Introduction to Topological Spaces and Set-Valued Maps

(Lecture Notes)

Abebe Geletu (Dr.)
Institute of Mathematics
Department of Operations Research & Stochastics
Ilmenau University of Technology

August 25, 2006

Contents

1 Preface

These notes are a result of a two semester course that I held at the technical university of Ilmenau during winter semester 2005 and summer semester 2006. These are simply lecture notes organized to serve as introductory course for advanced postgraduate and pre-doctoral students. The main objective is to give an introduction to topological spaces and set-valued maps for those who are aspiring to work for their Ph. D. in mathematics. It is assumed that measure theory and metric spaces are already known to the reader. Hence, only a review has been made of metric spaces. At the same time the topics on topological spaces are taken up as long as they are necessary for the discussions on set-valued maps. Here are to be found only basic issues on continuity and measurability of set-valued maps. Issues on selection functions, fixed point theory, etc. have not be dealt with due to time constraints.

The is not an original work of the writer. In many cases, I have attempted to mention the sources of theorems and statements. I have tried to supply my own versions of simplified proofs, whenever I felt necessary. These is not by far an all-inclusive introductory note. In fact, I leave it at the mercy of the criticisms, suggestions and comments of the reader. However, it is my belief that the material can serve as spring board to dive into the ocean of set-valued maps.

Abebe Geletu, August 2006.

2 Introduction to Metric Spaces

2.1 Introduction

Definition 2.1.1 (metric spaces). *Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}_+$ is called a **metric** on X if the following are satisfied*

M1: $\rho(x, y) \geq 0$*, for each* $x, y \in X$*;*

M2: $\rho(x, y) = 0$ *if and only if* $x = y$*;*

M3: $\rho(x, y) = \rho(y, x)$*;*

M4: for any $z \in X : \rho(x, y) \leq \rho(x, z) + \rho(z, y)$ *(triangle inequality).*

The set X together with the metric ρ is called a **metric space** *and this usually depicted by $< X, \rho >$.*

Example 2.1.2. *Some standard examples of metric spaces*

(a) $< \mathbb{R}^n, \rho >$ *with* $\rho(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^p)^{\frac{1}{p}}$*, where* $p > 0$*. Here, if* $p = 2$ *we obtain the Euclidean or the* l_2 *metric; if* $p = 1$ *we have the* l_1 *metric on* $\mathbb{R}^n$*, and so on.*

(b) $< \mathbb{R}^n, \sigma >$ *with* $\sigma(x, y) = \max_{i=1\ldots n} |x_i - y_i|$.

(c) $< \mathcal{C}[a, b], \rho_\infty >$ *with* $\rho_\infty(f, g) = \max_{t\in[a,b]} |f(t) - g(t)|$*; where* $\mathcal{C}[a, b]$ *represents the space of continuous functions on* $[a, b]$.

*(d) if X is any normed space with a norm $\| \cdot \|$, then $< X, \rho >$ will be a metric space if we define $\rho(x, y) := \|x - y\|$. In this case, ρ is called an **induced metric** - induced by this particular norm on X.*

From Exa. 2.1.2, it is obvious that there might be more than one metric on a given set.

In Def. 2.1.1 M1, M3 and M4 hold true, but M2 fails, then we call the metric ρ a **pseudometric** and the pair $< X, \rho >$ is called a **pseudometric space**. For instance the space L_p of functions is a pseudometric space; with metric induced by the L_p- norm. If all but M3, then $< X, \rho >$ is called a **quasi-metric space.** [1]

Furthermore, if $< X, \rho >$ is a metric space and $x, y \in X$, then the non-negative real number $\rho(x, y)$ can be interpreted as the **distance** between the elements x and y in the metric space $< X, \rho >$. In general, the distance between x and y can be different for a different metric. Note also that, in a metric space X, $x = y$ iff and only if the distance between x and y is 0.

Definition 2.1.3. *Let $< X, \rho >$ be a metric space and S any subset of X. Then the **diameter** of S w.r.t. the metric ρ is defined as:*

$$diamS := sup\{\rho(x, y) \mid x, y \in S\}.$$

*A set S is said to be **bounded** if $diamS < \infty$.*

[1] See Chap. 4 for an example of a quasi-metric.

Remark 2.1.4. *For a subset A of a metric space X, the following are easy to verify*

(i) if A is unbounded, then $diam(A) = \infty$.

(ii) if $cardA > 1$, then $diam(A) > 0$.

Definition 2.1.5 (distance between sets). *Let $< X, \rho >$ be a metric space and $A, B \subset X$ and $A, B \neq \emptyset$. Then the* ***distance between the sets*** *A and B with respect to the metric ρ is given by*

$$\begin{aligned} dist(A, B) \quad &:= \inf\{\rho(x, z) \mid x \in A, y \in B\} \\ &= \inf_{z \in A, x \in B} \rho(x, z). \end{aligned}$$

If $A = \{x\}$, then we write

$$dist(A, B) = dist(x, B).$$

Proposition 2.1.6. *If $A \subset B$, then $dist(A, B) = 0$.*

Definition 2.1.7 (product metric). *Let $< X_i, \rho_i >, i = 1, \ldots, n, n \geq 2$ be metric spaces. Then the product metric space is the Cartesian product*

$$\prod_{i=1}^{n} X_i := X_1 \times \ldots \times X_n$$

along with the metric given by

$$\rho(x, y) = \left[\sum_{i=1}^{n} \rho_i(x_i, y_i)^2 \right]^{\frac{1}{2}},$$

where $x = (x_1, \ldots, x_n) \in \prod_i^n X_i$ and $y = (y_1, \ldots, y_n) \in \prod_i^n X_i$.

