A moving Horizon Approach to a Chance Constrained Nonlinear Dynamic Optimization Problem

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A Chance Constrained Optimization Problem

\[(CCOPT) \quad \min f(u) \quad (1)\]
\[s.t. \quad g(u, y, \xi) = 0 \quad (2)\]
\[Pr\{G_i(u, y, \xi) \leq 0\} \geq \alpha_i, i = 1, \ldots, q, \quad (3)\]

where
- \(u \in \mathbb{R}^n\) is a vector of decision variables
- \(y \in \mathbb{R}^m\) is a vector of uncertain (output or state) variables
- \(\xi \in \mathbb{R}^p\) is a vector of uncertain (input) variables with known probability distribution;
- \(f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}\) is the objective (performance) function
- \(g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n\) represents the process model
- \(G_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}\) is the constraint function
- \(f, g, G\) are functions with continuous derivatives
Chance Constrained Optimization Problem

In addition

- $Pr(\cdot)$ is a probability measure
- $\alpha_i \in (0, 1), i = 1, \ldots, q$ - reliability level of satisfaction of constraints

Properties:

- less conservative as compared to deterministic constraints;
- reliability levels can be fixed by the user;
- but, higher reliability levels means higher cost; hence, compromise

Application Areas:

- finance and economics, water reservoir management, structural design, planning, chemical process engineering, etc.
Special form of Chance-Constraint (3)

\[ \Pr\{y_i^{\min} \leq y_i \leq y_i^{\max}\} \geq \alpha_i, \ i = 1, \ldots, q \leq m. \]  (4)

could specify, for example, the reliability level of

- meeting product specification
- availability of products or outputs
- risk tolerance
Chance Constrained Optimization Problem

**Numerical Solutions Techniques for (CCOPT)**
- through transform to deterministic optimization problems;
- the algorithms used are mainly gradient based.

**Difficulties Associated with (CCOPT)**
ξ is random \( \Rightarrow C_i = \{ y(u, \xi) \mid y_{i}^{min} \leq y(u, \xi) \leq y^{min} \} \) are random sets. Since \( g(u, y, \xi) \) is possibly non-linear,
- in general, the probability of the sets \( C_i \) is not trivial to determine; thus,
- values of active constraints and gradients are not easy to compute.
Back-Projection of Chance Constraints - a Monotonic Approach

Principle: if \( y \) and \( \xi \) are scalar random variables, \( t \) has known distribution, and \( y = \varphi(\xi) \), s.t. \( \varphi \) is strictly increasing (or decreasing), then \( \xi \uparrow \Rightarrow y \uparrow \) and

\[
Pr\{y_{\text{min}} \leq y \leq y_{\text{max}}\} \geq \alpha \iff Pr \left\{ \varphi^{-1}\left(y_{\text{min}}\right) \leq \xi \leq \varphi^{-1}\left(y_{\text{max}}\right) \right\} \geq \alpha
\]

To use this principle for (CCOPT) under (4) (Wendt et al. 2002):

- for each chance constrained \( y_i \), find some \( \xi_j \) such that
  \( \xi_j \uparrow \Rightarrow x_i \uparrow \) (or \( \xi_j \uparrow \Rightarrow x_i \downarrow \))
- requires an analysis of the model equations \( g(u, y, \xi) = 0 \)
- monotonicity holds for several model problems from chemical engineering
Problem: Optimal Control of a Buffer tank

- \( q(t) \): feed inflow rate
- \( C0(t) \): infeed concentration
- \( u(t) \): feed outflow rate

\[
V_{\text{min}} \leq V(t) \leq V_{\text{max}}
\]

\[
C_{\text{min}} \leq C(t) \leq C_{\text{max}}
\]

Figure: Buffer tank
Model Equations

Discrete-time consideration \( t = n \).

- \( q_n \) uncertain feed inflow rate \( \rightarrow q_n \sim N(\mu_n, \sigma_q^2) \);  
- \( C0_n \) uncertain feed concentration \( \rightarrow C0_n \sim N(\theta_n, \sigma_{C0}^2) \)

Model Equations:

\[
V_{n+1} = V_n + q_n - u_n, \\
C_{n+1} = C_n + \frac{q_n}{V_{n+1}} [C0_n - C_n],
\]
Moving Horizon Approach

Minimize \((\Delta u)^T \Delta u\)

Subject to:
\[ V_{n+1} = V_n + q_n - u_n \]
\[ C_{n+1} = C_n + \frac{q_n}{V_{n+1}} [C_0 - C_n] \]
\[ u_{\text{min}} \leq u_i \leq u_{\text{max}}, \quad i = n-1, \ldots, n+h-2 \]
\[ \Pr(V_{\text{min}} \leq V_i \leq V_{\text{max}}) \geq \alpha_1, \quad i = n \ldots, n+h-1 \]
\[ \Pr(C_{\text{min}} \leq C_i \leq C_{\text{max}}) \geq \alpha_2, \quad i = n, \ldots, n+h-1 \]

where
- \(\Delta u = (u_n - u_{n-1}, \ldots, u_{n+h-1} - u_{n+h-2})\)
- \(h\) is the prediction horizon
Calculating Probabilities

