These slides do not contain all the topics intended for discussion ..... Watch out errors are everywhere! In the meantime, I am happy to receive your suggestions, corrections and comments.

But, "I won’t leave any unfinished manuscripts” Harold Robbins - American author with 25 bestsellers.
Topics

- Basic Principles of the Interior Point (Barrier) Methods
- Primal-Dual Interior Point methods
- Primal-Dual Interior Point methods for Linear and Quadratic Optimization
- Primal-Dual-Interior Point methods for Nonlinear Optimization
- Current Issues
- Conclusion
- References and Resources
Basics of the Interior Point Method

Consider

\[(NLP) \quad \min_x f(x) \quad \text{s.t.} \]

\[g_i(x) \geq 0, \ i = 1, 2, \ldots, m_1; \]
\[h_j(x) = 0, \ j = 1, 2, \ldots, m_2; \]
\[x \geq 0, \]

where \(f, g_i, h_j : \mathbb{R}^n \rightarrow \mathbb{R}\) are at least once differentiable functions, \(x_{\min}, x_{\max} \in \mathbb{R}^n\) are given vectors.

Feasible set of NLP:

\[\mathcal{F} := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ i = 1, \ldots, m_1; \]
\[h_j(x) = 0, \ j = 1, \ldots, m_2; x \geq 0\}.\]
Basics of the Interior Point Method...

Idea of the interior point method:
• to iteratively approach the optimal solution from the interior of the feasible set

Figure: Feasible set $\mathcal{F}$

$F = \{x \in \mathbb{R}^n \mid g(x) \geq 0, h(x) = 0, x \geq 0\}$

$h(x) = 0$

$g(x) \geq 0$
Basics of the Interior Point Method...

Therefore (requirements for IPM):

- the interior of the feasible set should not be empty
- almost all iterates should remain in (the interior of the) feasible set

Question:
When is the interior of the feasible set non-empty?

Answer:

(i) if there is \( \overline{x} \in \mathbb{R}^n \) such that

\[
g_i(\overline{x}) > 0, \ i = 1, \ldots, m_1; \ h_j(\overline{x}) = 0, \ j = 1, \ldots, m_2; \ \overline{x} > 0.
\]

(ii) if the Mangasarian-Frmondovitz Constraint Qualification (MFCQ) is satisfied at a feasible point \( \overline{x} \),

then the interior of the feasible set of NLP is non-empty.
What is MFCQ?

Let $\bar{x} \in \mathcal{F}$; i.e. $\bar{x}$ is a feasible point of NLP.

Active constraints

- An inequality constraint $g_i(x)$ is said to be active at $\bar{x} \in \mathcal{F}$ if
  
  $$g_i(\bar{x}) = 0.$$  

- The set
  
  $$A(\bar{x}) = \{ i \in \{1, \ldots, m_1\} \mid g_i(\bar{x}) = 0 \}$$

  index set of active inequality constraints at $\bar{x}$.

\[
\begin{align*}
(NLP) \quad \min_x \{ f(x) = x_1^2 - x_2^2 \} \quad & \text{s.t.} \quad g_1(x) = x_1^2 + x_2^2 + x_3^2 + 3 \geq 0, \\
g_2(x) = 2x_1 - 4x_2 + x_3^2 + 1 \geq 0, \\
g_3(x) = -5x_1 + 3x_2 + 2 \geq 0, \\
x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0.
\end{align*}
\]
What is MFCQ ?...

The vector \( \bar{x}^\top = (1, 1, 1) \) is feasible to the NLP and

\[ g_2(\bar{x}) = 0 \text{ and } g_3(\bar{x}) = 0, \]

the active index set is \( \mathcal{A}(\bar{x}) = \{2, 3\} \).

**Mangasarian-Fromowitz Constraint Qualification**

Let \( \bar{x} \in \mathcal{F} \) (feasible point of NLP). Then MFCQ is said to be satisfied at \( \bar{x} \) if there is a vector \( d \in \mathbb{R}^n, d \neq 0 \), such that (i)

\[
(i) \quad d^\top \nabla g_i(\bar{x}) > 0, \quad i \in \mathcal{A}(\bar{x}), \quad \text{and}
\
(ii) \quad d^\top \nabla h_1(\bar{x}) = 0, \quad d^\top \nabla h_2(\bar{x}) = \ldots, \quad d^\top \nabla h_{m_2}(\bar{x}) = 0.
\]
What is MFCQ?

