Two-Dimensional DCT Pair:

\[
C_x(k_1, k_2) = \begin{cases} 
\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} 4x(n_1, n_2) \cos\left(\frac{\pi}{2N_1} k_1 (2n_1 + 1)\right) \cos\left(\frac{\pi}{2N_2} k_2 (2n_2 + 1)\right) & \text{for } 0 \leq k_1 \leq N_1 - 1, 0 \leq k_2 \leq N_2 - 1 \\
0 & \text{otherwise, resp.}
\end{cases}
\]

and otherwise, resp.

\[
x(n_1, n_2) = \begin{cases} 
1 \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} w_1(k_1)w_2(k_2)C_x(k_1, k_2) \cos\left(\frac{\pi}{2N_1} k_1 (2n_1 + 1)\right) \cos\left(\frac{\pi}{2N_2} k_2 (2n_2 + 1)\right) & \text{for } 0 \leq n_1 \leq N_1 - 1, 0 \leq n_2 \leq N_2 - 1 \\
0 & \text{otherwise, resp., with}
\end{cases}
\]

\[
w_i(k_i) = \begin{cases} 
0.5, & k_i = 0 \\
1, & 1 \leq k_i \leq N_i - 1
\end{cases}
\]

DCT’s equivalent to DFT’s Parseval Theorem: Energy Relationship:

\[
\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} |x(n_1, n_2)|^2 = \frac{1}{4N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} w_1(k_1)w_2(k_2)|C_x(k_1, k_2)|^2
\]
Filter Design By The Window Method

Goal: We would like to design a 2-dimensional filter, to filter our image, for instance to low pass filter it as a pre-processing step for down-sampling, or would would like to sharpen edges in the image, or we would like to detect edges in the image.

Problem: ideal filters are not realizable, infinitely long.

Approach: Shorten an ideal filter impulse response by multiplying it with a finite length window function.

Assume an ideal desired frequency response $H_d(\omega_1, \omega_2)$. Take the inverse Fourier Transform to obtain the impulse response of the desired (discrete) filter $h_d(n_1, n_2)$. In the window method, our FIR filter $h(n_1, n_2)$ is obtained by multiplying with a window $w(n_1, n_2)$:

$$h(n_1, n_2) = h_d(n_1, n_2)w(n_1, n_2).$$

What is the resulting frequency response of the windowed impulse response?

$$H(\omega_1, \omega_2) = H_d(\omega_1, \omega_2) \ast W(\omega_1, \omega_2)$$

Where $\ast$ is the circular convolution, i.e. the convolution with the periodically continued signal or sequence.
This resulting filter is then (2D) convolved with the entire image, to obtain the desired result.

Question here: What kind of window do we use? Rectangular window is simple, but often not the best. How do we extend it to 2 dimensions?

Possible Approaches: Take a separable window for 2D; or take a circular symmetric window for 2D.

Separable 2D window                       circular symmetric window
(inside is e.g. a 1, outside is the 0 of the window function).