones $\mathcal{D}(y) = C(\ell(y)) = \{u \in Y \mid \|u\| \leq \ell(y)(u)\}$ and the functionals (5) become

$$\begin{array}{rl} \theta(y) & := \ell(y)(y) \\ \eta_{\bar{y}}(y) & := \ell(y)(y - \bar{y}) \\ \gamma_{\bar{y}}(y) & := \ell(y)(y - \bar{y}) - \|y - \bar{y}\| \\ \xi_{\bar{y}}(y) & := \ell(y)(y - \bar{y}) + \|y - \bar{y}\| \end{array} \tag{8}$$

for each $y \in Y$.

3.4 Properties of the scalarizing functionals

Next, we study properties of the functionals (8). We show that these functionals inherit from ℓ such properties as continuity, lower semicontinuity, Lipschitzity and provide formula for their derivative or subdifferential. We also study the convexity of θ, $\eta_{\bar{y}}$ and $\xi_{\bar{y}}$. Let us begin with showing that the functionals (8) inherit the continuity from ℓ.

Proposition 3.13. *Suppose that ℓ is continuous near $y \in Y$. Then θ, $\eta_{\bar{y}}$, $\gamma_{\bar{y}}$ and $\xi_{\bar{y}}$ also are continuous near y.*

Proof. As ℓ is continuous near y, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\delta \leq \min\{\varepsilon/(2\|\ell(y)\|_*)\ ,\ \|y\|\ \}$ such that $\|\ell(y') - \ell(y)\|_* \leq \varepsilon/(4\|y\|)$ whenever $\|y' - y\| \leq \delta$. Then, for $y' \in Y$ satisfying $\|y' - y\| \leq \delta$, we have $\|y'\| \leq \|y' - y\| + \|y\| \leq \delta + \|y\| \leq 2\|y\|$ and therefore,

$$\begin{array}{rcl} |\theta(y') - \theta(y)| & = & |\ell(y')(y') - \ell(y)(y)| \\ & = & |(\ell(y') - \ell(y))(y') + \ell(y)(y' - y)| \\ & \leq & \|(\ell(y') - \ell(y)\|_*\ \|y'\| + \|\ell(y)\|_*\ \|y' - y\|\ . \\ & \leq & (\varepsilon/(4\|y\|))(2\|y\|) + \|\ell(y)\|_*\ \delta \\ & \leq & \varepsilon/2 + \varepsilon/2\ =\ \varepsilon \end{array}$$

Thus, the function θ is continuous near y. The continuity of the functions $\eta_{\bar{y}}$, $\gamma_{\bar{y}}$ and $\xi_{\bar{y}}$ follows from the continuity of the function θ and the norm. $\square$

In the case $Y = \mathbb{R}^n$ we can even speak about the lower semicontinuity of the functionals (8). Namely, we have

Proposition 3.14. *Let $Y = \mathbb{R}^n$ and $\ell = (\ell_1, \ldots, \ell_n)$. Suppose that ℓ_i $(i = 1, \ldots, n)$ are lower semicontinuous near $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then the functionals θ, $\eta_{\bar{y}}$, $\gamma_{\bar{y}}$ and $\xi_{\bar{y}}$ also are lower semicontinuous near y.*

Proof. Recalling that $\ell(y)(y) = y_1\ell_1(y) + \ldots + y_n\ell_n(y)$ one can derive from the lower semicontinuity of the functionals ℓ_i $(i = 1, \ldots, n)$ near y the one of the functionals θ, $\eta_{\bar{y}}$, $\gamma_{\bar{y}}$ and $\xi_{\bar{y}}$. $\square$

Next, we show that the functionals (8) inherit the lipschitzity from ℓ and we provide some formula of their derivative and subdifferential.

Proposition 3.15. *Suppose that ℓ is Lipschitz near $y \in Y$. Then θ, $\eta_{\bar{y}}$, $\gamma_{\bar{y}}$, $\xi_{\bar{y}}$ also are Lipschitz near y and one has*

$$\partial\gamma_{\bar{y}}(y) \subset \partial\eta_{\bar{y}}(y) + B_{Y^*} \text{ and } \partial\xi_{\bar{y}}(y) \subset \partial\eta_{\bar{y}}(y) + B_{Y^*}. \tag{9}$$

Moreover, if ℓ is Lipschitz near $\bar{y}$ then $\eta_{\bar{y}}$ is strictly differentiable at $\bar{y}$ and one has

$$\eta'_{\bar{y}}(\bar{y}) = \ell(\bar{y}). \tag{10}$$

and

$$\partial\gamma_{\bar{y}}(\bar{y}) \subset \{l(\bar{y})\} + B_{Y^*} \text{ and } \partial\xi_{\bar{y}}(\bar{y}) = \{l(\bar{y})\} + B_{Y^*}. \tag{11}$$

Proof. Suppose that ℓ is Lipschitz of rank K on the closed ball $B(y, \rho)$ centered at y with the radius ρ. Then for $y_1, y_2 \in B(y, \rho)$ we have

$$\begin{array}{rcl} |\theta(y_1) - \theta(y_2)| & = & |\ell(y_1)(y_1) - \ell(y_2)(y_2)| \\ & = & |(\ell(y_1) - \ell(y_2))(y_1) + \ell(y_2)(y_1 - y_2)| \\ & \leq & K\|y_1 - y_2\|\ \|y_1\| + \|\ell(y_2)\|_*\ \|y_1 - y_2\| \\ & = & (K\ \|y_1\| + \|\ell(y_2)\|_*)\ \|y_1 - y_2\| \\ & \leq & (K(\|y\| + \rho) + (\|\ell(y)\|_* + K\rho))\ \|y_1 - y_2\| \\ & = & (\|\ell(y)\|_* + 2K\rho + K\|y\|)\ \|y_1 - y_2\|. \end{array}$$

Thus, θ is Lipschitz near y. The lipschitzity of the functions $\eta_{\bar{y}}$, $\gamma_{\bar{y}}$ and $\xi_{\bar{y}}$ near y follows from the lipschitzity of the function θ and the norm near that point. It is easy to see that (9) follows from Proposition 2.5 (iv) and (vi).

Now assume that ℓ is Lipschitz of rank K on the ball $B(\bar{y}, \rho)$. Our next aim is to prove that $\eta_{\bar{y}}$ is strictly differentiable at $\bar{y}$ and (10) holds. By Proposition 2.4, it suffices to show that for each v in Y one has

$$\lim_{y' \to \bar{y},\ t \downarrow 0} \frac{\eta_{\bar{y}}(y' + tv) - \eta_{\bar{y}}(y')}{t} = l(\bar{y})(v). \tag{12}$$

By the definition of the functional $\eta_{\bar{y}}$ it holds $\eta_{\bar{y}}(y' + tv) - \eta_{\bar{y}}(y') = \ell(y' + tv)(y' + tv - \bar{y}) - \ell(y')(y' - \bar{y})$. Further, as ℓ is Lipschitz near $\bar{y}$, one has

$$\begin{aligned}
&\left| \frac{\eta_{\bar{y}}(y' + tv) - \eta_{\bar{y}}(y')}{t} - l(\bar{y})(v) \right| \\
= &\left| \frac{\ell(y' + tv)(y' + tv - \bar{y}) - \ell(y')(y' - \bar{y})}{t} - \ell(\bar{y})(v) \right| \\
= &\left| \frac{(\ell(y' + tv) - \ell(y'))(y' - \bar{y})}{t} + \frac{(\ell(y' + tv)(tv)}{t} - \ell(\bar{y})(v) \right| \\
\leq &\left| \frac{(\ell(y' + tv) - \ell(y'))(y' - \bar{y})}{t} \right| + |(\ell(y' + tv) - \ell(\bar{y}))(v)| \\
\leq &\ K\|v\|\, \|y' - \bar{y}\| + K\, \|y' + tv - \bar{y}\|\, \|v\|
\end{aligned}$$

which yields (12).