Figure: A Mangasarian-Fromowitz Vector $d$

- $d$ forms an acute angle ($< 90^0$) with each $\nabla g_i(\bar{x})$, $i \in A(\bar{x})$. 
What is MFCQ ...?

An implications of the MFCQ:

There is $\alpha$ such that

- $\bar{x} + \alpha d > 0$.
- $g(\bar{x} + \alpha d) \approx g(\bar{x}) + \alpha d^\top \nabla g_i(\bar{x}) > 0$, $i = 1, \ldots, m_1$,
- $h_j(\bar{x} + \alpha d) \approx h_j(\bar{x}) + \alpha d^\top \nabla h_j(\bar{x}) = 0$, $j = 1, \ldots, m_2$.

$\Rightarrow \bar{x} + \alpha d$ is in the interior of the feasible set $\mathcal{F}$.

$\Rightarrow$ The interior of the feasible set is not empty.

Example (continued...)

- $\nabla g_2(\bar{x}) = (2, -4, 2)$ and $\nabla g_3(\bar{x}) = (-5, 3, 0)$.
- For $d^\top = (-1, 0, 2)$ we have $d^\top \nabla g_2(\bar{x}) > 0$ and $d^\top \nabla g_3(\bar{x}) > 0$; and
- $x = (1, 1, 1) + \frac{1}{10} (-1, 0, 2) > 0$.

MFCQ guarantees that the interior of $\mathcal{F}$ is not empty.
Forcing iterates remain in the interior of $\mathcal{F}$

**Question:**
How to force almost all iterates remain in the interior of the feasible set $\mathcal{F}$?

**Answer:**
Use barrier functions?

A well-known barrier function is the **logarithmic barrier function**

$$B(x, \mu) = f(x) - \mu \left( \sum_{i=1}^{m_1} \log(g_i(x)) + \sum_{l=1}^{n} \log(x_l) \right)$$

where $\mu$ is known as **barrier parameter**.

- The logarithmic terms $\log(g_i(x))$ and $\log(x_l)$ are defined at points $x$ for which $g_i(x) > 0$ and $x_l > 0$, $l = 1, \ldots, n$. 

Introduction to Interior Point Methods
Basics of the Interior Point Method...

- Instead of the problem NLP, consider the parametric problem

\[(NLP)_\mu \quad \min_{x} B(x, \mu) \]

\[s.t. \quad h_j(x) = 0, j = 1, \ldots, m_2.\]

- To find an optimal solution \(x_\mu\) of \((NLP)_\mu\) for a fixed value of the barrier parameter \(\mu\).

**Lagrange function of \((NLP)_\mu\):**

\[
\mathcal{L}_\mu(x, \lambda) = f(x) - \mu \left( \sum_{i=1}^{m_1} \log(g_i(x)) + \sum_{l=1}^{n} \log(x_l) \right) - \sum_{j=1}^{m_2} \lambda_j h_j(x).
\]
Basics of the Interior Point Method...

Necessary optimality (Karush-Kuhn-Tucker) condition:

for a given \( \mu \), a vector \( x_\mu \) is a minimum point of \((\text{NLP})_\mu\) if there is a Lagrange parameter \( \lambda_\mu \) such that, the pair \((x_\mu, \lambda_\mu)\) satisfies:

\[
\nabla_\lambda \mathcal{L}_\mu(x, \lambda) = 0 \\
\nabla_x \mathcal{L}_\mu(x, \lambda) = 0 
\]

⇒ Thus we need to solve the system

\[
-h(x) = 0 \\
-\nabla f(x) - \mu \left( \sum_{i=1}^{m_1} \frac{1}{g_i(x)} \nabla g_i(x) + \sum_{l=1}^{m_1} \frac{1}{x_l} e_l \right) + \sum_{j=1}^{m_2} \lambda_j \nabla h_j(x) = 0
\]

• Commonly, this system is solved iteratively using the Newton Method.
Basics of the Interior Point Method...

