**FRIEDRICH-ALEXANDER-UNIVERSITÄT ERLANGEN-NÜRNBERG**
**INSTITUT FÜR ANGEWANDTE MATHEMATIK**

# Cone-valued maps in optimization

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No. 348 2011

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ISSN 1435-5833, 2011, No. 348

# Cone-valued maps in optimization

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June 24, 2011

**Abstract**

Cone-valued maps are special set-valued maps where the image sets are cones. Such maps play an important role in optimization, for instance in optimality conditions or in the context of Bishop-Phelps cones. In vector optimization with variable ordering structures, they have recently attracted even more interest. We show that classical concepts for set-valued maps as cone-convexity or monotonicity are not appropriate for characterizing cone-valued maps. For instance, every convex or monotone cone-valued map is a constant map. Similar results hold for cone-convexity, sublinearity, upper semicontinuity or the local Lipschitz property. We also propose therefore new concepts for cone-valued maps.

**Keywords:** cone-valued map; set-valued map; convex; linear; semicontinuous; monotone

**AMS Subject Classification:** 90C30;90C29.

## 1 Introduction

Cone-valued maps are special set-valued maps where the images are cones. Thus, in the following, we consider set-valued maps $F: S \to 2^Y$ with $X$, $Y$ real linear topological spaces and $S$ a nonempty subset of $X$, such that $y \in F(x)$ and $\lambda \geq 0$ imply $\lambda y \in F(x)$ for all $x \in S$. These maps play an important role in optimization. For instance, optimality conditions based on contingent cones [19, Theorem 3.48] associate to each $x \in S$, with $S$ a nonempty subset of a real normed space, the contingent cone $T(S,x)$ to the set $S$ at $x$. The operation taking the polar cone, also a cone-valued map, was studied in [24]. In [6] the continuity of the cone-valued map associating to each $x$ the normal cone to the related level set of a function $f$ is discussed. If $f$ is convex and finite, then this cone-valued map equals the map associating to each $x$ the cone generated by the subgradient of $f$ at $x$. Also, the map associating to any sublinear functional its epigraph is a set-valued map with each image a convex cone. In a recent work [18] in optimization Bishop-Phelps

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cones in a real normed space $(Y, \|\cdot\|_Y)$ were examined: for an arbitrary continuous linear functional $\phi$ on $Y$ the convex cone

$$\mathcal{C}(\phi) := \{y \in Y \mid \|y\|_Y \leq \phi(y)\} \tag{1}$$

is denoted a Bishop-Phelps cone. This again describes a cone-valued map $\mathcal{C}\colon Y^* \to 2^Y$.

From a more applied point of view, in [19, p. 373] an example of a cone-valued map is given which describes the emission cone of ultrasonic sensors of autonomous transportation robots used for determining distances to obstacles.

Cone-valued maps are also of special importance in vector optimization with a variable ordering structure [8, 9, 12, 13, 11]. There, one assumes that the preferences of a decision maker are given by a cone-valued map $\mathcal{D}\colon Y \to 2^Y$ which associates to each element $y$ in the image space of the vector optimization problem a convex cone of dominated or preferred directions: $\mathcal{D}(y) = \{d \in Y \mid y - d \text{ is preferred to } y\} \cup \{0_Y\}$ or $\mathcal{D}(y) = \{d \in Y \mid y + d \text{ is dominated by } y\} \cup \{0_Y\}$. For studying characterizations of optimal elements and scalarizations for such vector optimization problems, it is important to have information about the properties of these cone-valued maps describing the ordering relation. For instance, in [8] the linearity of a cone-valued map was assumed for such considerations, or in [13, Prop. 3.17], for a cone-valued map with images Bishop-Phelps cones, linearity and monotonicity was required for the map defining the linear functionals $\phi$ of the Bishop-Phelps cones. Such cone-valued maps appear also in the context of vector variational inequalities [1, 25, 7], vector complementarity problems [16] and vector equilibrium problems [21], see also Section 2.1 in [11] and the references therein.

We discuss in this paper for several well-known properties defined for set-valued maps whether they are also applicable for cone-valued maps, i.e. whether assuming them is a too strong or a too weak assumption. For instance, convexity of a cone-valued map implies that the map is constant, i.e. this assumption is too strong. In case of non-appropriateness of the notions we propose new, more applicable, concepts. By that we provide tools for describing cone-valued maps which is necessary for instance when dealing with variable ordering structures. Similar considerations for set-valued maps with bounded images or with the property $F(0_X) = \{0_Y\}$ have been taken in [5] and [15], see also [22] and the references therein, where it was shown, for instance, that such maps are single-valued under properties as convexity or linearity.

In the mentioned applications it is generally assumed – additionally to $F(x)$ being a cone – that $F(x)$ is convex, i.e. $F(x) + F(x) \subset F(x)$, and closed for all considered $x$. Thus we pay especially attention to such maps. We also provide several examples of cone valued maps $F\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ as they would be of interest for modeling preferences in biobjective optimization with a variable ordering structure, i.e. with convex images and with the nonnegative orthant, or, at least, with directions pointing towards the nonnegative orthant, included in the images. Further, we concentrate especially on those properties of cone-valued maps which are of interest for the study of variable ordering structures. In Section 2 we discuss convexity and linearity properties as cone-convexity and sublinearity. In Section 3 continuity is examined while in Section 4 the property of monotonicity is discussed.

# 2 Convexity and linearity of cone-valued maps

Let $X$, $Y$ be real linear topological spaces with $S$ a nonempty algebraically open convex subset of $X$, if not stated otherwise, and $C \subset Y$ a convex cone. If we consider set-valued maps $F$ on a set $S$ or a space $X$ we always assume $F(x)$ to be nonempty for any $x \in S$ or $x \in X$, respectively. $F(S)$ denotes the image of $S$ under $F$, i.e. $F(S) = \bigcup_{x \in S} F(x)$.

## 2.1 Cone-convex cone-valued maps

We start by recalling the definition of a $C$-convex set-valued map for $C$ a convex cone, see for instance, [20, Def. 2.1],[2, Lemma 2.1.2] as well as [3] and the references therein.