Finally, (11) follows from (10) and Proposition 2.5 (i), (iv) and (vi). $\square$

Note that Proposition 3.15 can be applied for instance to the the case when ℓ is the function considered in Example 3.6.

Below we prove a result on the convexity of the functional $\xi_{\bar{y}}$ that will be used to formulate sufficient conditions for the existence of weakly nondominated solutions. We shall use the following notion related to ℓ:

Definition 3.16. *We say that ℓ is monotone if $(\ell(y_1) - \ell(y_2))(y_1 - y_2) \geq 0\ \forall y_1, y_2 \in Y$.*

Proposition 3.17. *Suppose that ℓ is linear and monotone. Then the functionals θ, $\eta_{\bar{y}}$ and $\xi_{\bar{y}}$ are convex.*

Proof. Let $y_1, y_2 \in Y$ and $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$. We have

$$\begin{aligned}
\theta(\lambda_1 y_1 + \lambda_2 y_2) &= \ell(\lambda_1 y_1 + \lambda_2 y_2)(\lambda_1 y_1 + \lambda_2 y_2) \\
&= \lambda_1 \ell(y_1)(\lambda_1 y_1 + \lambda_2 y_2) + \lambda_2 \ell(y_2)(\lambda_1 y_1 + \lambda_2 y_2) \\
&= \lambda_1 \ell(y_1)(y_1) + \lambda_1 \lambda_2 \ell(y_1)(y_2 - y_1) + \lambda_2 \ell(y_2)(y_2) + \lambda_1 \lambda_2 \ell(y_2)(y_1 - y_2) \\
&= \lambda_1 \theta(y_1) + \lambda_2 \theta(y_2) - \lambda_1 \lambda_2 (\ell(y_1) - \ell(y_2))(y_1 - y_2) \\
&\leq \lambda_1 \theta(y_1) + \lambda_2 \theta(y_2).
\end{aligned}$$

Thus, θ is convex. The convexity of the functions $\eta_{\bar{y}}$ and $\xi_{\bar{y}}$ follows from the convexity of the function θ and the norm, and the linearity of the map ℓ. $\square$

Example 3.18. *Let $Y = R^n$, M be a real positive semidefinite $n \times n$ matrix and $\ell(y) := My$ for all $y \in Y$. Then ℓ is linear, monotone and according to Proposition 3.17, the functionals θ, $\eta_{\bar{y}}$ and $\xi_{\bar{y}}$ are convex.*

3.5 Existence result

We state an existence result for weakly nondominated elements and nondominated elements.

Theorem 3.19. *Suppose that $A \subset Y$ is a nonempty compact set and ℓ is Lipschitz of rank K satisfying $K\|a\| \leq 1$ for all $a \in A$.*

(i) If $\|\ell(a)\|_* > 1$ for all $a \in A$ then A has a weakly nondominated element w.r.t. $\mathcal{D}$.

(ii) If $K\|a\| < 1$ for all $a \in A$ then A has a nondominated element w.r.t. $\mathcal{D}$.

(iii) If θ has a strict minimizer $\bar{a}$ on A then $\bar{a}$ is a nondominated element w.r.t. $\mathcal{D}$ of A.

Proof. Since ℓ is Lipschitz, Proposition 3.15 yields that θ is also Lipschitz. Therefore, θ attains its minimum on the compact set A at, say $\tilde{a} \in A$, i.e.

$$\ell(a)(a) \geq \ell(\tilde{a})(\tilde{a}), \ \forall \ a \in A \setminus \{\tilde{a}\}$$

which implies

$$\begin{aligned} \xi_{\tilde{a}}(a) \ & = \ell(a)(a - \tilde{a}) + \|a - \tilde{a}\| \\ & \geq \ell(\tilde{a})(\tilde{a}) - \ell(a)(\tilde{a}) + \|a - \tilde{a}\| \\ & \geq -\|\ell(\tilde{a}) - \ell(a)\|_* \|\tilde{a}\| + \|a - \tilde{a}\| \\ & \geq -K\|\tilde{a} - a\|\|\tilde{a}\| + \|a - \tilde{a}\| \\ & = (1 - K\|\tilde{a}\|)\|a - \tilde{a}\| \geq 0, \quad \text{for all } \ a \in A \setminus \{\tilde{a}\}, \end{aligned}$$

i.e. $\xi_{\tilde{a}}(a) \geq 0$ for all $a \in A \setminus \{\tilde{a}\}$. When either $K\|a\| < 1$ for all $a \in A$ or $\tilde{a}$ is a strict minimizer of θ i.e. $\ell(a)(a) > \ell(\tilde{a})(\tilde{a})$ for all $a \in A \setminus \{\tilde{a}\}$, one can use a similar argument to get $\xi_{\tilde{a}}(a) > 0$ for all $a \in A \setminus \{\tilde{a}\}$. This means according to Theorem 3.7 that $\tilde{a}$ is a weakly nondominated element of A in the case (i) and a nondominated element of A in the cases (ii)-(iii). ☐

Let us illustrate Theorem 3.19 by the following example.

Example 3.20. *Let $Y = \mathbb{R}^2$, the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $\mathcal{D}(y) := C(\ell(y))$ with $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ as in (4) and*

$$A := \left\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, \ y_2 \geq 0 \ \|y\| \leq 2\right\}$$

with $\|.\|$ the Euclidean norm. Then ℓ is Lipschitz of rank $K = 1/2$ and it holds $K\|a\| \leq 1$ for all $a \in A$. Since

$$\theta(y_1, y_2) = \underbrace{\left(\frac{3 + \sin y_1}{2}\right)}_{>0} y_1 + \underbrace{\left(\frac{3 + \cos y_2}{2}\right)}_{>0} y_2 > 0 = \theta(0, 0) \textit{ for all } (y_1, y_2) \in A \setminus \{(0, 0)\},$$

Theorem 3.19 implies that $(0, 0)$ is a nondominated element of A w.r.t. $\mathcal{D}$.

4 Optimality conditions for nondominated solutions of vector optimization problems

In these sections we consider vector optimization problems with the image space equipped with a variable ordering structure. We formulate the Fermat rule for the unconstrained case and the Lagrange multiplier rule for the case with constraints.