2.2 Open and Closed Sets

Definition 2.2.1 (open set). *Let $< X, \rho >$ be a metric space. A set O is called* ***open*** *in X if*

$$\forall x \in O, \exists r > 0 : \{y \in X \mid \rho(x, y) < r\} \subset O.$$

The set

$$\mathbf{B}_r(x) := \{y \in X \mid \rho(x, y) < r\}$$

is called the **open ball** of radius r and center at x. Hence, a set $O \subset X$ is open if for each $x \in X$ there is an open ball $\mathbf{B}_r(x)$ such that $B_r(x) \subset O$.

• The sets X and $\emptyset$ are open. For $x \in X$ and $r > 0$, the open ball $\mathbf{B}_r(x)$ is an open set.

Definition 2.2.2 (neighborhood). *We say that a set $U \subset X$ is a* ***neighborhood*** *of a point x in X iff there is an open set O such that*

$$x \in O \subset U.$$

Definition 2.2.3 (interior of a set). *Let $< X, \rho >$ be a metric space and $A \subset X$.*

(i) A point $x \in A$ is called an ***interior*** *point of A iff*

$$\exists r > 0 : \mathbf{B}_r(x) \subset A;$$

(ii) The collection of all interior points of a set A is known as the ***interior*** *of A and is denoted by $intA$.*

Remark 2.2.4. *For any set A:*

(i) we have $intA \subset A$ and $intA$ is an open set.

(ii) A is an open set iff $A = intA$; i.e. for an open set, all of its elements are in its interior.

Definition 2.2.5 (closed set)**.** *Let $< X, \rho >$ be a metric space and $F \subset X$. Then F is a **closed** set iff $X \setminus F$ is an open set.*

Hence, the complement of an open set is closed and the complement of a closed set is open.

Proposition 2.2.6.

(i) The intersection of any finite number *of open sets is an open set;*

(ii) the union of any collection *of open sets is open.*

(iii) The union of a finite collection of closed sets is closed;

(iv) the intersection of an arbitrary collection of closed sets is closed.

Definition 2.2.7 (accumulation point, closure of a set)**.** *Let $< X, \rho >$ be a metric space and $A \subset X$.*

*(i)A point $\overline{x} \in X$ is an **accumulation** point of A iff*

$$\forall r > 0 : \mathbf{B}_r(\overline{x}) \cap A \neq \emptyset.$$

We denote set of all accumulation points *of a set A by A', and A' is sometimes called the **derived set** of A.*

(ii) The closure of a set A, denoted by clA, is defined as

$$clA := A \cup A'$$

Proposition 2.2.8. *Let A be a subset of a metric space. Then*

(i) $A \subset clA$;

(ii) clA is a closed set;

(iii) A is a closed set iff $A = clA$.

Excercises 2.2.9. *Verify the following properties for the interior and closure of sets A and B.*

(i) $int(A \cup B) \supset intA \cup intB$;

(ii) $cl(A \cup B) = clA \cup clB$;

(iii) $int(A \cap B) = intA \cap intB$;

(iv) $cl(A \cap B) \subset clA \cap clB$;

(v) $diam(A) = diam(clA)$.

(vi) $dist(x, A) = 0 \Leftrightarrow x \in cl\ A$. Consequently, for any set A, we have $dist(A, clA) = 0$.

Definition 2.2.10 (boundary)**.** *Let $A \subset X$ be non-empty. Then* ***boundary*** *∂A of A is defined as*

$$\partial A := clA \setminus intA.$$

Note that, if A is a closed set, then $\partial A \subset A$.

Definition 2.2.11 (dense set)**.** *A subset D of a metric space $< X, \rho >$ is dense in X iff*

$$clD = X.$$

That is, a set is dense if its closure is the whole space.

Proposition 2.2.12. *The following are easy to verify:*

(i) a set D is a dense set iff $int(X \setminus D) = \emptyset$;

(ii) if D is a dense in set and $D \subset B$, then B is also a dense set.

Remark 2.2.13 (On the importance of dense sets). *Note that the density of a set D in a set X (w.r.t. a metric ρ) implicity contains the possibility of the* ***approximatability*** *of the elements of X by the elements of D. In fact, if x_0 is any element of X, we can find an element d_0 of D which is arbitrarily close to x w.r.t. ρ. Specifically, for any $\varepsilon > 0$, the density of D in X w.r.t. ρ implies*

$$\mathbf{B}_\varepsilon(x_0) \cap D \neq \emptyset \Rightarrow \exists d_0 \in D : d_0 \in \mathbf{B}_\varepsilon(x_0) \Rightarrow \exists d_0 \in D : d_0 : \rho(x_0, d_0) < \varepsilon.$$

In this respect, one of the well known results of Karl Weierstrass guarantees that: the set of all polynomials is dense in $C[a, b]$ w.r.t. the metric ρ_∞ *(see. Example 2.1.2). Implying that, every continuous function on $[a, b]$ can be approximated by a polynomial on $[a, b]$ (see for instance Meinardus[18]).*

Definition 2.2.14 (separable metric space)**.** *A metric space X is called separable if it has a countable dense subset; i.e. if there is $D \subset X$ such that D is countable and $clD = X$.*

The Euclidean space $\mathbb{R}^n$ is a standard example of a separable metric space, since $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$, where $\mathbb{Q}$ stands for the set of rational numbers.