**Definition 2.1.** *A set-valued map $F\colon S \to 2^Y$ is called $C$-convex, if for all $x^1, x^2 \in S$ and $\lambda \in [0, 1]$*

$$\lambda F(x^1) + (1 - \lambda)F(x^2) \subset F(\lambda x^1 + (1 - \lambda)x^2) + C. \tag{2}$$

*If $F$ is $\{0_Y\}$-convex, then $F$ is called convex.*

Assuming the convexity of the cone-valued map defining the variable ordering structure of a vector optimization problem, the convexity of a nonlinear scalarization functional [11, Subsection 4.4.2.2], a modification of the Tammer-Weidner functional [14], can be shown. However, we will see that convexity of a cone-valued map is a too strong assumption. We start by some first characterizations.

**Lemma 2.2.** *If a set-valued map $F\colon S \to 2^Y$ is $C$-convex, then $F(x) + C$ is convex for all $x \in S$.*

*Proof.* For any $x \in S$ we obtain from (2) by choosing $x^1 = x^2 = x$ and by using that $C$ is a convex cone, that

$$\lambda(F(x) + C) + (1 - \lambda)(F(x) + C) \subset F(x) + C \quad \text{for all } \lambda \in [0, 1].$$

□

**Lemma 2.3.** *Let $F\colon S \to 2^Y$ be a cone-valued map. If $F$ is $C$-convex, then*

$$F(x) + C = F(S) + C \qquad \textit{for all } x \in S. \tag{3}$$

*Proof.* It holds $\lambda F(x) = F(x)$ and $0_Y \in F(x)$ for all $x \in S$ and all $\lambda > 0$. Thus (2) implies

$$F(x^2) \subset F(\lambda x^1 + (1 - \lambda)x^2) + C \tag{4}$$

for all $x^1, x^2 \in S$, $\lambda \in (0, 1)$. Let $\bar{x} \in \operatorname{cor}(S) = S$ be given and choose $x \in S$ arbitrarily. Then there is some $\mu > 0$ such that $x^1 := \bar{x} + \mu(\bar{x} - x) \in S$. For $x^2 := x$ and $\lambda := 1/(1+\mu)$, (4) implies

$$F(x) \subset F\left(\frac{1}{1+\mu}(\bar{x} + \mu(\bar{x} - x)) + \frac{\mu}{1+\mu}x\right) + C = F(\bar{x}) + C$$

and thus

$$F(S) \subset F(\bar{x}) + C.$$

As $S$ is algebraically open, there is some $\nu > 0$ such that $x^1 := x + \nu(x - \bar{x}) \in S$. For $x^2 := \bar{x}$ and $\lambda := 1/(1+\nu)$, (4) implies

$$F(\bar{x}) \subset F\left(\frac{1}{1+\nu}\,(x + \nu(x - \bar{x})) + \frac{\nu}{1+\nu}\bar{x}\right) + C = F(x) + C,$$

i.e. $F(S) + C \subset F(\bar{x}) + C \subset F(x) + C \subset F(S) + C$ and the assertion is proven. $\square$

The following theorem states that depending on the cone $C$ the $C$-convexity of a cone-valued map might be a too strong assumption.

**Theorem 2.4.** *Let $F\colon S \to 2^Y$ be a cone-valued map with $F(x)$ a convex cone for all $x \in S$ and let*

$$C \subset \bigcap_{x \in S} F(x).$$

*If $F$ is $C$-convex, then $F$ is a trivial cone-valued map, i.e. $F(x) = K$ for all $x \in S$ and for some convex cone $K \subset Y$.*

*Proof.* For any $x \in S$ it holds, because of the assumption on $C$ and as $F(x)$ is a convex cone and $0_Y \in C$,

$$F(x) + C \subset F(x) + F(x) \subset F(x) \subset F(x) + C,$$

i.e. $F(x) = F(x) + C$. Using the result of Lemma 2.3 this implies

$$F(x) = F(x) + C = F(S) + C = \bigcup_{x \in S} (F(x) + C) = \bigcup_{x \in S} F(x) = F(S) =: K$$

for all $x \in S$. $\square$

**Corollary 2.5.** *Let $F\colon S \to 2^Y$ be a cone-valued map. If $F$ is convex, then $F$ is a trivial cone-valued map, i.e. $F(x) = K$ for all $x \in S$ and for some convex cone $K \subset Y$.*

*Proof.* By Lemma 2.2, as $F$ is convex, i.e. $F$ is $\{0_Y\}$-convex, $F(x)$ is a convex cone for all $x \in S$ and with Theorem 2.4 the result follows. $\square$

On the other hand, for $F(x) \subset C$ for all $x \in S$, $F$ is always $C$-convex, i.e. $C$-convexity is a redundant assumption.

**Theorem 2.6.** *Let $F\colon S \to 2^Y$ be a cone-valued map and let $F(x) \subset C$ for all $x \in S$. Then $F$ is $C$-convex.*

*Proof.* For any $x^1, x^2 \in S$, $\lambda \in (0,1)$ it holds

$$\begin{aligned} \lambda\, F(x^1) + (1-\lambda)\, F(x^2) &= F(x^1) + F(x^2) \subset C + C \subset C \subset \{0_Y\} + C \\ &\subset F(\lambda x^1 + (1-\lambda)x^2) + C. \end{aligned}$$

$\square$

The following lemma states that a modification of (2) in the sense of a restriction on bounded subsets of the cones – the cones intersected with the unit ball –, instead of the unbounded cones, delivers still not an appropriate concept for $C$-convexity of a cone-valued map $F$ with convex cones as image sets and $C \subset F(x)$ for all $x \in S$.