4.1 Nondominated solutions and their scalar characterization

Assume that X and Y are topological spaces and Y is equipped with a variable ordering structure defined by a cone-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a convex cone. Let $F : X \to 2^Y$ be a given set-valued map and $S \subset X$ a nonempty set. Denote $F(S) = \bigcup_{x\in S} F(x)$. Consider the following vector optimization problem

$$\text{Minimize} \quad F(x) \;\; \text{subject to} \quad x \in S\ . \tag{VP}$$

The various notions of nondominated (and minimal) elements w.r.t. the ordering map $\mathcal{D}$ for sets naturally induce corresponding notions of solutions to the optimization problem (VP) as follows.

Definition 4.1. *Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. Then a pair $(\bar{x}, \bar{y})$ is called a* "N" solution *of the problem (VP) w.r.t. the ordering map $\mathcal{D}$, if $\bar{y}$ is a "N" element of the image set $F(S)$ respectively. Here, "N" may be (local, weakly, strongly, max-) nondominated, (local, weakly, strongly) minimal or maximal.*

When F is a single-valued map $f\colon X \to Y$, we put $\bar{y} = f(\bar{x})$ in Definition 4.1.

From now on unless otherwise stated we always make a *convention* that X is a nonempty set, Y is a normed space, $\ell : Y \to Y^*$ is a map and Y is equipped with a variable ordering structure defined by a cone-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y) = C(\ell(y))$. The main result of this subsection is the following scalar characterization for nondominated solutions of the problem (VP).

Theorem 4.2. *Let $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$.*

(i) $(\bar{x}, \bar{y})$ is a strongly nondominated solution of the problem (VP) w.r.t. the ordering map $\mathcal{D}$ if and only if the functional $\gamma_{\bar{y}}$ attains its minimum over $F(S)$ at $\bar{y}$, which means that

$$\gamma_{\bar{y}}(y) \geq \gamma_{\bar{y}}(\bar{y}) = 0,\ \forall y \in F(S).$$

(ii) $(\bar{x}, \bar{y})$ is a nondominated solution of the problem (VP) w.r.t. the ordering map $\mathcal{D}$ if and only if the functional $\xi_{\bar{y}}$ attains its strict minimum over $F(S)$ at $\bar{y}$, which means that

$$\xi_{\bar{y}}(y) > \xi_{\bar{y}}(\bar{y}) = 0,\ \forall y \in F(S) \setminus \{\bar{y}\}.$$

(iii) Supposing that $\|\ell(y)\|_* > 1$ (and hence int $\mathcal{D}(y) \neq \emptyset$) for all $y \in F(S)$, $(\bar{x}, \bar{y})$ is a weakly nondominated solution of the problem (VP) w.r.t. the ordering map $\mathcal{D}$ if and only if the functional $\xi_{\bar{y}}$ attains its minimum over $F(S)$ at $\bar{y}$, which means that

$$\xi_{\bar{y}}(y) \geq \xi_{\bar{y}}(\bar{y}) = 0, \ \forall y \in F(S).$$

Proof. This result is a direct consequence of the definition of of (strongly, weakly) nondominated solutions of (VP) and Theorem 3.7. □

4.2 Fermat rule for the unconstrained optimization problem

Consider the unconstrained vector optimization problem

$$\text{Minimize} \quad F(x) \quad \text{subject to} \quad x \in X \ , \tag{UVP}$$

where $F : X \to 2^Y$ is a set-valued map. Our aim is to establish necessary and sufficient conditions for (UVP) in the form of the Fermat rule. In this subsection we assume additionally that X and Y are Banach spaces. The main result reads as follows.

Theorem 4.3. *Assume that F is closed (i.e. the graph of F is closed) and ℓ is Lipschitz on Y. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$.*

(i) (Necessary condition) If $(\bar{x}, \bar{y})$ is a nondominated solution of (UVP) w.r.t. the ordering map $\mathcal{D}$ then

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) \text{ for some } y^* \in \{\ell(\bar{y})\} + B_{Y^*}. \tag{13}$$

(ii) (Necessary and sufficient conditions) Suppose that $\|\ell(y)\|_* > 1$ (hence, int $\mathcal{D}(y) \neq \emptyset$) for all $y \in F(X)$. Then (13) is necessary for $(\bar{x}, \bar{y})$ to be a weakly nondominated solution of (UVP) w.r.t. the ordering map $\mathcal{D}$. If we assume that the graph of F is convex and ℓ is linear and monotone, then (13) also is sufficient for $(\bar{x}, \bar{y})$ to be a weakly nondominated solution of (UVP) w.r.t. the ordering map $\mathcal{D}$.

Proof. (i) By Definition 4.1, $\bar{y}$ is a nondominated element of $F(X)$ w.r.t. the ordering map $\mathcal{D}$. Then Theorem 3.7 implies that the functional $\xi_{\bar{y}}$ attains its minimum over $F(X)$ at $\bar{y}$. Hence, the functional $\tilde{\xi}_{\bar{y}} := \xi_{\bar{y}} + \chi_{\mathrm{gr} F} : X \times Y \to \overline{\mathbb{R}}$ defined by

$$\tilde{\xi}_{\bar{y}}(x, y) = \xi_{\bar{y}}(y) + \chi_{\mathrm{gr} F}(x, y)$$

attains its minimum over $X \times Y$ at $(\bar{x}, \bar{y})$, i.e.

$$\tilde{\xi}_{\bar{y}}(x, y) \geq \tilde{\xi}_{\bar{y}}(\bar{x}, \bar{y}) = 0, \ \forall \ (x, y) \in X \times Y.$$

Note that the function $\chi_{\mathrm{gr} F}$ is lower semicontinuous as the graph of F is closed and that the functional $\xi_{\bar{y}}$ is Lipschitz on Y according to Proposition 3.15. We can now apply Proposition 2.5 (iii)-(v) to get

$$\begin{array}{rl} (0,0) & \in \partial\tilde{\xi}_{\bar{y}}(\bar{x}, \bar{y}) = \partial(\xi_{\bar{y}}(.) + \chi_{\mathrm{gr}\ F}(.,.))(\bar{x}, \bar{y}) \\ & \subset (\{0\} \times \partial\xi_{\bar{y}})(\bar{y}) + N((\bar{x}, \bar{y}); \mathrm{gr}\ F). \end{array} .$$

Hence, there exists $y^* \in \partial\xi_{\bar{y}}(\bar{y})$ with $(0, -y^*) \in N((\bar{x}, \bar{y}); \mathrm{gr}\ F)$. This implies $0 \in D^*F(\bar{x}, \bar{y})(y^*)$. Taking into account the relation (11) of Proposition 3.15 we obtain (13).