Proposition 2.2.15. *A metric space X is separable iff for each open set $O \subset X$ there is a countable family $\{O_k\}$ of open sets such that*

$$O = \bigcup_{O_k \subset O} O_k.$$

2.3 Subspaces of Metric Spaces

Recall that, a metric $\rho : X \times X \to \mathbb{R}_+ = [0, +\infty)$ is a mapping. Hence,

Definition 2.3.1 (subspace of a metric space)**.** *Let $< X, \rho >$ be a metric space and $S \subset X$. The restriction ρ_S of ρ to $S \times S$ is a metric on S and the pair $< \rho_S, S >$ is called a* ***subspace*** *of $< X, \rho >$.*

A set which is closed relative to S may not be closed in X. For instance, consider the space $X = [0, 1]$ with the absolute valued metric $\rho(x, y) = |x - y|$ and $S = (0, 1]$. The set $A = (0, \frac{1}{2}]$ is closded w.r.t. to $< S, \rho_S >$, but not in X.

Note that, a set $A \subset S$ is open relative to S iff, for each $x \in A$, there is $r > 0$ such that

$$x \in \mathbf{B}_r^S(x) := \{y \in S \mid \rho(x,y) < r\} \subset A.$$

This is the same as

$$\exists r > 0 : x \in \mathbf{B}_r(x) \cap S \subset A,$$

for each $x \in A$. With this observation, the following can be easily verified:

Proposition 2.3.2. *Let $< X, \rho >$ be a metric space, S is a subspace of X and $A \subset S$. Then*

(i) if A is open relative to S, then there exists an open set O in X such that $A = O \cap S$;

(ii) if A is closed relative to S, then there exists a closed set F in X such that $A = F \cap S$.

Proof. (i) Let $A \subset S$ be open relative to S implies that, for each $x \in A$, there is $r(x) > 0$ such that $x \in B_{r(x)}(x) \cap S \subset A$. Hence

$$A = \bigcup_{x \in A} \left(B_{r(x)}(x) \cap S\right) = \bigcup_{x \in A} \left(B_{r(x)}(x)\right) \cap S$$

Set $O := \bigcup_{x \in A} \left(B_{r(x)}(x)\right)$. Then O is an open set in X and

$$A = O \cap S.$$

(ii) Exercise!

□

Proposition 2.3.3. *Every subspace of a separable metric space is separable.*

2.4 Sequences, Convergence and Complete Metric Spaces

Given a metric space X, a function $f : \mathbb{N} \to X$ is called a sequence such that for each $n \in \mathbb{N}, f(n) = x_n \in X$. Traditionally, a sequence is represented by the image set $\{x_n\}_{n \in \mathbb{N}}$ (note that $f(\mathbb{N}) = \{x_n\}_{n \in \mathbb{N}}$). In fact, we can drop '$n \in \mathbb{N}$' (when not required) and simply write $\{x_n\}$ for the sequence $\{x_n\}_{n \in \mathbb{N}}$.

Definition 2.4.1 (convergence)**.** *We say that a sequence $\{x_n\} \subset X$* ***converges to*** *a point $\overline{x} \in X$ iff*

$$\forall \varepsilon > 0, \exists N : \rho(x_n, \overline{x}) < \varepsilon, \forall n \geq N.$$

In this case we write

$$x_n \to \overline{x}$$

and we call $\overline{x}$ the ***limit*** *of the sequence $\{x_n\}$.*

Thus, a sequence $\{x_n\}$ converges to $\overline{x}$ iff

$$\forall \varepsilon > 0, \exists N : x_n \in B_\varepsilon(\overline{x}), \forall n \geq N.$$

Equivalently, a sequence $\{x_n\}$ converges to $\overline{x}$ iff

$$\rho(x_n, \overline{x}) \to 0;$$

i.e. the distance between $\overline{x}$ and x_n goes to zero as n goes to infinity.

It is easy to verify that, in a metric space, a convergent sequence has a unique limit point.

Proposition 2.4.2. *Let X be a metric space and $B \subset X$. Then*

(i) a point $\overline{x}$ is in the closure of B iff there is a sequence $\{x_n\} \subset B$ such that $x_n \to \overline{x}$;

(ii) the set B closed iff for every sequence $\{x_n\} \subset B$ and $x_n \to \overline{x}$ implies $\overline{x} \in B$.

Definition 2.4.3 (subsequence)**.** *Let $\{x_n\}$ be a sequence in a metric space X. A subset $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ is called **subsequence** of $\{x_n\}$;*

Corollary 2.4.4. *A point $\overline{x} \in X$ is an accumulation of a sequence $\{x_n\}$, then there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to $\overline{x}$.*

Obviously, *the limit* of a sequence is an accumulation point.

2.4.1 Complete Metric Spaces

Definition 2.4.5 (Cauchy sequence)**.** *Let $< X, \rho >$ be a metric space. A sequence $\{x_n\} \subset X$ is a **Cauchy sequence** iff*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \rho(x_n, x_m) < \varepsilon, \forall n, m \geq N.$$

It is easy to verify that, in every metric space a convergent sequence is always a Cauchy sequence. However, the converse of this statement is not always true. That is, there are metric spaces in which Cauchy sequences may not converge.

Definition 2.4.6 (complete metric space)**.** *A metric space $< X, \rho >$ is said to be a **complete metric space** iff every Cauchy sequence $\{x_n\} \subset X$ converges in X.*

Example 2.4.7. *(Examples of complete metric spaces)*

- *The euclidean space $\mathbb{R}^n, n \in \mathbb{N}$, is a complete metric space.*
- *The space of real-valued continuous functions $\mathcal{C}[a,b] = \{f \mid f : [a,b] \to \mathbb{R}$ and f is continuous$\}$ is complete w.r.t. the metric ρ_∞.*

Proposition 2.4.8. *Let $< X, \rho >$ be a complete metric space and $S \subset X$. Then*

(i) if S is a closed set in X, then $< S, \rho_S >$ is a complete metric subspace of X, conversely;

(ii) if $< S, \rho_S >$ is a complete metric subspace, then S is a closed set in X.