**Lemma 2.7.** *Let $(Y, \|\cdot\|_Y)$ be a normed space and let $F\colon S \to 2^Y$ be a cone-valued map with $F(x)$ a convex cone and $C \subset F(x)$ for all $x \in S$. If $F$ satisfies for all $x^1, x^2 \in S$ and $\lambda \in [0,1]$*

$$\lambda \left(F(x^1) \cap B_Y\right) + (1-\lambda) \left(F(x^2) \cap B_Y\right) \subset \left(F(\lambda x^1 + (1-\lambda)x^2) \cap B_Y\right) + C \tag{5}$$

*with $B_Y \subset Y$ the closed unit ball, then $F(x) = K$ for all $x \in S$ and for some convex cone $K \subset Y$.*

*Proof.* $0_Y \in F(x) \cap B_Y$ for any $x \in S$, (5) and the assumption on $C$ imply for all $x^1, x^2 \in S$, $\lambda \in (0,1)$:

$$\begin{array}{rcl}(1-\lambda)(F(x^2) \cap B_Y) & \subset & (F(\lambda x^1 + (1-\lambda)x^2) \cap B_Y) + C \\ & \subset & F(\lambda x^1 + (1-\lambda)x^2) + F(\lambda x^1 + (1-\lambda)x^2) \\ & \subset & F(\lambda x^1 + (1-\lambda)x^2).\end{array}$$

Considering the cone generated by the set $(1-\lambda)(F(x^2) \cap B_Y)$ yields

$$F(x^2) \subset F(\lambda x^1 + (1-\lambda)x^2).$$

With the same arguments as in the proof of Lemma 2.3 we obtain for an arbitrary $\bar{x} \in S$ that $F(\bar{x}) = F(S) =: K$. ☐

For overcoming these drawbacks, we propose to modify the definition of $C$-convexity of a cone-valued map $F$ as follows: instead of considering the cones $F(x)$ we give a definition via the bases of the cones $F(x)$, to be more concrete, via the vector-valued map associating to each $x \in S$ the linear functional $\ell_x$ defining a base of $F$. Recall that a nonempty convex subset $B$ of a nontrivial convex cone $K \subset Y$ is called a base of $K$ if every $y \in K \setminus \{0_Y\}$ has a unique representation $y = \lambda b$ for some $\lambda > 0$ and some $b \in B$. Any cone having a base is pointed and a subset $B$ of $K$ is a base of $K$ if and only if there is a continuous linear functional $\ell \in K^{\#} := \{y^* \in Y^* \mid y^*(y) > 0 \;\text{ for all } y \in K \setminus \{0_Y\}\}$ with $B = \{y \in K \mid \ell(y) = 1\}$. According to the Krein-Rutman-Theorem every nontrivial closed pointed convex cone in a real separable normed space has a base. Recall also that a vector-valued map $f\colon S \to Y$ is $C$-convex for some convex cone $C \subset Y$ if for all $x^1, x^2 \in S$ and $\lambda \in [0,1]$

$$\lambda f(x^1) + (1-\lambda) f(x^2) \in \{f(\lambda x^1 + (1-\lambda)x^2)\} + C$$

and that the dual cone $K^*$ to a cone $K \subset Y$ is defined by $K^* := \{y^* \in Y^* \mid y^*(y) \geq 0 \text{ for all } y \in K\}$.

**Definition 2.8.** *Let $S$ be a nonempty convex subset of $X$ and let $F\colon S \to 2^Y$ be a cone-valued map with $F(x)$ a pointed convex cone having a base for each $x \in S$. If there is a $C^*$-convex map $\ell\colon S \to Y^*$ with*

$$B(x) := \{y \in F(x) \mid \ell(x)(y) = 1\} \tag{6}$$

*is a base of $F(x)$ for all $x \in S$, then $F$ is called $C$-baseconvex.*

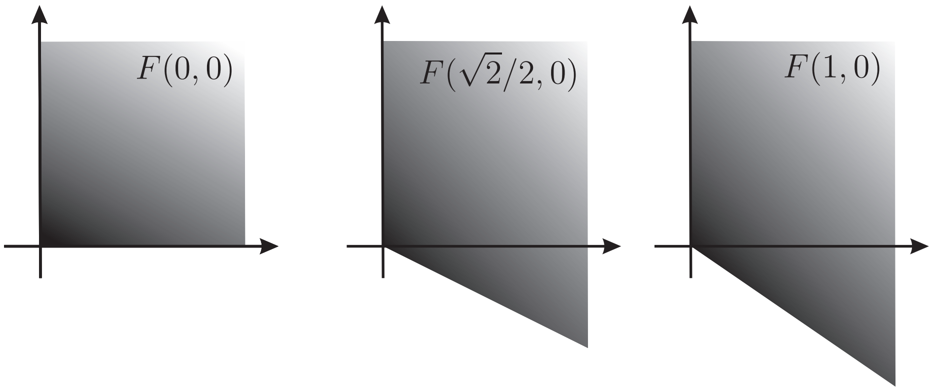


Figure 1: The cones $F(0,0)$, $F(\sqrt{2}/2,0)$ and $F(1,0)$ of Example 2.9.

**Example 2.9.** *Let $X$ be a real linear topological space with $S$ a nonempty convex subset and let $(Y, \|\cdot\|_Y)$ be a normed space. Let $\ell\colon S \to Y^*$ be an arbitrary $C^*$-convex map with $\|\ell(x)\|_* > 1$ for all $x \in S$. Here, $\|\cdot\|_*$ denotes the dual norm of $\|\cdot\|_Y$. We consider the cone-valued map $F\colon S \to 2^Y$ with images Bishop-Phelps cones, cf. (1), defined by*

$$F(x) = \mathcal{C}(\ell(x)) = \{y \in Y \mid \|y\|_Y \leq \ell(x)(y)\} \text{ for all } x \in S.$$

*The sets $F(x)$ are nontrivial closed pointed convex cones [18, Prop. 2.2 and 2.3]. According to [18, Prop. 2.18], $\ell(x)$ defines by (6) a base for $F(x)$ and thus $F$ is $C$-baseconvex on $S$.*

*For instance, let $X = Y = \mathbb{R}^2$ be Euclidean spaces with $C = \mathbb{R}^2_+ = C^*$. Define $\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\ell(x_1,x_2) = (1+x_1^2, 1+x_2^2)^\top$ for all $(x_1,x_2) \in \mathbb{R}^2$. Then the cones $F(x) = \mathcal{C}(\ell(x))$ have a base defined by $\ell(x)$ and $\ell$ is $C^*$-convex and thus $F$ is $C$-baseconvex on $\mathbb{R}^2$. As $\|\ell(x)\|_2 > 1$ for all $x \in \mathbb{R}^2$, the cones $F(x)$ have a nonempty interior [18, Theorem 2.5.(b)]. See Figure 1 for an illustration of some of the cones. Additionally, $C = \mathbb{R}^2_+ \subset F(x)$ for all $x \in \mathbb{R}^2$. As $F(x) \not\equiv K$ for some cone $K$, according to Theorem 2.4, $F$ is not $C$-convex.*

The relevance of properties of cone-valued maps defined via the vector-valued function determining a base of the cones becomes clearer by the notions of baselinear and basemonotone, which will be introduced in the next sections.