(ii) Since the proof of the necessary condition is similar to that of the assertion (i), we shall prove only the sufficient condition. By the definition of the coderivative, (13) is equivalent to $(0, -y^*) \in N((\bar{x}, \bar{y}); \mathrm{gr}F)$. Since the graph of F is convex, Proposition 2.5 (v) implies that the normal cone is understood in the sense of convex analysis and therefore, one has

$$\langle (0, -y^*), (x - \bar{x}, y - \bar{y}) \rangle \leq 0 \quad \forall (x, y) \in \mathrm{gr}F$$

($\langle ., . \rangle$ means dual pairing), which yields

$$y^*(y - \bar{y}) \geq 0 \quad \forall y \in F(X). \tag{14}$$

Further, since ℓ is linear and monotone, Proposition 3.17 implies that the functional $\xi_{\bar{y}}$ is convex. Therefore, by Proposition 2.5 (ii), the subdifferential of $\xi_{\bar{y}}$ is understood in the sense of convex analysis and we get

$$y^* \in \partial\xi_{\bar{y}}(\bar{y}) \Leftrightarrow y^*(y - \bar{y}) \leq \xi_{\bar{y}}(y) - \xi_{\bar{y}}(\bar{y}).$$

This and (14) imply that $\xi_{\bar{y}}(y) - \xi_{\bar{y}}(\bar{y}) \geq 0$ for all $y \in F(X)$. According to Theorem 4.2, $(\bar{x}, \bar{y})$ is a weakly nondominated solution of (UVP) w.r.t. the ordering map $\mathcal{D}$. $\square$

Note that according to Remark 3.8, the assertion in the above theorem also is necessary for $(\bar{x}, \bar{y})$ to be a strongly nondominated solution for the problem (UVP).

Remark 4.4. *(Nontriviality of y^* in (13)) As $0 \in D^*F(\bar{x}, \bar{y})(0)$ holds for any pair $(\bar{x}, \bar{y})$, it is of interest to know whether one can find nonzero y^* in (13). Such a question has also been posed for the case with a non-variable ordering structure and a positive answer is obtained for weakly optimal solutions under the assumption that the interior of the ordering cone is nonempty. Returning to the assumptions stated in Theorem 4.3, we see that a similar positive answer do exist for a weakly nondominated solution w.r.t. variable ordering structures. Namely, since $\|\ell(y)\|_* > 1$ (hence, int $\mathcal{D}(y) \neq \emptyset$) for all $y \in F(X)$, we have in particular $\|\ell(\bar{y})\|_* > 1$ which together with $y^* \in \{\ell(\bar{y})\} + B_{Y^*}$ implies that $\|y^*\|_* > 0$ or $y^* \neq 0$.*

One can derive from Theorem 4.3 various versions of the Fermat rule for the problem (UVP) with the objective map being a single-valued map $f : X \to Y$. We illustrate this by formulating just one version for the case when f is strictly differentiable.

Theorem 4.5. *Consider the vector optimization problem*

$$\text{Minimize} \quad f(x) \quad \text{subject to} \quad x \in X\ . \tag{15}$$

Assume that f is strictly differentiable on X and ℓ is Lipschitz on Y. If $(\bar{x}, \bar{y})$ is a nondominated solution w.r.t. the ordering map $\mathcal{D}$ of the problem (15) then

$$0 = [f'(\bar{x})]^*(y^*) \text{ for some } y^* \in \{\ell(\bar{y})\} + B_{Y^*}. \tag{16}$$

Proof. According to Proposition 2.6 we have $D^*f(\bar{x},\bar{y})(y^*) = [f'(\bar{x})]^*(y^*)$. The assertion follows from this and Theorem 4.3. $\square$

We illustrate Theorem 4.5 by the following examples.

Example 4.6. *Let the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $\mathcal{D}(y) = C(\ell(y))$ with $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ as in (4), and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) := (x_1^2, x_2^2)$ for all $(x_1, x_2) \in \mathbb{R}^2$. One can check that $(\bar{x}, \bar{y}) = ((0, 0), (0, 0))$ is a nondominated solution of the problem (15) w.r.t. the ordering map $\mathcal{D}$ and the assumptions of Theorem 4.5 are satisfied. Moreover, since*

$$f'(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = [f'(\bar{x})]^*$$

we get $[f'(\bar{x})]^*(y^*) = 0$ for all $y^* \in \mathbb{R}^2$, which means that (16) in Theorem 4.5 holds.

Example 4.7. *Let the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $\mathcal{D}(y) = C(\ell(y))$ with $\ell : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$\ell(y) := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} y \quad \text{for all} \quad y \in \mathbb{R}^2,$$

and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x) := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ \pi \end{pmatrix} \quad \text{for all} \quad x \in \mathbb{R}^2.$$

Then $f(\mathbb{R}^2) = \{(t, -t + \pi) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$ and ℓ is linear and monotone with $\|\ell(y)\|_* \geq \pi > 1$ for all $y \in f(\mathbb{R}^2)$. One can check that $(\bar{x}, \bar{y})$, with $\bar{x} = (0, 0)$ and $\bar{y} = f(\bar{x}) = (0, \pi)$, is a (weakly) nondominated solution of the problem (15) w.r.t. the ordering map $\mathcal{D}$. For $y^* := (\pi, \pi)$ we have

$$y^* \in \{\ell(\bar{y})\} + B_{\mathbb{R}^2} = \{(\pi, \pi)\} + B_{\mathbb{R}^2}$$

which together with

$$[f'(\bar{x})]^* = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^\top$$

yields that $0 = [f'(\bar{x})]^* y^*$. Thus, (16) in Theorem 4.5 holds.

4.3 Lagrange multiplier rule for the constrained optimization problem

Our next purpose is to establish the Lagrange multiplier rule for the constrained set optimization problem

$$\text{Minimize} \;\; F(x) \;\; \text{subject to} \;\; x \in X \text{ and } G(x) \cap \mathcal{C} \neq \emptyset \;, \tag{CVP}$$

where F and G are set-valued maps from a Banach space X respectively into Banach spaces Y and Z, $\mathcal{C} \subset Z$ is a nonempty set (not necessarily a convex cone). Note that one can consider also an additional geometric constraint $x \in \Omega$ but for the sake of simplicity we restrict ourselves to the case $\Omega = X$. Denote by S the *feasible set* of (CVP), i.e.

$$S = \{x \in X \mid G(x) \cap \mathcal{C} \neq \emptyset\}.$$

We will need the following assumption.

Assumption 4.8. *Let $\bar{x} \in S$, $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x}) \cap \mathcal{C}$. Let the set $\mathcal{C}$ be closed; let F and G be closed and pseudo-Lipschitz around $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{z})$ resp.; and let G be metrically regular around $(\bar{x}, \bar{z})$ relatively to $X \times \mathcal{C}$.*

Recall that F is pseudo-Lipschitz around $(\bar{x}, \bar{y})$ [1] if there exist scalars $r > 0$ and $\gamma > 0$ such that for all $x, x' \in \bar{x} + rB_X$

$$(\bar{y} + rB_Y) \cap F(x) \subset F(x') + \gamma \|x - x'\| B_Y$$

and that G is metrically regular around $(\bar{x}, \bar{z})$ relatively to $X \times \mathcal{C}$ [12] if there exist scalars $r > 0$ and $\gamma > 0$ such that for all $(x, z) \in [(\bar{x} + rB_X) \times (\bar{z} + rB_Z)] \cap (X \times \mathcal{C})$

$$d((x, z); (X \times \mathcal{C}) \cap \text{gr } G) \leq \gamma d(z; G(x)).$$

Our version of the Lagrange multiplier rule for (CVP) reads as follows.