Newton method to solve the system of nonlinear equations $F_\mu(x, \lambda) = 0$ for a fixed $\mu$, where

$$F_\mu(x, \lambda) = \left( \nabla f(x) - \mu \left( \sum_{i=1}^{m_1} \frac{1}{g_i(x)} \nabla g_i(x) + \sum_{l=1}^{m_1} \frac{1}{x_l} e_l \right) + \sum_{j=1}^{m_2} \lambda_j \nabla h_j(x) \right)$$

Algorithm:

**Step 0:** Choose $(x_0, \lambda_0)$.

**Step k:**

- Find $(\Delta^k_x, \Delta^k_\lambda) = d$ by solving the linear system $J_{F_\mu}(x_k, \lambda_k)d = -F_\mu(x_k, \lambda_k)$
- Determine a step length $\alpha_k$
- Set $x_{k+1} = x_k + \alpha_k \Delta^k_x$ and $\lambda_{k+1} = x_k + \alpha_k \Delta^k_\lambda$

STOP if convergence is achieved; otherwise CONTINUE.
Basics of the Interior Point Method...

- For each give $\mu$, the above algorithm can provide a minimal point $x_\mu$ of the problem $(\text{NLP})_\mu$.

Question: What is the relation between the problem NLP and $(\text{NLP})_\mu$?

Question: How to choose $\mu$’s?

Answer (a general strategy): choose a sequence $\{\mu_k\}$ of decreasing, sufficiently small non-negative barrier parameter values

- to obtain associated sequence $\{x_{\mu_k}\}$ optimal solutions of $(\text{NLP})_{\mu_k}$.

Properties

- The optimal solutions $x_\mu$ lie in the interior of the feasible set of NLP.
- The solutions $x_{\mu_k}$ converge to a solution $x^*$ of NLP; i.e.

\[
\lim_{\mu \to 0^+} x_\mu = x^*.
\]
Drawbacks of the primal barrier interior

\[ J_{F\mu}(x, \lambda) = \begin{pmatrix} \frac{J_h(x)}{H(x)} - \mu \left( \sum_{i=1}^{m_1} \frac{1}{g_i(x)} \left[ \nabla g_i(x) \nabla g_i(x)^\top + G_i(x) \right] - \sum_{i=1}^{m_1} \frac{1}{x_i^2} e_i \right) + \sum_{j=1}^{m_2} \lambda_j \nabla h_j(x) [J_h(x)]^\top \end{pmatrix}, \]

where, \( H(x) \) is the Hessian matrix of \( f(x) \), \( J_h(x) \) is the Jacobian matrix of \( h(x)^\top = (h_1(x), h_2(x), \ldots, h_{m_2}(x)) \), \( G_i(x) \) is the Hessian matrix of \( g_i(x) \), \( H_j(x) \) is the Hessian matrix of \( h_j(x) \).

**Drawback:** as the values of \( \mu \) get closer to 0 the matrix \( D \) can become ill-conditioned.

**Example (continued):**

For our example we have

\[
D(x) = \frac{1}{g_1(x)} \begin{bmatrix} 4x_1^2 + 2 & 4x_1x_2 & 4x_1x_3 \\ 2x_1x_2 & 4x_2^2 + 2 & 4x_1x_3 \\ 4x_1x_3 & 4x_1x_3 & 4x_2^2 + 2 \end{bmatrix} + \frac{1}{g_2(x)} \begin{bmatrix} 4 & -8 & 4x_3 \\ -8 & 16 & -8x_3 \\ 4x_3 & -8x_3 & 4x_3 + 2 \end{bmatrix} + \frac{1}{g_3(x)} \begin{bmatrix} 25 & -15 & 0 \\ -15 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} - X^{-2}
\]

where \( X = \text{diag}(x) \). For example, at the feasible interior point \( x^\top = (1, 2, 8) \) we have \( \text{cond}(D) \approx 113.6392 \), which is large.
Drawbacks of the primal barrier interior

Note that:
- the matrix $\nabla g(x) [\nabla g(x)]^\top$ is of rank 1, so not invertible and has large condition number.
- the expression $\frac{1}{g(x)}$ gets larger as $g(x)$ gets smaller, usually near to the boundary of the feasible region.