## 2.2 Convex like and quasiconvex cone-valued maps

In the following we discuss two weaker convexity concepts which are appropriate also for cone-valued maps. In [20, Def. 2.1] a convex like set-valued map $F\colon S \to 2^Y$ w.r.t. a convex cone $C \subset Y$ was defined by the following:

**Definition 2.10.** *A set-valued map $F\colon S \to 2^Y$ is called convex like w.r.t. $C$ if*

$$\lambda F(x^1) + (1-\lambda)F(x^2) \subset F(S) + C \quad \text{for all } x^1, x^2 \in S \text{ and } \lambda \in (0,1).$$

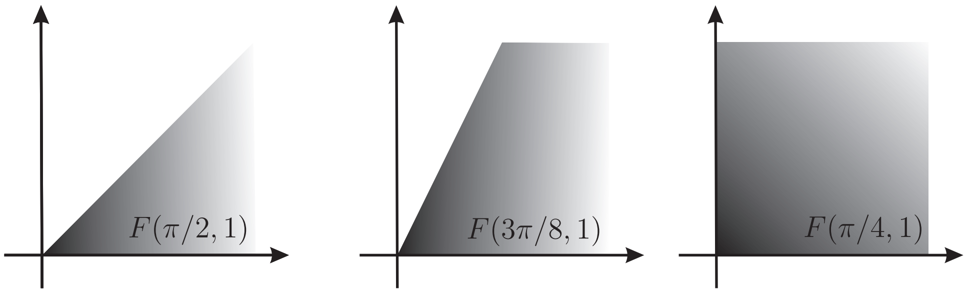


Figure 2: The cones $F(\pi/2,1)$, $F(3\pi/8,1)$ and $F(\pi/4,1)$ of Example 2.11.

According to [20, Prop. 2.2], $F$ is convex like w.r.t. $C$ if and only if $F(S)+C$ is a convex set, which is not a strong assumption if we assume $F$ to be a cone-valued map with $F(x)$ a convex cone for all $x \in S$. In the next example a convex like cone-valued map is given:

**Example 2.11.** *For the cone-valued map $F\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by*

$$F(x_1,x_2) := \begin{cases} \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0,\pi/4]\} & \text{if } x_1 \geq \pi/2, \\ \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, \frac{\pi}{2}+\frac{\pi}{4}-x_1]\} & \text{if } x_1 \in (\pi/4,\pi/2), \\ \mathbb{R}^2_+ & \text{if } x_1 \leq \pi/4, \end{cases}$$

*see Figure 2, it holds $F(\mathbb{R}^2) = \mathbb{R}^2_+$ and thus $F$ is, for instance, convex-like w.r.t. any convex cone $C \subset \mathbb{R}^2_+$ or $\mathbb{R}^2_+ \subset C$.*

Also the quasiconvexity [20] of a cone-valued map is an appropriate notion:

**Definition 2.12.** *A set-valued map $F\colon S \to 2^Y$ is called $C$-quasiconvex if*

$$(F(x^1)+C) \cap (F(x^2)+C) \subset F(\lambda x^1 + (1-\lambda)x^2) + C \quad \text{for all } x^1, x^2 \in S \text{ and } \lambda \in (0,1).$$

If $F$ is $C$-convex then it is also $C$-quasiconvex [20, Prop. 2.1]. Thus, by Theorem 2.6, a cone-valued map $F$ with $F(x) \subset C$ for all $x \in S$ is also $C$-quasiconvex. In contrast to $\{0_Y\}$-convex cone-valued maps also nontrivial $\{0_Y\}$-quasiconvex maps exist:

**Example 2.13.** *Consider $X$, $Y$ and $F$ as specified in Example 2.11. For arbitrary $u = (u_1,u_2), v = (v_1,v_2) \in \mathbb{R}^2$ with $u_1 \leq v_1$ it holds $F(v) \subset F(u)$. Thus, for arbitrary $u, v \in \mathbb{R}^2$ and arbitrary $\lambda \in (0,1)$ we have w.l.o.g. $u_1 \leq v_1$ and then also $\lambda u_1 + (1-\lambda)v_1 \leq v_1$ and we conclude*

$$F(u) \cap F(v) = F(v) \subset F(\lambda u + (1-\lambda)v)$$

*and thus $F$ is $\{0_Y\}$-quasiconvex.*

## 2.3 Linear cone-valued maps

In [8, p.3] the linearity of a cone-valued map $F\colon Y \to 2^Y$ with $F(y)$ a convex cone for all $y \in Y$ was presumed for being able to show the convexity of a nonlinear scalarization functional in vector optimization with variable ordering structures. This is a stronger assumption than convexity and according to Corollary 2.5 it is a too strong assumption. However note that linearity is already a strong assumption for set-valued maps. For instance, $F\colon X \to 2^Y$ linear implies $F(X)$ to be a subspace and $F(0_X) = \{0_Y\}$ and if $F$ is constant and linear, then $F(x) \equiv \{0_Y\}$.

**Lemma 2.14.** *Let $F\colon X \to 2^Y$ be a cone-valued map and assume*

$$\mu F(x^1) + \lambda F(x^2) = F(\mu x^1 + \lambda x^2) \quad \text{for all} \quad x^1, x^2 \in X \text{ and } \mu, \lambda \in \mathbb{R}. \tag{7}$$

*Then $F(x) = \{0_Y\}$ for all $x \in X$.*

*Proof.* Condition (7) implies that $F$ is $C$-convex with $C = \{0_Y\}$, i.e. convex. Corollary 2.5 implies $F(x) = K$ for all $x \in X$ with $K$ a convex cone. Then (7) reads as $\mu K + \lambda K = K$ for all $\mu, \lambda \in \mathbb{R}$ and $\mu = \lambda = 0$ imply $K = \{0_Y\}$. □

Therefore, analogously to Definition 2.8, we modify the concept of linearity and adapt it to cone-valued maps by defining linearity via the linearity of a vector-valued map defining a base for each cone $F(x)$.