Theorem 4.9. *Let Assumption 4.8 be satisfied and suppose that ℓ is continuous near $\bar{y}$.*

(i) (Necessary condition) If $(\bar{x}, \bar{y})$ is a nondominated solution of (CVP) w.r.t. the ordering map $\mathcal{D}$ then

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*G(\bar{x}, \bar{z})(z^*) \text{ for some } y^* \in \partial \xi_{\bar{y}}(\bar{y}) \text{ and } z^* \in N(\bar{z}; \mathcal{C}). \quad (17)$$

(ii) (Necessary and sufficient conditions) Suppose that $\|\ell(y)\|_* > 1$ (hence, int $\mathcal{D}(y) \neq \emptyset$) for all $y \in F(S)$. Then (17) is necessary for $(\bar{x}, \bar{y})$ to be a weakly nondominated solution of (CVP) w.r.t. the ordering map $\mathcal{D}$. If we assume that the graphs of F and G are convex, ℓ is linear and monotone and $\mathcal{C}$ is convex, then (17) also is sufficient for $(\bar{x}, \bar{y})$ to be a weakly nondominated solution of (UVP) w.r.t. the ordering map $\mathcal{D}$.

(iii) If ℓ is Lipschitz near $\bar{y}$, then one can choose y^* in (17) such that $y^* \in \{\ell(\bar{y})\} + B_{Y^*}$.

Proof. The proof is similar to that of Theorem 3.7 in [12].

(i) Since $(\bar{x}, \bar{y})$ is a nondominated solution of (CVP) w.r.t. the ordering map $\mathcal{D}$, $\bar{y}$ is a nondominated element of $F(S)$. Theorem 3.7 yields that $\xi_{\bar{y}}$ attains its minimum over $F(S)$ at $\bar{y}$. Therefore, $(\bar{x}, \bar{y}, \bar{z})$ is a minimizer of the problem

$$\text{Minimize } q(x, y, z) \text{ subject to } (x, y) \in \text{gr } F \text{ and } (x, z) \in (X \times \mathcal{C}) \cap \text{gr } G,$$

where $q(x, y, z) := \xi_{\bar{y}}(y)$. Then by the Clarke penalization (Proposition 2.4.3 in [9]), regularity assumption and Proposition 2.6 in [12], for some integer $l > 0$ large enough, $(\bar{x}, \bar{y}, \bar{z})$ is an unconstrained minimizer of

$$(x, y, z) \mapsto q(x, y, z) + l\, d((x, y); \text{gr } F) + l\, d((x, z); \text{gr } G) + l\, d((x, z); X \times \mathcal{C}).$$

Observe that the functions $d((x, y); \text{gr } F)$ and $d((x, z); X \times \mathcal{C})$ are Lipschitz. By Proposition 2.5 (iii)-(vi), 0 is in the sum of the subdifferentials and there exist

$$y_1^* \in \partial q(\bar{x}, \bar{y}, \bar{z}) = \partial \xi_{\bar{y}}(\bar{y}),$$

$$(x_2^*, y_2^*) \in l\, \partial d((\bar{x}, \bar{y}); \text{gr } F) \subset N((\bar{x}, \bar{y}); \text{gr } F)$$

$$(x_3^*, z_3^*) \in l\, \partial d((\bar{x}, \bar{z}); \text{gr } G) \subset N((\bar{x}, \bar{z}); \text{gr } G)$$

and

$$(0, z_4^*) \in l\, \partial d((\bar{x}, \bar{z}); X \times \mathcal{C}) \subset N((\bar{x}, \bar{z}); X \times \mathcal{C})$$

such that

$$0 = x_2^* + x_3^*, \qquad 0 = y_1^* + y_2^* \text{ and } 0 = z_3^* + z_4^*.$$

Putting $y^* = y_1^* = -y_2^*$ and $z^* = z_4^* = -z_3^*$, we obtain

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*G(\bar{x}, \bar{z})(z^*)$$

and (17) holds.

(ii) Since the proof of the necessary condition is similar to that of the assertion (i), we shall prove only the sufficient condition. Suppose that the graphs of F, G and the set $\mathcal{C}$ are convex, ℓ is linear and monotone and (17) holds. By (17) one can find elements $x_1^*, x_2^* \in X^*$, $y^* \in \partial \xi_{\bar{y}}(\bar{y})$ and $z^* \in N(\overline{z}; \mathcal{C})$ such that $(x_1^*, -y^*) \in N((\overline{x}, \overline{y}); \text{gr } F)$, $(x_2^*, -z^*) \in N((\overline{x}, \overline{z}); \text{gr } G)$ and

$$x_1^* + x_2^* = 0. \tag{18}$$

Since the graphs of F, G and the set $\mathcal{C}$ are convex, according to the definition of the normal cone of convex analysis, we have

$$\langle (x_1^*, -y^*), (x, y) - (\overline{x}, \overline{y}) \rangle \leq 0 \text{ for all } (x, y) \in \text{gr } F \tag{19}$$

$$\langle (x_2^*, -z^*), (x, z) - (\overline{x}, \overline{z}) \rangle \leq 0 \text{ for all } (x, z) \in \text{gr } G \tag{20}$$

and

$$\langle z^*, z - \overline{z} \rangle \leq 0 \text{ for all } z \in \mathcal{C}. \tag{21}$$

Summarizing (19)-(21) and taking account of (18) we obtain

$$\langle y^*, y - \overline{y} \rangle \geq 0 \text{ for all } y \in F(S). \tag{22}$$

Further, since ℓ is linear and monotone, Proposition 3.17 implies that the functional $\xi_{\bar{y}}$ is convex. Hence the definition of subdifferential in the sense of convex analysis yields

$$y^* \in \partial \xi_{\bar{y}}(\bar{y}) \Leftrightarrow y^*(y - \bar{y}) \leq \xi_{\bar{y}}(y) - \xi_{\bar{y}}(\bar{y})$$

which together with (22) implies $\xi_{\bar{y}}(y) - \xi_{\bar{y}}(\bar{y}) \geq 0$ for all $y \in F(S)$. Theorem 4.2 then implies that $(\bar{x}, \bar{y})$ is a weakly nondominated solution of (CVP) w.r.t. the ordering map $\mathcal{D}$.

(iii) It is a consequence of Proposition 3.15. □

We would like to mention that the functionals (8) and techniques of the works [14, 15] can be used for establishing other optimality conditions for the problem (CVP) and that similar to Remark 4.4, one can find nonzero y^* in (17) under the assumptions that ℓ is Lipschitz and $\|\ell(y)\|_* > 1$ (hence, int $\mathcal{D}(y) \neq \emptyset$) for all $y \in F(S)$. When both the objective and constraint maps are single-valued, one can apply techniques of scalar optimization to deduce the Lagrange multiplier rule for (CVP) with variable ordering structure. We illustrate this by considering the following lipschitz finite-dimensional case. In the theorem and example below, the subdifferential is understood in the sense of Clarke.