Advise: Do not use the constraint function $g_i(x) \geq 0, i = 1, \ldots, m_1$ directly with the logarithmic barrier function.

Instead, introduce slack variables $s = (s_1, s_2, \ldots, s_{m_1})$ for inequality constraints so that:

$$g_i(x) - s_i = 0, s_i \geq 0, i = 1, \ldots, m_1.$$ 

(That is, we lift the problem into a higher dimension by adding new variables, so that we have to work with $z = (x, s) \in \mathbb{R}^{n+m_1}$. Frequently, in higher dimensions, we may have a better point of view.)
The Primal-Dual Interior Point Method

This leads to the problem

$$(NLP)_\mu \min_{(x,s)} \left\{ f(x) - \mu \left( \sum_{l=1}^{n} \log(x_l) + \sum_{i=1}^{m_1} \log(s_i) \right) \right\}$$

s.t.

$$g_i(x) - s_i = 0, \quad i = 1, \ldots, m_1$$
$$h_j(x) = 0, \quad j = 1, \ldots, m_2.$$ only with equality constraints and objective function with barrier terms on the variables.

$$(NLP)_\mu \min_{(x,s)} \left\{ f(x) = \left( x_1^2 - x_2^2 \right) - \mu \left[ \sum_{i=1}^{3} \left( \log s_i + \log x_i \right) \right] \right\} \quad (1)$$

s.t.

$$g_1(x) = x_1^2 + x_2^2 + x_3^2 + 3 - s_1 = 0,$$
$$g_2(x) = 2x_1 - 4x_2 + x_3^2 + 1 - s_2 = 0,$$
$$g_3(x) = -5x_1 + 3x_2 + 2 - s_3 = 0.$$
Primal-dual Interior Method for LOPs

- Consider a standard linear optimization problem

\[(LOP) \quad \begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax = b, \\
& \quad x \geq 0
\end{align*}\]

where \(A\) is \(m \times n\) matrix, \(b \in \mathbb{R}^n\).

- The dual problem to LOP is:

\[(LOP)_D \quad \begin{align*}
\max_{\lambda,s} & \quad b^T \lambda \\
\text{s.t.} & \quad A^T \lambda + s = c.
\end{align*}\]

Here, \(s\) is slack variable.
Primal-dual Interior Method for LOPs

The Lagrange function of LOP:

\[ \mathcal{L}(x, \lambda, s) = c^\top x - \lambda^\top (Ax - b) - \sum_{i=1}^{m} s_i x_i, \]

where:

- \( \lambda^\top = (\lambda_1, \ldots, \lambda_m) \) is a vector of Lagrange multipliers associated with the equality constraints \( Ax = b \), and
- \( s = (s_1, \ldots, s_n) \) is a vector of Lagrange-multipliers associated with \( x \geq 0 \); hence \( s \geq 0 \).

- Here, the Lagrange-multiplier vector \( s \) is same as the slack variable \( s \) in the dual problem \((\text{LOP})_D\).
Primal-dual Interior Method for LOPs...

- The optimality criteria for $x^*$ to be a solution of the primal problem (P) and $(\lambda^*, s^*)$ to be a solution of dual problem (D) is that $(x^*, \lambda^*, s^*)$ should satisfy:

\[
\begin{align*}
    c - A^T \lambda - s &= 0 \\  
    Ax &= b \\  
    XSe &= 0 \\  
    (x, s) &\geq 0
\end{align*}
\]  

(4) 
(5) 
(6) 
(7)

where:

\[
    X = \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
    \end{bmatrix},
    S = \begin{bmatrix}
    s_1 \\
    s_2 \\
    \vdots \\
    s_n
    \end{bmatrix},
    e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
\]
Primal-dual Interior Method ...