Lemma 2.4.9. *Let $\{x_n\}$ be a sequence. If, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that*

$$\rho(x_n, x_{n+1}) < \varepsilon, \forall n \geq N,$$

then $\{x_n\}$ is a Cauchy sequence.

Proposition 2.4.10 (Cantor's Theorem)**.** *Let $< X, \rho >$ be a complete metric space and $\{F_n\}$ a sequence of subsets of X. If, for each n, F_n is a non-empty closed set,*

$$F_n \supset F_{n+1} \text{ and } \lim_{n\to\infty} diam(F_n) = 0,$$

then $\bigcap_{n=1}^{\infty} F_n$ is non-empty and contains only one element.

Proposition 2.4.11. *Let $< X, \rho >$ be a metric space. If every sequence of closed balls $\{B_{r_n}(x_n)\}$, with the property that $B_{r_n}(x_n) \supset B_{r_{n+1}}(x_{n+1})$ and $r_n \to 0$, has a non-empty intersection, then $< X, \rho >$ is a complete metric space.*

Excercises 2.4.12. *Prove the following*

1. *If a Cauchy sequence $\{x_n\}$ has a accumulation point $\overline{x}$, then $\{x_n\}$ converges to $\overline{x}$.*
2. *If $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in a metric space $< X, \rho >$, then $\rho(x_n, y_n)$ is a convergent sequence of real numbers.*
3. *Given a sequence $\{x_n\}$. Then $\{x_n\}$ is a Cauchy sequence iff the sequence $\{dimA_n\}$ is a decreasing sequence of real number and $dimA_n \downarrow 0$; where*
$$A_n := \{x_k : k \geq n\}.$$
4. *The product $\prod_{i=1}^{n} X_i$ of metric spaces is complete iff each of the metric spaces $X_i, i = 1, \ldots, n$, is complete.*

2.5 Baire Category

Baire had categorized sets in to two groups as: sets of first category and second category.

Definition 2.5.1 (nowhere dense)**.** *Let $< X, \rho >$ be a metric space. A set $E \subset X$ is said to be* ***nowhere dense*** *if $X \setminus clE$ is a dense set in X.*

If E is nowhere dense and $A \subset E$, then A is also nowhere dense.

Proposition 2.5.2. *A subset E of a metric space is nowhere dense if and only if $int(clE) = \emptyset$; i.e. clE contains no open ball.*

Proof.

$$\begin{aligned} \text{E is nowhere dense} \quad & \Leftrightarrow X \setminus clE \text{ is dense} \\ & \Leftrightarrow int\,[X \setminus (X \setminus clE)] = \emptyset \;\; \text{(Rem. 2.2.12)} \\ & \Leftrightarrow int(clE) = \emptyset. \end{aligned}$$

□

Consequently, given a set A, the boundary ∂A is a nowhere dense set.

Theorem 2.5.3 (Baire)**.** *Let X be a complete metric space and $\{O_n\}_{n \in \mathbb{N}}$ a countable collection of dense open subsets of X. Then $\cap O_n$ is dense in X.*

Proof. See for instance P. 158 of Royden[21]. □

Definition 2.5.4 (sets of first and second category)**.** *A set A is of* ***first category*** *if A is a union of a countable collection of nowhere dense sets. If A is not of first category, then it is of* ***second category***.[2]

A nowhere dense set is of first category.

[2]sometimes sets of first category are called meager while those of first category are called nonmeager

Corollary 2.5.5 (Baire Category Theorem). *Any complete metric space is of second category.*

Proof. Follows from Thm. 2.5.3 and Def. 2.5.4. (See also P. 89 of Shirali & Vasudeva[23] for a direct proof). □

Proposition 2.5.6. *Every subset of a set of first category is of first category.*

Proposition 2.5.7. *The union of a countable collection of sets of first category is a gain of first category.*

Corollary 2.5.8. *If X is a complete metric space, then every non-empty open subset of X is of second category; i.e. non-empty open subsets of X cannot be given by a union of a countable number of nowhere dense sets.*

Proof. Let $\emptyset \neq U \subset X$ be an open set. Assume that there is a countable collection $\{E_n\}_{n\in\mathbb{N}}$ of nowhere dense subsets of X such that $U = \bigcup_{n\in\mathbb{N}} E_n$. Then, for each $n \in \mathbb{N}$, $O_n := X \setminus clE_n$ is a dense open subset of X. By Thm. 2.5.3, it follows that

$$\bigcap_{n\in\mathbb{N}} O_n$$

dense in X. Since U is an open set, there is $x \in U \cap \bigcap_{n\in\mathbb{N}} O_n \neq \emptyset$. This implies $x \notin X \setminus \bigcap_{n\in\mathbb{N}} O_n = \bigcup_{n\in\mathbb{N}} E_n$. But this contradicts that $U = \bigcup_{n\in\mathbb{N}} E_n$. Therefore, U is not of first category. □

Remark 2.5.9. *Let X be a complete metric space and $\{E_n\}$ be a countable collection of nowhere dense sets. Then, for any non-empty open set U in X, there is an element $x_0 \in U$ which does not belong to $\bigcup_{n\in\mathbb{N}} E_n$.*

Theorem 2.5.10 (Baire-Hausdorff). *If X is a complete metric space and $A \subset X$ a set of first category, then $X \setminus A$ is dense in X.*

Proof. Let

$$A = \bigcup_{n\in\mathbb{N}} E_n, \text{ where } E_n \text{ is a nowhere dense, for each } n \in \mathbb{N}.$$

Then

$$X \setminus A = \bigcap_{n\in\mathbb{N}} (X \setminus E_n) \supset \bigcap_{n\in\mathbb{N}} (X \setminus clE_n).$$

But the sets $X \setminus clE_n$ are dense open subsets. Then Thm. 2.5.3 implies that $\bigcap_{n\in\mathbb{N}}(X \setminus clE_n)$ is a dense set. Therefore, $X \setminus A$ is a dense set. □

Proposition 2.5.11. *Let X be a metric space.*

(i) If O is an open and F is a closed sets in X, then the sets $clO \setminus O$ and $F \setminus intF$ are nowhere dense;

(ii) Let X be a complete metric space. If a set $F \subset X$ is closed and of first category, then F is nowhere dense.