**Definition 2.15.** *Let $S$ be a nonempty subset of $X$ and let $F\colon S \to 2^Y$ be a cone-valued map with $F(x)$ a pointed convex cone having a base for each $x \in S$. If there is a linear map $\ell\colon X \to Y^*$ with (6) is a base for $F(x)$ for all $x \in S$, then $F$ is called baselinear.*

Baselinear cone-valued maps are important for formulating sufficient optimality conditions for vector optimization problems with a variable ordering structure. In [13] such variable ordering structures defined by cone-valued maps with images Bishop-Phelps cones were studied. A scalarization functional was introduced which allows a complete characterization of optimal elements. Assuming baselinearity and basemonotonicity, see Definition 4.5 below, of the ordering map, the convexity of the scalarization functional was shown and by that sufficient optimality conditions of Fermat and Lagrange type were derived.

**Example 2.16.** *We consider again a cone-valued map $F$ with images Bishop-Phelps cones, compare Example 2.9. Let $X = Y = \mathbb{R}^2$ be Euclidean spaces and $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 1\}$. Define $\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\ell(x) = Lx$ for all $x \in \mathbb{R}^2$ with*

$$L := \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$$

*As $\ell$ is a linear map, $F\colon S \to 2^{\mathbb{R}^2}$ with*

$$F(x) = \{y \in \mathbb{R}^2 \mid \|y\|_2 \leq (Lx)^\top y\} \quad \text{for all } x \in S$$

*is baselinear.*

However, baselinear is still a strong assumption as it cannot be defined for any set $S$ with $0_X \in S$: As linearity of $\ell$ implies $\ell(0_X) = 0_{Y^*}$ this would mean $B(x) = \emptyset$ in contradiction to $B(x)$ defining a base.

## 2.4 Sublinear cone-valued maps

In this subsection we study for a cone-valued map $F\colon X \to 2^Y$ the properties of subadditivity and positive homogeneity, which are related to the notion of a convex process as defined in [2, Lemma 2.1.2],[9, Def. 1.70]. As before, let $C \subset Y$ be a convex cone.

**Definition 2.17.** *A set-valued $F\colon X \to 2^Y$ with*

$$F(x^1) + F(x^2) \subset F(x^1 + x^2) + C \quad \textit{for all} \quad x^1, x^2 \in X \tag{8}$$

*and*

$$F(\lambda x) = \lambda F(x) \quad \textit{for all} \quad \lambda > 0 \quad \textit{and}\ x \in X \tag{9}$$

*is called $C$-sublinear.*

A set-valued map satisfying (9) together with $0_Y \in F(0_X)$ is called a process.

**Definition 2.18.** *A $\{0_Y\}$-sublinear set-valued map $F\colon X \to 2^Y$ with $0_Y \in F(0_X)$ is called a convex process.*

It is obvious that a $C$-sublinear set-valued map is $C$-convex. Also, one can easily verify that the $C$-convexity of a cone-valued map together with (9) implies (8). With Theorem 2.4 and Corollary 2.5 we conclude:

**Corollary 2.19.** *Let $F\colon X \to 2^Y$ be a $C$-sublinear cone-valued map.*

*(a) If $F(x)$ is a convex cone and $C \subset F(x)$ for all $x \in X$, then $F(x) = K$ for all $x \in X$ and for some convex cone $K \subset Y$.*

*(b) If $C = \{0_Y\}$, i.e. $F$ is a convex process, then $F(x) = K$ for all $x \in X$ and for some convex cone $K \subset Y$.*

A cone-valued convex process $F$ satisfies (8) with $C = \{0_Y\}$ and (9), which implies that $F$ is convex and thus that $F$ is a constant map. But already (8) with $C = \{0_Y\}$ implies this fact for arbitrary set-valued maps with $0_Y \in F(x)$ for all $x \in X$ and thus in particular for cone-valued maps:

**Lemma 2.20.** *Let $F\colon X \to 2^Y$ be a set-valued map with $0_Y \in F(x)$ for all $x \in X$ and let $F$ satisfy (8) with $C = \{0_Y\}$. Then $F(x) = \hat{F}$ for all $x \in X$ and for some set $\hat{F} \subset Y$.*

*Proof.* For arbitrary $x \in X$ choose $x^1 := -x^2 := x$ in (8). Then $F(x) + F(-x) \subset F(0_X)$ and thus $F(x) \subset F(0_X)$ for all $x \in X$. We obtain

$$F(0_X) \subset F(X) =: \hat{F} \subset F(0_X).$$

This results in $F(0_X) = \hat{F}$. Setting $x^1 := x$, $x^2 := 0_X$, (8) leads to $F(x) + F(0_X) \subset F(x)$, i.e. $F(0_X) \subset F(x)$. Summarizing this we obtain $\hat{F} = F(0_X) \subset F(x) \subset \hat{F}$ for all $x \in X$. ∎

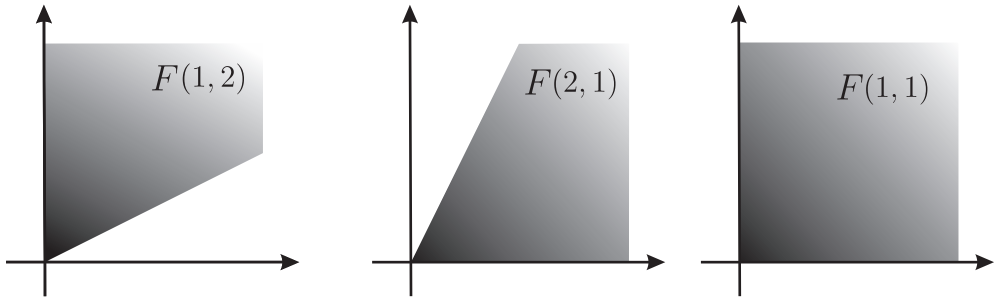


Figure 3: The cones $F(1,2)$, $F(2,1)$ and $F(1,1)$ of Example 2.21.