Theorem 4.10. *Let $f : \mathbb{R}^n \to \mathbb{R}^s$ and $g = (g_1, ..., g_t) : \mathbb{R}^n \to \mathbb{R}^t$ be single-valued maps. Consider the vector optimization problem*

$$\text{Minimize} \quad f(x) \quad \text{subject to} \quad g_j(x) \leq 0, \; j = 1, ..., t, \quad x \in \mathbb{R}^n. \tag{23}$$

Assume that f and g_j $(j = 1, ..., t)$ are Lipschitz on $\mathbb{R}^n$ and ℓ is Lipschitz on $\mathbb{R}^s$. If $(\bar{x}, \bar{y})$ (here, $\bar{y} = f(\bar{x})$) is a nondominated solution w.r.t. the ordering map $\mathcal{D}$ of the problem (23) then there exist scalars $\mu \geq 0$, $\lambda_j \geq 0$ $(j = 1, ..., t)$ not all zero such that

$$\sum_{j=1}^{t} \lambda_j g_j(\bar{x}) = 0 \tag{24}$$

and

$$0 \in \mu\, \partial(\xi_{\bar{y}} \circ f)(\bar{x}) + \sum_{j=1}^{t} \lambda_j \partial g_j(\bar{x}). \tag{25}$$

Proof. Since $(\bar{x}, \bar{y})$ is a nondominated solution of (23) w.r.t. the ordering map $\mathcal{D}$, $\bar{y}$ is a nondominated element of $f(S)$, where S is the set $\{x \in \mathbb{R}^n \mid \; g_j(x) \leq 0, \; j = 1, ..., t\}$. Theorem 3.7 yields that $\xi_{\bar{y}}$ attains its minimum over $f(S)$ at $\bar{y}$. Therefore, $(\bar{x}, \bar{y})$ is a minimizer of the scalar optimization problem

$$\text{Minimize} \quad (\xi_{\bar{y}} \circ f)(x) \quad \text{subject to} \quad g_j(x) \leq 0, \; j = 1, ..., t, \quad x \in \mathbb{R}^n. \tag{26}$$

One can prove that the composition map $\xi_{\bar{y}} \circ f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz on $\mathbb{R}^n$ using Proposition 3.15. The assertion now follows from Theorem 6.1.1 in [9] applied to the problem (26). □

Below we give an example illustrating Theorem 4.10.

Example 4.11. *Let the cone-valued map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $\mathcal{D}(y) = C(\ell(y))$ with $\ell : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$\ell(y) := \frac{1}{\pi} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} y \quad \text{for all} \quad y \in \mathbb{R}^2.$$

Consider the problem (23) with the maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x_1, x_2) := \left(x_1^2, x_2^2\right)^\top \quad \textit{for all} \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$g(x_1, x_2) := \begin{pmatrix} \pi - (x_1^2 + x_2^2) \\ x_1^2 + x_2^2 - 2\pi \end{pmatrix} \quad \textit{for all} \quad (x_1, x_2) \in \mathbb{R}^2.$$

Observe that f and g are strictly differentiable (hence, they are Lipschitz) and that ℓ is linear and Lipschitz on $\mathbb{R}^2$. Let $\bar{x} = (0, \sqrt{\pi})$ and $\bar{y} = (0, \pi)$. One can check that $(\bar{x}, \bar{y})$ is a nondominated solution of the constrained vector optimization problem (23) w.r.t. the ordering map $\mathcal{D}$. Further, let us calculate the subdifferentials figured in (25). It is easy to check that $\partial(\eta_{\bar{y}} \circ f)(\bar{x}) = \{(0, 2\sqrt{\pi})\}$ and $0 \in \partial\|f(.) - f(\bar{x})\|_{x=\bar{x}}$. Since $\xi_{\bar{y}}(y) = \eta_{\bar{y}}(y) + \|y - \bar{y}\|$, Proposition 2.5 (iv) implies that $(0, 2\sqrt{\pi}) \in \partial(\xi_{\bar{y}} \circ f)(\bar{x})$. We also have $\partial g_1(\bar{x}) = \{(0, -2\sqrt{\pi})\}$. Finally, it is easy to see that the relations (24) and (25) are satisfied with $\mu = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$.

5 Duality results

In this section we establish duality results for the constrained vector optimization problem

$$\text{Minimize} \quad f(x) \quad \text{subject to} \quad x \in \Omega \text{ and } g(x) \in -\mathcal{C}, \tag{CVP'}$$

where f and g are single-valued maps from a nonempty subset Ω of a linear space X respectively into a normed space Y and topological linear space Z, $\mathcal{C} \subset Z$ is a nonempty convex cone. We assume as before that the normed space Y is equipped with a variable ordering structure defined by a cone-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y) = C(\ell(y))$ a BP cone. Duality results for the case with non-variable ordering structure can be found in Chapter 8 of [18].

Denote by S the *feasible set* of (CVP'), i.e.

$$S = \{x \in \Omega \mid \; g(x) \in -\mathcal{C}\}$$

which we assume to be nonempty. We define the dual set by

$$D := \{\bar{y} \in Y \mid \exists u \in \mathcal{C}^* \text{ with } (\xi_{\bar{y}} \circ f + u \circ g)(x) \geq 0 \;\text{ for all } x \in \Omega\}, \tag{27}$$

where $\mathcal{C}^* := \{u \in Z^* \mid \; u(c) \geq 0 \;\forall c \in \mathcal{C}\}$ and $\xi_{\bar{y}}$ is as before the functional $\xi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) + \|y - \bar{y}\|$ for all $y \in Y$.

We first obtain a weak duality theorem which states that each element of the dual set given above is a "lower bound" for the values of the primal problem (CVP'). Here, "weak" means that these "lower bounds" is defined not by the ordering structure but through the corresponding scalarizing function $\xi_{\bar{y}}$.

Theorem 5.1. *Let $\bar{y} \in D$. Then for any $\hat{y} \in f(S)$ one has*

$$\xi_{\bar{y}}(\hat{y}) \geq \xi_{\bar{y}}(\bar{y}) = 0. \tag{28}$$

Moreover, if $\bar{y} \in D \cap f(S)$, i.e. $\bar{y} = f(\bar{x})$ for some $\bar{x} \in S$, and $\|\ell(y)\|_* > 1$ for all $y \in f(S)$, then $\bar{y}$ is a weakly nondominated element of $f(S)$ and thus $(\bar{x}, \bar{y})$ is a weakly nondominated solution of (CVP') w.r.t. the ordering map $\mathcal{D}$.

Proof. By the definition of the set D, there exists $u \in \mathcal{C}^*$ with

$$(\xi_{\bar{y}} \circ f + u \circ g)(x) \geq 0 \quad \text{for all } x \in \Omega.$$

Suppose that $\hat{y} = f(\hat{x})$ for some $\hat{x} \in S$. Then $g(\hat{x}) \in -\mathcal{C}$ and $u \circ g(\hat{x}) \leq 0$ and therefore, $\xi_{\bar{y}} \circ f(\hat{x}) \geq 0$ or $\xi_{\bar{y}}(\hat{y}) \geq 0$. Thus (28) holds. Further, if additionally $\bar{y} = f(\bar{x})$ for some $\bar{x} \in S$ and $\|\ell(y)\|_* > 1$ for all $y \in f(S)$, then (28) and Theorem 3.7 imply that $\bar{y}$ is a weakly nondominated element of the set $f(S)$ and the assertion follows. ☐

Under convexity and stability assumptions we have the following strong duality result.