Use monotony to transform the following events:

\[ q_{j-1} \uparrow \Rightarrow V_j \uparrow \]
\[ C_{0,j-1} \uparrow \Rightarrow C_j \]

Calculating probabilities:

\[
Pr(V_{\min} \leq V_j \leq V_{\max}) = Pr(\tilde{q}_{\min}(u) \leq q_{j-1} \leq \tilde{q}_{\max}(u))
\]
\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{\tilde{q}_{\min}(u)}^{\tilde{q}_{\max}(u)} \Phi(\xi) d\xi
\]

\[
Pr(C_{\min} \leq C_j \leq C_{\max}) = Pr(\tilde{C}_{0,\min}(u) \leq C_{0,j-1} \leq \tilde{C}_{0,\max}(u))
\]
\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{\tilde{C}_{0,\min}(u)}^{\tilde{C}_{0,\max}(u)} \Phi(\xi) d\xi
\]
Obtaining Gradients Of Probability Constraints

\[
\frac{\partial}{\partial u} Pr(V_{min} \leq V_j \leq V_{max}) \\
= \frac{\partial}{\partial u} Pr(\tilde{q}_{min}(u) \leq q_{j-1} \leq \tilde{q}_{max}(u)) \\
= \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{\tilde{q}_{min}(u)}^{\tilde{q}_{max}(u)} \Phi(\xi) d\xi \\
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \Phi(\xi_1, \ldots, \xi_{n-1}, \tilde{q}_{max}(u)) \frac{\partial}{\partial u} \tilde{q}_{max}(u) - \Phi(\xi_1, \ldots, \xi_{n-1}, \tilde{q}_{min}(u)) \frac{\partial}{\partial u} \tilde{q}_{min}(u) \right) d\xi_{n-1} \cdots d\xi_1
\]
Obtaining Gradients Of Probability Constraints (Continued)

\[
\frac{\partial}{\partial u} \Pr(C_{\min} \leq C_j \leq C_{\max})
\]

\[
= \frac{\partial}{\partial u} \Pr(\tilde{C}_{0\min}(u) \leq C_{0j-1} \leq \tilde{C}_{0\max}(u))
\]

\[
= \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\tilde{C}_{0\max}(u)} \Phi(\xi) d\xi
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \Phi(\xi_1, \ldots, \xi_{n-1}, \tilde{C}_{0\max}(u)) \frac{\partial}{\partial u} \tilde{C}_{0\max}(u) - \Phi(\xi_1, \ldots, \xi_{n-1}, \tilde{C}_{0\min}(u)) \frac{\partial}{\partial u} \tilde{C}_{0\min}(u) \right) d\xi_{n-1} \cdots d\xi_1
\]
Calculating Integrals

Recursive integration scheme (Prekopa 1995):

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(\xi) d\xi = \int_{-\infty}^{\infty} \Phi_1(\xi_1) d\xi_1,
\]

\[
\Phi_1(\xi_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(\xi_1, \ldots, \xi_t) d\xi_2 \cdots d\xi_t
\]

\[
= \int_{-\infty}^{\infty} \Phi_2(\xi_2) d\xi_2
\]

where

- \( \Phi : \mathbb{R}^t \to \mathbb{R} \)
- \( \Phi_j : \mathbb{R} \to \mathbb{R}, \quad j = 1, \ldots, t \)
- Integrals are computed through collocation
- Indefinite integrals are truncated onto the interval \([-4, 4]\)
Implementation and some Experiences

- **Algorithm and Implementation**
  - IpOpt (Version 3.4.2.) with the linear solver ma27
  - implementation under Matlab©
  - number of time intervals $n = 20$, length of horizon $h = 3$
  - integrals are calculated with max. absolute error of 0.002
  - termination tolerance for the NLP algorithm is 0.0001.

- **Experience**
  - computational effort increases exponentially with dimension of integrals
  - choose prediction horizon such, so that dimension of integrals $\leq 10$
  - IpOpt needs parameter tuning
  - initial point of optimization should be chose appropriately
Realizations

- initial values $V_0 = 160$, $C_0 = 50$;
- bounds $V_{\text{min}} = 130$, $V_{\text{max}} = 170$, $C_{\text{min}} = 49$, $C_{\text{max}} = 51$, $u_{\text{min}} = 0$, $u_{\text{max}} = +\infty$
- reliability levels $\alpha_1 = \alpha_2 = 0.8$;
- $\mu(q_t), \mu(CO_t), \sigma_t = \text{Var}(q_t) = 0.7$, $\varrho_t = \text{Var}(CO_t) = 1$, $t = 1, \ldots, n = 20$;
- 21 realizations.
Figure: Resulting tank concentration for the 20 time intervals
Realizations (Continued)

Resulting tank volume $V(k)$ for realized $q(k)$ and calculated $u(k)$

Figure: Realized tank volume
The framework is efficient to handle small-scale chance-constrained optimization problems.

Monotonicity relations can be verified for several practical problems.

The length of the prediction horizon $h$ is a tradeoff between control performance and speed of computation,

Future Work

- What to do if monotonicity is not available?
- What is the largest reliability level $\alpha$ that still keeps feasibility of the chance-constraints?
- How to compute integrals more efficiently?
- What is the effect of using a different (e.g. log-normal etc.) probability distribution?
References

THANK YOU!