For the cone $C$ quite large, $C$-sublinear cone-valued maps of course exist: if $F(x) \subset C$ for all $x \in X$, then (8) is always satisfied. Further one could define basesublinearity analogously to baselinearity. But note, nontrivial cone-valued maps which satisfy (9) exist:

**Example 2.21.** *Consider the cone-valued map* $F\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ *defined by*

$$F(x) := \begin{cases} \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [\bar{\varphi}_x - \pi/4, \bar{\varphi}_x + \pi/4] \cap [0, \pi/2]\} & \text{if } x \neq 0_{\mathbb{R}^2}, \\ \mathbb{R}^2_+ & \text{if } x = 0_{\mathbb{R}^2}, \end{cases}$$

*with* $\bar{\varphi}_x \in [0, \pi/2)$ *defined by*

$$x = (r_x \cos(l\bar{\varphi}_x), r_x \sin(l\bar{\varphi}_x)) \text{ for some } l \in \mathbb{N} \text{ and some } r_x \in \mathbb{R},\ r_x > 0.$$

*For an illustration of some of these cones* $F(x)$ *see Figure 3.* $F(x) = F(x/\|x\|_2)$ *for all* $x \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$*, thus* $F(x) = F(\lambda x)$ *for all* $\lambda > 0$ *and all* $x \in X$ *and this equals (9).*

# 3 Semicontinuity of cone-valued maps

In this section we examine the notion of semicontinuity for cone-valued maps as well as the local Lipschitz continuity property. In the following, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $S$ be a nonempty open subset of $X$.

## 3.1 Semicontinuous cone-valued maps

We start by recalling the definitions of upper and lower semicontinuous set-valued maps according to [4],[2, Section 1.4]:

**Definition 3.1.** *Let* $F\colon S \to 2^Y$ *be a set-valued map.*

*(a)* $F$ *is called lower semicontinuous at* $x^0 \in S$ *if for all open sets* $V \subset Y$ *with* $F(x^0) \cap V \neq \emptyset$ *a neighborhood* $U$ *of* $x^0$ *exists such that* $F(x) \cap V \neq \emptyset$ *for all* $x \in U$*.* $F$ *is said to be lower semicontinuous if it is lower semicontinuous at any* $x^0 \in S$*.*

*(b) $F$ is called upper semicontinuous at $x^0 \in S$ if for all open sets $V \subset Y$ with $F(x^0) \subset V$ a neighborhood $U$ of $x^0$ exists such that $F(x) \subset V$ for all $x \in U$. $F$ is said to be upper semicontinuous if it is upper semicontinuous at any $x^0 \in S$.*

For a survey over the different and related notions for upper and lower semicontinuity used in the literature we refer to [10].

In the following we give a simple example for an upper and a lower semicontinuous cone-valued map [23, Subsection 3.4.1].

**Example 3.2.** *Let $K_1 = \text{cone conv}\{(2,1),(1,2)\}$ and $K_2 = \mathbb{R}^2_+$. Here, cone$(\cdot)$ denotes the conic hull and conv$(\cdot)$ the convex hull. Define $F_1, F_2 \colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ by*

$$F_1(x) = \begin{cases} K_1 & \text{if } x \in \mathbb{R}^2 \setminus \mathbb{Z}^2, \\ K_2 & \text{if } x \in \mathbb{Z}^2, \end{cases} \quad \text{and } F_2(x) = \begin{cases} K_2 & \text{if } x \in \mathbb{R}^2 \setminus \mathbb{Z}^2, \\ K_1 & \text{if } x \in \mathbb{Z}^2. \end{cases}$$

*Then $F_1$ is upper but not lower semicontinuous and $F_2$ is lower but not upper semicontinuous.*

In [6] an example was provided showing that upper semicontinuity is in general not an appropriate notion for cone-valued maps.

**Example 3.3.** *Consider the cone-valued map $F \colon \mathbb{R}^n \to \mathbb{R}^2$ defined by*

$$F(x) = \begin{cases} K & \text{for all } x \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}, \\ (-\infty, 0] \times \{0\} & \text{for } x = 0_{\mathbb{R}^n} \end{cases}$$

*with $K$ some cone in $\mathbb{R}^2$. For the open set*

$$V = \big\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 < 1,\ |y_2| < \exp(y_1)\big\}$$

*it holds $F(0_{\mathbb{R}^n}) \subset V$. If $F$ is upper semicontinuous at $0_{\mathbb{R}^n}$, then there is a neighborhood $U$ of $0_{\mathbb{R}^n}$ on which $F$ can only take the values $F(0_{\mathbb{R}^n})$ or $\{0_{\mathbb{R}^2}\}$.*

We state this more general in the following lemma.

**Lemma 3.4.** *Let $(Y, \|\cdot\|_Y)$ be a real reflexive Banach space and $F \colon S \to 2^Y$ a cone-valued map with $F(x^0)$ a closed convex cone. $F$ is upper semicontinuous at $x^0$ if and only if there is a neighborhood $U$ of $x^0$ such that $F(x) \subset F(x^0)$ for all $x \in U$.*

*Proof.* The sufficiency of the condition is obvious. We show that the condition is also necessary. For that we assume there exists no neighborhood $U$ of $x^0$ such that $F(x) \subset F(x^0)$ for all $x \in U$. Choose $\varepsilon > 0$ and consider the open neighborhood $V_\varepsilon$ of $F(x^0)$ defined by

$$V_\varepsilon := \{z \in Y \mid \min_{y \in F(x^0)} \|z - y\|_Y < \varepsilon\}$$

(the above minimum over $F(x^0)$ always exists, cf. [17, Thm. 2.18]). Let $U$ be an arbitrary neighborhood of $x^0$. Then some $x \in U \setminus \{x^0\}$ exists such that $F(x) \not\subset F(x^0)$. Thus some $y_U \in F(x)$ exists with $y_U \notin F(x^0)$. Set

$$\mu := \min_{y \in F(x^0)} \|y_U - y\|_Y > 0. \tag{10}$$

As $F(x)$ is a cone also $sy_U \in F(x)$ for all $s \geq \frac{\varepsilon}{\mu} > 0$. Then

$$\min_{y \in F(x^0)} \|sy_U - y\|_Y = s \min_{y \in F(x^0)} \|y_U - \frac{1}{s}y\|_Y = s\mu \geq \varepsilon,$$

i.e., $sy_U \notin V_\varepsilon$, and thus $F(x) \not\subset V_\varepsilon$ for some $x \in U$. As $U$ was chosen arbitrarily this contradicts the upper semicontinuity of $F$ at $x^0$. $\square$

In the following we give an example showing that a result analogous to Lemma 3.4 does not hold for the lower semicontinuity of a cone-valued map.