Corollary 2.5.12. *The boundary ∂A of any set A is a nowhere dense set.*

Excercises 2.5.13. *Prove the following statements:*

1. *A closed set F is nowhere dense iff it contains no open set;*

2. *A set E is nowhere dense iff for any nonempty open set O there is a ball contained in $O \setminus E$;*
3. *Rem. 2.5.9;*
4. *Cor. 2.5.12;*
5. *If A and B are sets of second category, then what can you say about $A \cap B$, $A \cup B$ and $A \setminus B$?*
6. *If a set E is of first category, then any subset $A \subset E$ is also of first category;*
7. *If $\{E_n\}$ is a sequence of set of first category, then $\cup_{n \in \mathbb{N}} E_n$ is also of first category;*
8. *Let E be a subset of a complete metric space. Then if $X \setminus E$ is dense and $F \subset E$ is a closed set, then F is a nowhere dense set.*

2.6 Compact Metric Spaces

Definition 2.6.1 (compact set). *A subset K of a metric space is called* ***compact*** *if every sequence in K has a convergent subsequence in K; i.e. every sequence in K has an accumulation point in K. A metric space $< X, \rho >$ is said to be* ***a compact metric space*** *if X is a compact set.*[3]

Proposition 2.6.2 (a compact set is closed). *If K compact subset of a metric space, then K is a closed set.*

Proof. Use Prop. 2.4.2 and Def. 2.6.1. □

Let $< X, \rho >$ be a metric space and $S \subset X$. A family $\{A_\alpha \mid \alpha \in \Omega\}$ of subsets of X is a **covering** of S if

$$S \subset \bigcup_{\alpha \in \Omega} A_\alpha$$

for some index set Ω. If each $A_\alpha, \alpha \in \Omega$, is an open set, then $\{A_\alpha \mid \alpha \in \Omega\}$ is called an **open covering**. If there is $\Omega' \subset \Omega$ such that

$$S \subset \bigcup_{\alpha \in \Omega'} A_\alpha,$$

then $\{A_\alpha \mid \alpha \in \Omega'\}$ is a **subcovering** of $\{A_\alpha \mid \alpha \in \Omega\}$. If, in this case, the index set Ω' is finite, then we have a **finite subcovering**.

Lemma 2.6.3 (Lebesgue covering Lemma). *If K is a compact set and $\{O_\alpha \mid \alpha \in \Omega\}$ is an open covering of K, then there is $r > 0$[§] such that, for each $x \in K$ the open ball $B_r(x)$ is contained in an element of $\{O_\alpha \mid \alpha \in \Omega\}$; i.e. if K is compact, then, given $x \in K$,*

$$\exists r > 0, \exists \alpha \in \Omega : \mathbf{B}_r(x) \subset O_\alpha.$$

Proof. Let $\{O_\alpha \mid \alpha \in \Omega\}$ is an open covering of K. Assume that, given $x \in K$, for all $r > 0, \mathbf{B}_r(x)$ is not a subset of an element of $\{O_\alpha \mid \alpha \in \Omega\}$. Consequently, for $x_n \in K$

$$\mathbf{B}_{\frac{1}{n}}(x_n) \setminus O_\alpha \neq \emptyset, \forall \alpha \in \Omega \tag{*}$$

[3] The property that every sequence has a convergent subsequence is known as the **Bolzano-Weierstrass propery**.
[§] The number r is usually called the **Lebesgue number** of the set K.

Now, consider the sequence $\{x_n\} \subset K$. By Def. 2.6.1, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\} \subset \{x_n\}$ and $x_{n_k} \to \overline{x} \in K$. But since

$$K \subset \bigcup_{\alpha \in \Omega} O_\alpha,$$

it follows that $\overline{x} \in \bigcup_{\alpha \in \Omega} O_\alpha$. This implies that, for some $\alpha_0 \in \Omega, \overline{x} \in O_{\alpha_0}$. But, since O_{α_0} is an open set, there is $r > 0$ such that

$$\overline{x} \in \mathbf{B}_r(\overline{x}) \subset O_{\alpha_0}.$$

Thus, by the convergence of x_{n_k} to $\overline{x}$, there is a sufficiently large n_k such that

$$\frac{1}{n_k} < \frac{r}{2} \text{ and } x_{n_k} \in \mathbf{B}_{\frac{r}{2}}(\overline{x}).$$

Hence, for any $z \in \mathbf{B}_{\frac{1}{n_k}}(x_{n_k})$, we have

$$\rho(z, \overline{x}) \leq \rho(z, x_{n_k}) + \rho(x_{n_k}, \overline{x}) \leq \frac{1}{n_k} + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r$$

From this follows that

$$\mathbf{B}_{\frac{1}{n_k}}(x_{n_k}) \subset \mathbf{B}_r(\overline{x}) \subset O_{\alpha_0}.$$

But this is a contradiction to (*). Hence, the assumption is false and the claim of the lemma is justified. □

Theorem 2.6.4. *If K is a compact subset of a metric space, then, for every real number $r > 0$, there is a finite number of elements $x_1, \ldots, x_p$ of K such that the system of balls*

$$\{\mathbf{B}_r(x_k) \mid k = 1, \ldots, p\}$$

is an open covering of K.