**Question:**
Where is the relation with the interior point method?

- The barrier function associated to LOP is
  \[ B(x, \mu) = f(x) - \mu \sum_{i=1}^{m_1} \log(x_i) \]

- The barrier problem will be
  \[ (NLP)_\mu \min_x \left\{ f(x) - \mu \sum_{i=1}^{m_1} \log(x_i) \right\} \]
  \[ \text{s.t.} \]
  \[ Ax = b. \]

- The Lagrange function of the barrier Problem is
  \[ L_\mu(x, \lambda) = c^\top x - \lambda^\top (Ax - b) - \mu \sum_{i=1}^{n} \log(x_i). \]
For a given $\mu$, the pair $(x_\mu, \lambda_\mu)$ is a solution of the primal problem $\text{NLP}_\mu$ if it satisfies the optimality conditions:

\[
\nabla_x \mathcal{L}_\mu(x, \lambda) = 0 \quad (8)
\]

\[
\nabla_\lambda \mathcal{L}_\mu(x, \lambda) = 0 \quad (9)
\]

\[x > 0. \quad (10)\]

**KKT Conditions**

\[
\begin{align*}
\mathbf{c} - A^\top \lambda - \mu X^{-1} \mathbf{e} & = 0, \\
\mathbf{s} & = 0, \\
A\mathbf{x} & = \mathbf{b}, \\
x & > 0.
\end{align*}
\]

Where: \(\mathbf{s} = \mu X^{-1} \mathbf{e}\).
Primal-dual Interior Method for LOPs...

- It follows (since $x_i \neq 0$) that $s_i = \frac{\mu}{x_i} > 0 \Rightarrow s_i x_i = \mu, \ i = 1, \ldots, n.$

$$
\begin{bmatrix}
  s_1 x_1 \\
  s_2 x_2 \\
  \vdots \\
  s_n x_n \\
\end{bmatrix}
\begin{bmatrix}
  1 \\
  1 \\
  \vdots \\
  1 \\
\end{bmatrix}
= \mu
$$

$\Rightarrow$

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
= X
$$

$$
\begin{bmatrix}
  s_1 \\
  s_2 \\
  \vdots \\
  s_n \\
\end{bmatrix}
\begin{bmatrix}
  1 \\
  1 \\
  \vdots \\
  1 \\
\end{bmatrix}
= \mu
$$

$\Rightarrow \ XSe = \mu e.$
Primal-dual Interior Method for LOPs...

- Now, the optimality conditions, for the barrier problem NLP$_{\mu}$, given in (8) - (10) can be equivalently as:

\[
\begin{align*}
Ax &= b, \\
A^T \lambda + s &= c, \\
XSe &= \mu e \\
(x, s) &> 0.
\end{align*}
\]

- Note that, this system is the same as the equations (4) - (7), except the perturbation $XSe = \mu e$ and $(x, s) > 0$.
- For a given $\mu$, the system of nonlinear equations (11)-(14) provides a solution $(x_\mu, \lambda_\mu, s_\mu)$.
- $x_\mu$ lies in interior of the feasible set of LOP, while the pair $(\lambda_\mu, s_\mu)$ lies in the interior of the feasible set of LOP$_D$, due to $XSe = \mu e$ and $(x, s) > 0$. Furthermore,
Primal-dual Interior Method for LOPs...

- Furthermore, if

\[ x^* = \lim_{\mu \downarrow 0^+} x_\mu \text{ and } (\lambda^*, s^*) = \lim_{\mu \downarrow 0^+} (\lambda_\mu, s_\mu) \]

the \( x^* \) is a minimum point of LOP, while \((\lambda^*, s^*)\) is a maximum point of LOP\(_D\).

- Therefore, any algorithm that solves the system of nonlinear equations (11)-(14) is known as a **primal-dual interior point algorithm**.

- For a given \( \mu \), to determine the triple \((x_\mu, \lambda_\mu, s_\mu)\),

(1) solve the nonlinear system

\[
F_\mu(x, \lambda, s) = \begin{bmatrix}
Ax - b \\
A^T \lambda + s - c \\
XSe - \sigma \mu e
\end{bmatrix} = 0,
\]

(II) and guarantee always that \((x, s) > 0\).
Primal-dual Interior Method for LOPs...