**Example 3.5.** *Let $X$, $Y$ equal the Euclidean space $\mathbb{R}^2$ and let $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 1,\ x_2 > 1\}$ and $F\colon S \to 2^{\mathbb{R}^2}$,*

$$F(x_1, x_2) = \text{cone conv}\{(1,0), (x_1, x_2)\} \text{ for all } (x_1, x_2) \in S.$$

*We can easily verify that $F$ is lower semicontinuous on $S$: For that choose $x^0 \in S$ arbitrarily and an open set $V \subset \mathbb{R}^2$ with $F(x^0) \cap V \neq \emptyset$. Of interest are only sets $V$ with $\partial F(x^0) \cap V \neq \emptyset$ ($\partial$ denotes the boundary), otherwise the conclusion is obvious. Choosing $y^0 \in \partial F(x^0) \cap V$ arbitrarily there exists some $\delta > 0$ with*

$$B_{y^0,\delta} = \{y \in \mathbb{R}^2 \mid \|y - y^0\|_2 < \delta\} \subset V.$$

*As $y^0 \in \partial F(x^0)$, $y^0_2 = 0$ or $y^0 = sx^0$ for some $s > 0$. Only the second case is of interest. Then for $U = B_{x^0,\varepsilon} \cap S = \{x \in \mathbb{R}^2 \mid \|x - x^0\|_2 < \varepsilon\} \cap S$ with $\varepsilon = \delta/s$ it holds for any $x \in U$ that $x \in F(x)$ and thus also $sx \in F(x)$. As*

$$\|sx - sx^0\|_2 = s\|x - x^0\|_2 < s\varepsilon = \delta$$

*we obtain $sx \in B_{y^0,\delta} \subset V$, i.e. $sx \in F(x) \cap V$ and hence $F$ is lower semicontinuous in $x^0$. For more details we refer to [23, Subsection 3.4.1].*

For dealing with semicontinuity notions also for cone-valued maps $F$ one can just define upper/lower semicontinuity of $F$ via the map $F_B\colon S \to 2^Y$ defined by $F_B(x) = F(x) \cap B_Y$ for all $x \in S$ (with $B_Y$ the closed unit ball). The following example gives a cone-valued map $F$ with $F_B$ upper and lower semicontinuous:

**Example 3.6.** *Let $X$, $Y$ and $F$ be specified as in Example 3.5 and let $S = (0, 1/2) \times (0, 1/2)$. Then the map $F_B$ with*

$$F_B(x_1, x_2) = \text{cone conv}\{(1,0), (x_1, x_2)\} \cap B_Y \text{ for all } (x_1, x_2) \in S$$

*is upper and lower semicontinuous on $S$, cf. [23, Subsection 3.4.1].*

## 3.2 Lipschitz continuous cone-valued maps

We continue by recalling the definition of a locally Lipschitz map [2, Def. 1.4.5].

**Definition 3.7.** *A set-valued map $F\colon S \to 2^Y$ is called locally Lipschitz at $\bar{x} \in S$ with constant $\alpha > 0$ if there exists some neighborhood $U(\bar{x}) \subset S$ of $\bar{x}$ such that*

$$F(x^1) \subset F(x^2) + \alpha \|x^1 - x^2\|_X B_Y \quad \text{for all } x^1, x^2 \in U(\bar{x}). \tag{11}$$

**Theorem 3.8.** *Let $F\colon S \to 2^Y$ be a cone-valued map with $F(x)$ a closed cone for all $x \in S$. If $F$ is locally Lipschitz at $\bar{x} \in S$ with constant $\alpha > 0$, then there is some neighborhood $U(\bar{x}) \subset S$ of $\bar{x}$ with $F(x) = K$ for all $x \in U(\bar{x})$ and with $K \subset Y$ a closed cone.*

*Proof.* Let $U(\bar{x}) \subset S$ and $\alpha > 0$ be given such that (11) is satisfied. Let $x^1, x^2 \in U(\bar{x})$, $x^1 \neq x^2$ be arbitrarily chosen and set $l := \alpha\|x^1 - x^2\|_X > 0$. Then

$$F(x^1) \subset F(x^2) + lB_Y. \tag{12}$$

Assuming $F(x^1) \neq F(x^2)$ there exists w.l.o.g. $\bar{y} \in F(x^1)$ with $\bar{y} \notin F(x^2)$. Then $\bar{y} \neq 0_Y$, as the images of $F$ are cones. As $F(x^1)$ is a cone, also $\lambda\bar{y} \in F(x^1)$ for all $\lambda > 0$ and with (12) we get $\lambda\bar{y} \in F(x^2) + lB_Y$ for all $\lambda > 0$. This yields, because $F(x^2)$ is a cone, $\bar{y} \in F(x^2) + \frac{l}{\lambda}B_Y$. For $\lambda$ to infinity this implies $\bar{y} \in \mathrm{cl}(F(x^2)) = F(x^2)$, which is a contradiction. ☐

Therefore, the local Lipschitz property is generally not an appropriate concept for cone-valued maps. Instead, we can define the local Lipschitz property of a cone-valued map $F$ via the map $F_B\colon S \to 2^Y$ defined by $F_B(x) = F(x) \cap B_Y$ for all $x \in S$ (with $B_Y$ again the closed unit ball), compare Example 3.6. The following example gives a cone-valued map $F$ with $F_B$ locally Lipschitz continuous.

**Example 3.9.** *Let $X$ and $Y$ be the Euclidean space $\mathbb{R}^2$ and consider the cone-valued map $F\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by*

$$\begin{aligned} F(x_1, x_2) = \{(y_1, y_2) \in \mathbb{R}^2 \mid \quad & y_1 = r\cos\varphi,\ y_2 = r\sin\varphi,\ r \geq 0, \\ & \varphi \leq \tfrac{\pi}{2} + \min\left\{|x_2|, \tfrac{\pi}{6}\right\},\ \varphi \geq -\min\left\{|x_1|, \tfrac{\pi}{6}\right\}\right\}. \end{aligned}$$

*Here, $\mathbb{R}^2_+ \subset F(x)$ for all $x \in \mathbb{R}^2$, see Figure 4. We show that the map $F_B$ defined by $F_B(x) = F(x) \cap B_{\mathbb{R}^2}$ for all $x \in \mathbb{R}^2$ is locally Lipschitz continuous on $\mathbb{R}^2$. For that we have to show that there is for any $\bar{x} \in \mathbb{R}^2$ some neighborhood $U(\bar{x})$ and some $\alpha > 0$ such that*

$$F(x^1) \cap B_{\mathbb{R}^2} \subset F(x^2) \cap B_{\mathbb{R}^2} + \alpha\|x^1 - x^2\|_2 B_{\mathbb{R}^2} \quad \text{for all } x^1, x^2 \in U(\bar{x}).$$