Theorem 5.2. *Let $\bar{y} = f(\bar{x})$ for some $\bar{x} \in S$. Assume that Ω is convex, f is linear, g is $\mathcal{C}$-convex (i.e. for $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$ one has $g(\lambda x_1 + (1-\lambda)x_2) \in \lambda g(x_1) + (1-\lambda)g(x_2) - \mathcal{C})$), ℓ is linear and monotone and $\|\ell(y)\|_* > 1$ for all $y \in f(S)$. Assume further that the scalar optimization problem*

$$\inf_{x \in S} \xi_{\bar{y}}(f(x)) \tag{29}$$

is stable, i.e.

$$\inf_{x \in S} (\xi_{\bar{y}} \circ f)(x) = \sup_{u \in \mathcal{C}^*} \inf_{x \in \Omega} (\xi_{\bar{y}} \circ f + u \circ g)(x)$$

and the problem on the right hand side has at least one solution. If $(\bar{x}, \bar{y})$ is a weakly nondominated solution of (CVP') w.r.t. $\mathcal{D}$, i.e. $\bar{y}$ is a weakly nondominated element of $f(S)$ w.r.t. $\mathcal{D}$, then $\bar{y}$ is also a weakly maximal element of D w.r.t. $\mathcal{D}$.

Proof. The proof is similar to that of Theorem 8.7 in [18]. Note that under these assumptions the set S is convex. According to Proposition 3.17 the functional $\xi_{\bar{y}}$ is convex and one can easily prove that the composite map $\xi_{\bar{y}} \circ f$ is convex. Since $\bar{y}$ is a weakly nondominated element of $f(S)$, Theorem 3.7 implies

$$0 \leq \xi_{\bar{y}}(f(x)) \quad \text{for all} \quad x \in S.$$

Hence, $\bar{x}$ is a minimal solution of the convex optimization problem (29) which is assumed to be stable. Therefore, one can find $\bar{u} \in \mathcal{C}^*$ with

$$\inf_{x \in S} (\xi_{\bar{y}} \circ f)(x) = \inf_{x \in \Omega} (\xi_{\bar{y}} \circ f + \bar{u} \circ g)(x)$$

and thus

$$(\xi_{\bar{y}} \circ f + \bar{u} \circ g)(\hat{x}) \geq \inf_{x \in S} (\xi_{\bar{y}} \circ f)(x) = \xi_{\bar{y}}(\bar{y}) = 0 \quad \text{for all } \hat{x} \in \Omega,$$

i.e. $\bar{y} \in D$ and hence $\bar{y} \in f(S) \cap D$.

Next we show that $\bar{y}$ is a weakly maximal element of D, i.e., there is no $d \in D$ such that $d - \bar{y} \in \text{int}\mathcal{D}(\bar{y})$. Indeed, according to Theorem 5.1, for $\bar{y} \in f(S)$ and any $d \in D$ one has $\xi_d(\bar{y}) \geq 0$, i.e.

$$\|d - \bar{y}\| \geq \ell(\bar{y})(d - \bar{y}),$$

which implies with Proposition 2.3 that $d - \bar{y} \notin \text{int}\mathcal{D}(\bar{y})$. Therefore, $d - \bar{y} \notin \text{int}\mathcal{D}(\bar{y})$ for all $d \in D$, as it was to be shown. □

Note that in contrast to Theorems 5.1 and 5.2, the duality results in [11] involve linear scalarization functionals but they concern only weakly minimal solutions of the primal problem (CVP') w.r.t. $\mathcal{D}$ and these solutions are related to weakly max-nondominated elements of the dual set.

6 Conclusions

In this work we present for the first time scalarizing functionals completely characterizing *nondominated elements* of a set (Theorem 3.7) and *nondominated solutions* of a vector optimization problem w.r.t. a variable ordering structure for a special case when the cones describing this ordering structure are Bishop-Phelps cones. We showed that these scalarizing functionals have nice properties which allowed us to obtain necessary and sufficient optimality conditions in the form of Fermat rule and Lagrange multiplier rule and some duality results for single- and set-valued vector optimization problems.

Such a characterization for nondominated solutions was unknown in a general case and, in opposition to the notion of minimal solutions w.r.t. a variable ordering structure, known scalarizing functionals for non-variable ordering structures as e.g. examined in [10] cannot be modified to characterize nondominated solution.

We would like to mention that a complete characterization of a minimal element $\bar{y}$ w.r.t. a variable ordering structure similar to that in Theorem 3.7 can be formulated with the help of the following functionals

$$\begin{aligned} \bar{\xi}_{\bar{y}}(y) &:= \ell(\bar{y})(y - \bar{y}) + \|y - \bar{y}\|_{\bar{y}} \quad \text{for all} \quad y \in Y, \\ \bar{\gamma}_{\bar{y}}(y) &:= \ell(\bar{y})(y - \bar{y}) - \|y - \bar{y}\|_{\bar{y}} \quad \text{for all} \quad y \in Y \end{aligned}$$

for some $\bar{y} \in Y$. Hence most of the results presented here can be adapted to the case of minimal elements w.r.t. a variable ordering structure.

Acknowledgment The research was initiated during the second author's stay at the Chair of Applied Mathematics II of the University of Erlangen-Nürnberg under the Georg Forster grant of the Alexander von Humboldt Foundation and was supported by the National Foundation for Science and Technology Development, Vietnam. Especially the second author thanks Prof. J. Jahn for hospitality, advices and help in her work and the authors thank Prof. B. Mordukhovich and Prof. A. Lewis for providing the formulation of the subdifferential in Proposition 2.5(vi) and Prof. F. Heyde for valuable discussions.

References

[1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis* (Wiley, New York, 1984).

[2] D. Baatar and M.M. Wiecek, "Advancing equitability in multiobjective programming", *Comput. Math. Appl.* 52 (2006) 225–234.

[3] K Bergstresser, A. Charnes and P.L. Yu, "Generalization of domination structures and nondominated solutions in multicriteria decision making", *J. Optim. Theory Appl.* 18 (1976) 3–13.

[4] E. Bishop and R.R. Phelps, "The support functionals of a convex set", *Proc. Sympos. Pure Math.* 7 (1962) 27–35.

[5] G.Y. Chen, "Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia Theorem", *J. Optim. Theory Appl.* 74 (1992) 445–456.

[6] G.Y. Chen and X.Q. Yang, "Characterizations of variable domination structures via nonlinear scalarization", *J. Optim. Theory Appl.* 112(1) (2002) 97–110.

[7] G.-Y. Chen, X. Huang and X. Yang, *Vector Optimization, Set-Valued and Variational Analysis* (Springer, Berlin, 2005).

[8] K.L. Chew, "Domination structures in abstract spaces", *Southeast Asian Bulletin of Mathematics*, Proc. first franco-southeast Asian Math. Conference (1979) 190–204.

[9] F.H. Clarke, *Optimization and nonsmooth analysis*, (SIAM, New York, 1990), Reprint, originally published: Wiley, New York, 1983.

[10] G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization* (Springer, Berlin, 2008).