Proof. Assume that there is $r > 0$ such that, for any finite number of elements $x_1, \ldots, x_n$ of K, the system $\{\mathbf{B}_r(x_p) \mid k = 1, \ldots, p\}$ does not cover K. This implies, given $x_1 \in K$, then $\mathbf{B}_r(x_1)$ does not cover K. Hence, $\exists x_2 \in K \setminus \mathbf{B}_r(x_1)$. Again $\{\mathbf{B}_r(x_1), B_r(x_2)\}$ does not cover K. This implies, $\exists x_3 \in K \setminus (\mathbf{B}_r(x_1) \cup \mathbf{B}_r(x_2))$. Proceeding in this way, given $n \in \mathbb{N}$,

$$\exists x_n \in K \setminus \bigcup_{k=1}^{n-1} \mathbf{B}_r(x_k)$$

Hence, we have constructed a sequence $\{x_n\} \subset K$ with the property that $\rho(x_n, x_m) \geq r$ whenever $m \neq n$. Since, K is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Hence, there is n_{k_0} such that

$$\rho(x_{n_k}, x_{n_l}) < r, \forall k, l \geq n_{k_0}.$$

But this contradicts the fact that $\rho(x_{n_k}, x_{n_l}) \geq r$ for $n_k \neq n_l$. Hence, the assumption is false and the claim of the theorem holds true. □

The theorem next gives an alternative definition for compactness of a set in a metric space. In fact, the following characterization is used to define compactness of a set in general topological spaces in terms of coverings.

Theorem 2.6.5. *Let K be a subset of a metric space. Then the following statements are equivalent:*

(i) The set K is compact;

(ii) Every open covering of K has a finite subcovering.

Proof. "$(i) \Rightarrow (ii)$": Let K be a compact set and $\{O_\alpha \mid \alpha \in \Omega\}$ is an open covering of K. Let $r > 0$ be the Lebesgue number of K (as given by Lem. 2.6.3). Hence, by Thm. 2.6.4, there is a finite number of elements $x_1, \dots, x_n \in K$ such that the system

$$\{B_r(x_k) \mid k = 1, \dots, n\}$$

is a covering of K. Again, by Lem. 2.6.3, each of the balls $B_r(x_k)$ is contained in some element of $\{O_\alpha \mid \alpha \in \Omega\}$; say $B_r(x_k) \subset O_{\alpha_k}$ for some $\alpha_k \in \Omega, k = 1, \dots, n$. Hence

$$K \subset \bigcup_{k=1}^{n} B_r(x_k) \subset \bigcup_{k=1}^{n} O_{\alpha_k}$$

Consequently, the covering $\{O_\alpha \mid \alpha \in \Omega\}$ has a finite subcovering of K.

"$(ii) \Rightarrow (i)$": Let $\{x_n\}$ be any sequence in K. Define the following family of open subsets of K

$$\mathcal{O} := \{O \subset K \mid O \text{ open and } O \text{ contains a finite number of elements of the sequence } \{x_n\}\}.$$

Then $\mathcal{O}$ cannot be a covering of K.(Otherwise, there will be a finite number of open sets $O_1, \dots, O_n$ such that $K \subset \bigcup_{k=1}^{n} O_k$. From this follows that $\{x_n\} \subset \bigcup_{k=1}^{n} O_k$. This implies that, there is at least one $O_{k_0}, 1 \leq k_0 \leq n$, that contains infinitely many element of the sequence $\{x_n\}$, but this contradicts the definition of $\mathcal{O}$). Hence,

$$\exists \overline{x} \in K \setminus \bigcup_{O \in \mathcal{O}} O.$$

Now, for each $k \in \mathbb{N}$, the open ball $B_{\frac{1}{k}}(\overline{x})$ does not belong to $\mathcal{O}$ or cannot be a subset of any of the elements of $\mathcal{O}$. Consequently, for each $k \in \mathbb{N}, B_{\frac{1}{k}}(\overline{x})$ contains infinitely many elements of $\{x_n\}$. Hence, there is a subsequence of $\{x_n\}$ that converges to $\overline{x}$. Therefore, K is a compact set. □

2.6.1 Bounded Sets and Totally Bounded Metric Spaces

Recall that, by Def. 2.1.3, a set B is bounded if $diam\ B$ is a finite real number. Equivalently, B is a bounded set if there exists a real number $M > 0$ such that

$$\rho(x, y) \leq M, \forall x, y \in B.$$

Trivially,

Lemma 2.6.6. *A set B is bounded in a metric space X iff there is an open (or closed) set of finite diameter that contains B.*

Proposition 2.6.7. *A closed subset of a compact metric space is compact. A compact subset of a metric space is both closed and bounded.*

Proof. (a) Let $F \subset K$ be a closed set and $\{O_\alpha \mid \alpha \in \Omega\}$ be an open covering of F. Then the family

$$\{X \setminus F, O_\alpha, \alpha \in \Omega\}$$

is an open covering of K. This implies that, there is a finite subcovering $\{X \setminus F, O_1, \dots, O_n\}$ of K. Consequently, $\{O_1, \dots, O_n\}$ must cover F. Hence, F is a compact set.

(b) Let K be a compact subset of a metric space. The closedness of K has been given by Prop. 2.6.2. Hence, it remains to show the boundedness. Then, by Thm. 2.6.4, there exists a real number $r > 0$ and finite elements $x_1, \ldots, x_N$ such that

$$K \subset \bigcup_{k=1}^{N} \mathbf{B}_r(x_k).$$

Now define

$$O := \bigcup_{k=1}^{N} \mathbf{B}_r(x_k) \text{ and } r_0 := \underbrace{r + \ldots + r}_{N \text{ times}} + \max_{1 \le i,j \le n} \rho(x_i, x_j).$$

Then $diam\ O \le r_0$ and $K \subset O$. Consequently, K is a bounded set.