*This can be done by straightforward calculations using that cosinus and sinus are Lipschitz continuous functions and by using $\|x\|_2 \leq \|x\|_1$ for all $x \in \mathbb{R}^2$. For more details we refer to [23, Subsection 3.4.2].*

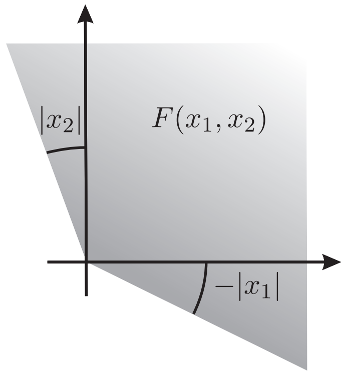


Figure 4: Images $F(x)$ for $x \in (-\frac{\pi}{6}, \frac{\pi}{6}) \times (-\frac{\pi}{6}, \frac{\pi}{6})$ of Example 3.9.

If $\bar{y}$ is given in $F(\bar{x})$ in Definition 3.7 then $F$ is called locally pseudo-Lipschitz at $(\bar{x}, \bar{y})$ with constant $\alpha > 0$ [2, Def. 1.4.5] if there exists some neighborhood $U(\bar{x}) \subset S$ of $\bar{x}$ and $V(\bar{y}) \subset Y$ of $\bar{y}$ such that

$$F(x^1) \cap V(\bar{y}) \subset F(x^2) + \alpha \|x^1 - x^2\|_X B_Y \quad \text{for all } x^1, x^2 \in U(\bar{x}). \tag{13}$$

This property is also known as Aubin property. A cone-valued map $F$ can be locally pseudo-Lipschitz without being locally constant:

**Example 3.10.** *Let $F\colon S \to 2^Y$ be a cone-valued map with $\bar{y} \in int(\bigcap_{x \in S} F(x))$. Here, int($\cdot$) denotes the interior. Then $F$ is locally pseudo-Lipschitz at any $(\bar{x}, \bar{y})$ with arbitrary constant $\alpha > 0$ and arbitrary $\bar{x} \in X$.*

Also, any cone-valued map $F$ with $F_B$ being locally Lipschitz continuous at $\bar{x}$, is also locally pseudo-Lipschitz at $(\bar{x}, \bar{y})$ for any $\bar{y} \in \mathrm{int} B_Y$.

# 4 Monotonicity of cone-valued maps

In this section we discuss the notion of monotonicity for cone-valued maps. A modification of this notion has already proved to be important in the study of cone-valued maps in the context of vector optimization problems with a variable ordering structure.

Let $(Y, \langle \cdot, \cdot \rangle)$ be a real Hilbert space identified with its dual, if not stated otherwise, and let $S$ be an open nonempty subset of $Y$. Recall that a monotone set-valued map $F\colon S \to 2^Y$ is defined by [2, Def. 3.5.1]

**Definition 4.1.** *A set-valued map $F\colon S \to 2^Y$ is called monotone if for all $y^1, y^2 \in S$ and all $u^1 \in F(y^1)$, $u^2 \in F(y^2)$*

$$\langle u^1 - u^2, y^1 - y^2 \rangle \geq 0. \tag{14}$$

It turns out that all monotone cone-valued maps $F$ are constant trivial-valued maps.

**Theorem 4.2.** *Let $F\colon S \to 2^Y$ be a monotone set-valued map with $0_Y \in F(y)$ for all $y \in S$. Then $F(y) = \{0_Y\}$ for all $y \in S$.*

*Proof.* Let $y \in S$ be arbitrarily chosen. As $S$ is an open set, for any $h \in Y$ there exists some $\lambda > 0$ such that $y \pm \lambda h \in S$. Choose any $u \in F(y)$. First, set $y^1 := y + \lambda h$. Then according to (14) it holds for all $u^1 \in F(y^1)$

$$\langle u^1 - u, \lambda\, h\rangle \geq 0.$$

Choosing $u^1 := 0_Y \in F(y^1)$ we get $\langle u, h\rangle \leq 0$. Next, set $y^2 := y - \lambda h$. Then with (14) it holds for all $u^2 \in F(y^2)$, $\langle u - u^2, \lambda\, h\rangle \geq 0$. Again, as $0_Y \in F(y^2)$, we obtain $\langle u, h\rangle \geq 0$ and thus $\langle u, h\rangle = 0$ for all $h \in Y$ and hence $u = 0_Y$. □

**Corollary 4.3.** *Let $F\colon S \to 2^Y$ be a monotone cone-valued map. Then $F(y) = \{0_Y\}$ for all $y \in S$.*

Analogously to $C$-baseconvex and baselinear, basemonotonicity can be defined. For that we need the following definition:

**Definition 4.4.** *Let $Y$ be a real linear topological space. We say that a vector valued map $f\colon S \to Y^*$ is monotone if $(f(y^1) - f(y^2))(y^1 - y^2) \geq 0$ for all $y^1, y^2 \in S$.*

**Definition 4.5.** *Let $Y$ be a real linear topological space and $S \subset Y$ a nonempty subset. Let $F\colon S \to 2^Y$ be a cone-valued map with $F(y)$ a pointed convex cone having a base for any $y \in S$. If there is a monotone map $\ell\colon S \to Y^*$ with*

$$B(y) = \{z \in Y \mid \ell(y)(z) = 1\}$$

*is a base of $F(y)$ for any $y \in S$, then $F$ is called basemonotone.*

**Example 4.6.** *We consider again the cone-valued map $F$ with images Bishop-Phelps cones as defined in Example 2.16. As $L$ is a real positive semidefinite matrix, the map $\ell$ defined by $\ell(y) = Ly$ for all $y \in \mathbb{R}^2$ is monotone and thus $F$ is also basemonotone.*

As mentioned earlier, see the comment after Definition 2.15, basemonotone and baselinear cone-valued maps play an important role in vector optimization with a variable ordering structure [13].

# Acknowledgements

The author is grateful to Truong Xuan Duc Ha and Marco Pruckner for valuable discussions.

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