[11] G. Eichfelder, "Vector optimization with a variable ordering structure", *Preprint series of the Institute of Applied Mathematics, Univ. Erlangen-Nürnberg*, No. 328 (2009).

[12] B. El Abdouni, L. Thibault, Optimality conditions for problems with set-valued objectives, J. Applied Analysis 2 (1996) 183–201.

[13] A. Engau, "Variable preference modeling with ideal-symmetric convex cones", *J. Global Optim.* 42 (2008) 295–311.

[14] T.X.D. Ha, "Lagrange multipliers for set-valued optimization problems associated with coderivatives", *J. Math. Anal. Appl.* 311 (2005) 647–663.

[15] T.X.D. Ha, "Optimality conditions for several types of efficient solutions of set-valued optimization problems", in: *Nonlinear Analysis and Variational Problems, Eds. P. Pardalos, Th.M. Rassias, A.A. Khan, Springer* (2009) 305–324.

[16] N.J. Huang, X.Q. Yang and W.K. Chan, "Vector complementarity problems with a variable ordering relation", *European Journal of Operational Research* 176 (2007) 15–26.

[17] A.D. Ioffe, "Approximate subdifferentials and applications 3: Metric theory", *Mathematica* 36 (1989) 1–38.

[18] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions*

[19] J. Jahn, "Bishop-Phelps Cones in Optimization", *International Journal of Optimization: Theory, Methods and Applications* 1 (2009) 123-139.

[20] R. Kasimbeyli,"A nonlinear cone separation theorem and scalarization in nonconvex vector optimization", *SIAM J. Optim.* 20 (2010) 1591–1619.

[21] M.A. Krasnoselskii, *Positive solutions of operator equations* (Noordhoff, Groningen, 1964).

[22] S.J. Li and M.H. Li, "Levitin-Polyak well-posedness of vector equilibrium problems", *Math. Meth. Oper. Res.* 69 (2009) 125–140.

[23] B.S. Mordukhovich, "Maximum principle in problems of time optimal control with nonsmooth contraints", *J. Appl. Math. Mech.* 40 (1976), 960–969.

[24] B.S. Mordukhovich, "Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems", *Soviet Math. Dokl.* 22 (1980) 526–530.

[25] B.S. Mordukhovich, *Variational Analysis and Generalized differentiation, I: Basic Theory, Grundlehren Series (Fundamental Principles of Mathematical Sciences) 330* (Springer, Berlin, 2006).

[26] B.S. Mordukhovich, *Variational Analysis and Generalized differentiation, II: Applications, Grundlehren Series (Fundamental Principles of Mathematical Sciences) 331* (Springer, Berlin, 2006).

[27] M. Petschke, "On a theorem of Arrow, Barankin, and Blackwell", *SIAM J. Control Optim.* 28 (1990) 395–401.

[28] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization* (Academic Press, London, 1985).

[29] M. Wacker, *Multikriterielle Optimierung bei Registrierung medizinischer Daten* (Diplomarbeit, Univ. Erlangen-Nürnberg, 2008).

[30] P. Weidner, "Dominanzmengen und Optimalitätsbegriffe in der Vektoroptimierung", *Wissenschatl. Zeitschrift Ilmenau* 31 (1985) 133–146.

[31] M.M. Wiecek, "Advances in cone-based preference modeling for decision making with multiple criteria", *Decision Making in Manufacturing and Services* 1 (2007) 153–173.

[32] P.L. Yu, "Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives", *J. Optim. Theory Appl.* 14 (1974) 319–377.

[33] P.L. Yu, *Multiple-criteria Decision Making: Concepts, Techniques and Extensions* (Plenum Press, New York, 1985).

PREPRINTS

DES INSTITUTS FÜR ANGEWANDTE MATHEMATIK
DER UNIVERSITÄT ERLANGEN-NÜRNBERG

ZULETZT ERSCHIENENE BEITRÄGE:

314 M. Bischoff , K. Dächert: *Allocation Search Methods for a Generalized Class of Location-Allocation Problems*

315 P. Knabner: *Open Questions and Research Directions in Parameter Identification for Multicomponent Reactive Transport in Porous Media*

316 G. Eichfelder , J. Jahn: *Set-Semidefinite Optimization*

317 M. Stingl , M. Kočvara , G. Leugering: *A Sequential Convex Semidefinite Programming Algorithm for Multiple-Load Free Material Optimization*

318 J. Jahn , E. Schaller: *A Global Solver for Multiobjective Nonlinear Bilevel Optimization Problems*

319 M. Stingl , M. Kočvara , G. Leugering: *Free Material Optimization with Control of the Fundamental Eigenfrequency*

320 G. Eichfelder: *Solving Nonlinear Multiobjective Bilevel Optimization Problems with Coupled Upper Level Constraints*

321 O. Museyko , G. Leugering , M. Stiglmayr , K. Klamroth: *On the application of the Monge-Kantorovich problem to image registration*

322 C. Hopfgartner , I. Scholz , M. Gugat , G. Leugering , J. Hornegger: *Intensity based Three-Dimensional Reconstruction with Nonlinear Optimization*

323 P. Hastreiter , G. Leugering , O. Museyko: *A free-discontinuity problem for the registration of images with incomplete information*

324 N. Suciu , C. Vamoş , H. Vereecken , K. Sabelfeld , P. Knabner: *Dependence on Initial Conditions, Memory Effects, and Ergodicity of Transport in Heterogeneous Media*

325 J. Jahn: *Bishop-Phelps Cones in Optimization*

326 J. Hoffmann: *Results of the GdR MoMaS Reactive Transport Benchmark with RICHY2D*

327 A. Khludnev , G. Leugering: *On Elastic Bodies with Thin Rigid Inclusions and Cracks*

328 G. Eichfelder: *Vector Optimization with a Variable Ordering Structure*

329 M. A. Fontelos , G. Grün , S. Jörres: *On a Phase-Field Model for Electrowetting and Other Electrokinetic Phenomena*

330 J. Haslinger , G. Leugering , M. Kočvara , M. Stingl: *Multidisciplinary Free Material Optimization*

331 B. Schmidt , M. Stingl , D. A. Berry , M. Döllinger: *Material parameter optimization in a multi-layered vocal fold model*

332 M. Kaiser, A. Thekale: *Solving nonlinear feasibility problems with expensive functions*

333 I. Bomze , G. Eichfelder: *Copositivity detection by difference-of-convex decomposition and ω-subdivision*

334 M. Prechtel, G. Leugering, P. Steinmann, M. Stingl: *Towards optimization of crack resistance of composite materials by adjustment of fiber shapes*

335 A. M. Khludnev , G. Leugering: *Optimal control of cracks in elastic bodies with thin rigid inclusions*

336 M. Prechtel, P. Leiva Ronda, R. Janisch, A. Hartmaier, G. Leugering, P. Steinmann, M. Stingl: *Simulation of fracture in heterogeneous elastic materials with cohesive zone models*

337 E. Marchand: *Combined Deterministic-Stochastic Sensitivity Analysis; Application to Uncertainty Analysis.*

338 G. Eichfelder , T.X.D. Ha: *Optimality conditions for vector optimization problems with variable ordering structures*