□

Definition 2.6.8 (total boundedness)**.** *A metric space X is said to be totally bounded iff, for each $\varepsilon > 0$, there is a finite collection $\{x_1, \ldots, x_n\}$ of elements of X such that*

$$\forall x \in X, \exists x_k \in \{x_1, \ldots, x_n\} : \rho(x, x_k) < \varepsilon.$$

Totally bounded metric spaces are also alternatively known as **pre-compact** metric spaces.

Proposition 2.6.9. *If X is a totally bounded metric space, then every sequence in X contains a Cauchy subsequence.*

Proposition 2.6.10. *A metric space X is compact if and only if it is both complete and totally bounded.*

Proof. If K is compact and $r > 0$, then $\{B_r(x) \mid x \in K\}$ has a finite subcover. □

Obviously, a compact metric space is pre-compact (totally bounded). But, for a pre-compact metric to be compact, it needs to be complete.

Excercises 2.6.11. *Prove that*

1. *The intersection of any collection of compact sets is again compact; and the union of a finite number of compact sets is compact.*

2. *If K is a compact subset of a metric space X, then there is a countable family $\{A_n \mid n \in \mathbb{N}\}$ of subsets of X, such that*

$$cl\left(\bigcup_{n \in \mathbb{N}} A_n\right) = K.$$

3. *Every compact metric space is separable.*

4. *If $< X_i, \rho_i >, i = 1, \ldots, n$, are compact metric spaces, then the product $\prod_{i=1}^{n}$ is also a compact metric space. Moreover, if $\{< X_k, \rho_k >\}_{k \in \mathbb{N}}$ is a countable collection of compact metric spaces, then $\prod_{i=1}^{\infty}$ is also a compact metric space.*

5. *Let $< X, \rho >$ be a metric space. A collection $\mathcal{F}$ of subsets of X is said to have the* ***finite intersection property*** *iff every finite subset of $\mathcal{F}$ has a non-empty intersection. Then prove that: if X be a compact, then every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property has a nonempty intersection.*

6. *A metric space X is compact iff every countable collection $\{F_n\}$ of non-empty closed sets in X, with the property $F_n \supset F_{n+1}$ (i.e. $\{F_n\}$ is a* ***nested sequence****), has a non-empty intersection; i.e. $\cap F_n \neq \emptyset$.*
7. *Show that a totally bounded metric space is second countable (has a countable basis).*
8. *The product metric space $\prod_{i=1}^{n} X_i$ is totally bounded iff each $X_i, i = 1, \ldots, n$, is totally bounded.*

2.7 Functions

2.7.1 Continuity of Functions

Let $< X, \rho >$ and $< Y, \sigma >$ be two metric spaces. We consider a function $f : X \to Y$, which associates, to each $x \in X$, a unique element $y \in Y$ such that $f(x) = y$. Sometimes, the terms "function" and "mapping" are used interchangeably.

Definition 2.7.1 (continuous function). *Let f be a function from a metric space $< X, \rho >$ to a metric space $< Y, \sigma >$, written $f : X \to Y$. Then*

(i) f is said to be **continuous at a point** *$x_0 \in X$, if for every $\varepsilon > 0$, there is a $\delta > 0$¶ such that*

$$\sigma(f(x), f(x^0)) < \varepsilon, \text{ for each } x \text{ with } \rho(x, x_0) < \delta;$$

(ii) f is said to be **a continuous function** *(on X) if it is continuous at every point x in X*

Equivalently, f is a continuous function at x_0 iff

$$\forall \varepsilon > 0, \exists \delta > 0 : f(x) \in B_\varepsilon(f(x^0)), \ \forall x \in B_\delta(x_0).$$

This is the same as

$$\forall \varepsilon > 0, \exists \delta > 0 : B_\delta(x_0) \subset f^{-1}\left(B_\varepsilon(f(x^0))\right).$$

Let $f : X \to Y$ be a function, $S \subset X$ and $M \subset Y$. Then

- the **image** of the set S under the mapping f is give by

$$f(S) := \{f(x) \in Y \mid x \in S\};$$

- the **inverse image of M under f** is given by

$$f^{-1}(M) = \{x \in X \mid f(x) \in M\}.$$

Thus we can easily verify the following:

Proposition 2.7.2 (inverse image of an open set). *Let X and Y be metric spaces and $f : X \to Y$ be a function. Then f is a continuous function iff, for every open set $O \subset Y$, $f^{-1}(O)$ is an open set in X.*

Note, that if $f : X \to Y$ is continuous, then, for any closed set $F \subset Y$, $f^{-1}(F)$ is a closed set in X.

Proposition 2.7.3. *The image of a compact set under a continuous mapping is again a compact set.*

¶Properly speaking, δ depends on x_0 and ε; i.e. for a different x_0 we may have a different δ and to show this dependence it is usually written $\delta(x_0, \varepsilon)$.

Proof. Let X and Y be arbitrary metric spaces, $f : X \to Y$ and $K \subset X$ be a compact set. Let $\{O_\alpha \mid \alpha \in \Omega\}$ be an open covering of $f(K)$ in Y. That is,

$$f(K) \subset \bigcup_{\alpha \in \Omega} O_\alpha.$$

$\Rightarrow$

$$K \subset \bigcup_{\alpha \in \Omega} f^{-1}(O_\alpha).$$

Hence, by the continuity of f the collection $\{f^{-1}(O_\alpha) \mid \alpha \in \Omega\}$ is an open covering of K (see. Prop. 2.7.2). But, since K is compact, there is a finite collection $\{O_1, \ldots, O_p\}$ such that

$$K \subset \bigcup_{k=1}^{p} f^{-1}(O_k).$$

Consequently,

$$f(K) \subset \bigcup_{k=1}^{p} O_k.$$

From this we conclude that the image set $f(K)$ is compact. $\square$

Continuity of functions can also be characterized in terms of sequences, as indicated next.