- The set of 
  \[ C = \{(x(\mu), \lambda(\mu), s(\mu)) \mid F_\mu(x(\mu), \lambda(\mu), s(\mu)) = 0, (x(\mu), s(\mu)) > 0\} \]
  is known as the central path.

  (I) To solve the system

  \[
  F_\mu(x, \lambda, s) = \begin{bmatrix}
  Ax - b \\
  A^\top \lambda + s - c \\
  XSe - \sigma \mu e
  \end{bmatrix} = 0
  \]

  use a Newton method.

- For a given \( \mu \) and feasible point \((x, \lambda, s)\), determine 
  \( d = (\Delta x, \Delta \lambda, \Delta s) \) by solving 
  \( J_\mu(x, \lambda, s)d = -F_\mu(x, \lambda, s) \); i.e.,

  \[
  \begin{bmatrix}
  A & 0 & 0 \\
  0 & A^\top & I \\
  X & 0 & S
  \end{bmatrix}
  \begin{bmatrix}
  \Delta x \\
  \Delta \lambda \\
  \Delta s
  \end{bmatrix}
  = -
  \begin{bmatrix}
  Ax - b \\
  A^\top \lambda + s - c \\
  XSe - \sigma \mu e
  \end{bmatrix}
  \]

- Next iterate 
  \( (x^+, \lambda^+, s^+) = (x, \lambda, s) + \alpha(\Delta x, \Delta \lambda, \Delta s) \).
II: Question

How to guarantee that \((x_\mu, s_\mu) > 0\)?

Answer

We know that \(x_i s_i = \mu, i = 1, \ldots, n\). Hence,

\[
x^\top s = x_1 s_1 + x_2 s_2 + \ldots + x_n s_n = n\mu \Rightarrow \frac{x^\top s}{n} = \mu
\]

Therefore, choose \(\mu\) so that \(\frac{x^\top s}{n} > 0\).

Importance of the central path

- Additionally, for \((x_\mu, \lambda(\mu), s_\mu) \in C\) we have \(\frac{x^\top(\mu)s(\mu)}{n} = \mu\).
- Fast convergence of a PDIPM algorithm is achieved if iterates lie on the central path.
- The parameter \(\sigma\) is known as a centering parameter. Thus, \(\sigma\) is chosen to force iterates remain closed to (or on) the central path.
Primal-dual Interior Method for LOPs...

A primal-dual interior point algorithm (PDIPM):

Step 0: ● Give an initial point \((x_0, \lambda_0, s_0)\) with \((x_0, s_0) > 0\).
  ● Set \(k \leftarrow 0\) and \(\mu_0 = \frac{x_0^T s_0}{n}\)

Repeat:
  ● Choose \(\sigma_k \in (0, 1]\);
  ● Solve the linear system (16) with \(\mu = \mu_k\) and \(\sigma = \sigma_k\)
    to obtain \((\Delta x_k, \Delta \lambda_k, \Delta s_k)\);
  ● Choose step-length \(\alpha_k \in (0, 1]\)
  ● and set

\[
\begin{align*}
  x_{k+1} &= x_k + \alpha_k \Delta x_k \\
  \lambda_{k+1} &= \lambda_k + \alpha_k \Delta \lambda_k \\
  s_{k+1} &= s_k + \alpha_k \Delta s_k.
\end{align*}
\]

Until: Some termination criteria is satisfied.
Primal-dual Interior Method for LOPs...

Questions:
Q1: How to determine the step length $\alpha_k$?
Q2: How to choose the centering parameter $\sigma_k$?
Q3: What is a suitable termination criteria?
Q4: How to solve the system of linear equations (16)?

Some strategies for step-length selection:
(a) Use $\alpha_k = 1, k = 1, 2, \ldots$. But, generally, not advised.
(b) Choose $\alpha_k$ so that

$$
x_k + \alpha_k \Delta x_k > 0
$$
$$
s_k + \alpha_k \Delta s_k > 0.
$$

Compute the largest $\alpha$ that satisfies these conditions

$$
\alpha_{max} = \min \left\{ \min \left\{ \frac{x_{k,i}}{-\Delta x_{k,i}} \mid \Delta x_{k,i} < 0 \right\}, \min \left\{ \frac{s_{k,i}}{-\Delta s_{k,i}} \mid \Delta s_{k,i} < 0 \right\} \right\}
$$

Then choose $\alpha_k = \min\{1, \eta_k \cdot \alpha_{max}\}$. Typically $\eta_k = 0.999$. 

Introduction to Interior Point Methods
Primal-dual Interior Method for LOPs...

(c) Different step lengths for $x$ and $s$ may provide a better accuracy. So choose

$$\alpha_{k,x} = \min\{1, \eta_k \cdot \alpha_{\text{max},x}\} \quad \text{and} \quad \alpha_{k,s} = \min\{1, \eta_k \cdot \alpha_{\text{max},s}\}$$

Use the following update $x_{k+1} = x_k + \alpha_{k,x} \Delta x_k$ and $(\lambda_{k+1}, s_{k+1}) = (\lambda_k, s_k) + \alpha_{k,s} (\Delta \lambda_k, \Delta s_k)$.

Some strategies for choice of centering parameter:

(a) $\sigma_k = 0$, $k = 1, 2, \ldots$, - affine-scaling approach;
(b) $\sigma_k = 1$, $k = 1, 2, \ldots$,
(c) $\sigma_k \in [\sigma_{\text{min}}, \sigma_{\text{max}}] = 1$, $k = 1, 2, \ldots$ Commonly, $\sigma_{\text{min}} = 0.01$ and $\sigma_{\text{max}} = 0.75$ (path following method)
(d) $\sigma_k = 1 - \frac{1}{\sqrt{n}}$, $k = 1, 2, \ldots$, (with $\alpha_k = 1$ - short-step path-following method)
Some termination criteria:

- Recall that, at a solution \((x, s, \lambda)\) equation (12) should be satisfied
  \[
  c = A^T \lambda + s.
  \]
  This is equivalent to
  \[
  c^T = \lambda^T A + s^T.
  \]
  Multiplying both sides by \(x\), we obtain
  \[
  c^T x = \lambda^T A x + s^T x.
  \]
  \[
  \Rightarrow c^T x = b^T x + s^T x.
  \]
  Hence, \(s^T x = c^T x - b^T x\).

- Hence,
  \[
  s^T x = c^T x - b^T x
  \]
  \(s^T x\) is a **measure of gap** between the primal objective function \(c^T x\) and the dual objective function \(b^T \lambda\).
The optimality condition LOP’s demands that: optimal solutions should satisfy $c^\top x = b^\top x$.

So the expression $\mu = \frac{s^\top x}{n} = \frac{c^\top x - b^\top x}{n}$ is known as a measure of the duality gap between LOP and LOP$_D$.

**Termination**

The algorithm can be terminated at iteration step $k$ if the duality gap $\mu_k = \frac{x_k^\top s_k}{n}$ is sufficiently small, say $\mu_k < \varepsilon$. 
Primal-dual Interior Method for LOPs...

Solution strategies for the system of linear equations

\[
\begin{bmatrix}
A & 0 & 0 \\
0 & A^T & I \\
X & 0 & S
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta s
\end{bmatrix}
= \begin{bmatrix}
b - Ax \\
c - A^T \lambda - s \\
\mu e - XSe
\end{bmatrix}
\]

(16)

- The efficiency of the primal-dual interior point methods is highly dependent on the algorithm used to solve this \(2n + m\) linear system.
- The choice of an algorithm depends on the structure and properties of the coefficient matrix \( \begin{bmatrix}
A & 0 & 0 \\
0 & A^T & I \\
X & 0 & S
\end{bmatrix} \).

Sometimes it may be helpful first to eliminate \(\Delta x\) and \(\Delta s\) and solve for \(\Delta \lambda\) from the reduced system

\[
\left(AX^{-1}XA^T\right) \Delta \lambda = AX^{-1} S \left(c - \mu X^{-1} \lambda\right) + b - Ax,
\]

(17)

then to directly compute \(\Delta s = c - A^T \lambda - s - A^T \Delta \lambda\) and \(\Delta x = X^{-1} \left(\mu e - XSe - S\Delta s\right)\).