Proposition 2.7.4. *Let X and Y be metric spaces, $f : X \to Y$ and $x_0 \in X$. Then f is continuous at x_0 iff and only for each sequence $\{x_n\}$ that converges to x_0 in X the sequence $\{f(x_n)\}$ converges to $f(x_0)$ in Y.*

Proposition 2.7.5 (continuity of compositions)**.** *Let $< X, \rho >, < Y, \sigma >$ and $< Z, \varrho >$ be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be functions. If f is continuous in X and g is continuous on Y, then the composition $g \circ f$ is a continuous function on X.*

Proposition 2.7.6 (continuity on a product space)**.** *Let $< X_i, \rho_i >, i = 1, \ldots, n,$ be metric spaces and*

$$X = \prod_{i}^{n} X_i,$$

with the product metric ρ on X and $f : X \to Y$ be a function and $< Y, \sigma >$ is a metric space. The function f is continuous at $x^0 = (x_1^0, \ldots, x_n^0)$ iff each of the functions

$$f_i(\cdot) := f(x_1^0, \ldots, x_{i-1}^0, \cdot, x_{i+1}^0, \ldots, x_n^0),$$

are continuous on X_i at $x_i^0, i = 1, \ldots, n.$

Corollary 2.7.7. *A function $f := X \times Y \to Z$ is continuous iff, both functions $f_y(\cdot) : X \to Z$, for each fixed $y \in Y$; and $f_x(\cdot) : Y \to Z$, for each fixed $x \in X$, are continuous; where*

$$f_y(\cdot) := f(\cdot, y) \;\; \textit{and} \;\; f_x(\cdot) := f(x, \cdot).$$

Corollary 2.7.8. *Let $< X, \rho >$ be a metric space. Then the metric $\rho : X \times X \to \mathbb{R}$ is a continuous function.*

Excercises 2.7.9. *Prove the following statements.*

(a) Let $\pi_x : X \times Y \to X$ be the projection mapping and Y be a compact metric space. If $F \subset X \times Y$ is closed, then $\pi_x(F)$ is a closed set in X.

(b) Let $f : X \to Y$ be a function and Y is a compact metric space. If the graph of f

$$Graph(f) := \{(x, y) \in X \times Y \mid y = f(x)\}$$

is a closed set in $X \times Y$, then f is a continuous function. (Hint: for $B \subset Y$, show that $f^{-1}(B) = \pi_x(\pi_y^{-1}(B) \cap Graph(f))$. Use this for a closed set B and apply excercise (a))

2.7.2 Real Valued Functions

Given a metric space X, a function $f : X \to \mathbb{R}$ is a **real valued** function. Continuous real valued functions posses some very important properties.

Proposition 2.7.10. *Let X be a metric space and $f : X \to \mathbb{R}$. Then the following hold true:*

(i) if f is a continuous function, then $-f$ is also a continuous function;

(ii) if f is a continuous function and $\alpha \in \mathbb{R}$, then the set

$$\{x \in X \mid f(x) < \alpha\}$$

is an open set.

From Prop. 2.7.10, we observe that the set

$$\{x \in X \mid f(x) > \alpha\}$$

is also an open set.

Corollary 2.7.11. *If f is a continuous real valued function on a metric space X and $\alpha \in \mathbb{R}$, then the sets*

$$\{x \in X \mid f(x) \le \alpha\}, \{x \in X \mid f(x) \ge \alpha\}$$

and

$$\{x \in \mid f(x) = \alpha\}$$

are closed sets.

Definition 2.7.12 (upper and lower semi-continuous functions)**.** *Let $f : X \to \mathbb{R}$ and $x_0 \in X$. Then*

*(i) f is said to be **upper semi-continuous** at x_0 if*

$$\forall \varepsilon > 0, \exists \delta > 0 : f(x) - f(x_0) < \varepsilon, \forall x \in B_\delta(x_0).$$

*Moreover, f is said to be **an upper semi-continuous function** on X if f is upper semi-continuous at every $x \in X$;*

*(ii) f is said to be **lower semi-continuous**(l.s.c.) if $-f$ is upper semi-continuous.*

Proposition 2.7.13. *Let $f : X \to \mathbb{R}$. Then* *(i) if f is an **upper semi-continuous**(u.s.c) function, then, for every real number $\alpha \in \mathbb{R}$, the set*

$$\{x \in X \mid f(x) > \alpha\}$$

is an open set set.

*(ii) if f is an **lower semi-continuous**(l.s.c) function, then, for every real number $\alpha \in \mathbb{R}$, the set*

$$\{x \in X \mid f(x) < \alpha\}$$

is an open set set.

Corollary 2.7.14. *A real valued continuous function is both lower and upper semi-continuous.*

Corollary 2.7.15. *Let $f : X \to \mathbb{R}$ and let $\alpha \in \mathbb{R}$ be any. Then*

(i) if f is upper semi-continuous, then the set

$$\{x \in X \mid f(x) \leq \alpha\}$$

is a closed set; and

(ii) if f is lower semi-continuous, then the set

$$\{x \in X \mid f(x) \geq \alpha\}$$

is a closed set.

Proposition 2.7.16. *Let $f : X \to \mathbb{R}$ and let $\{x_n\}$ be a sequence that converges to $x_0 \in X$. Then*

(i) if f is u.s.c. at x_0, then

$$\limsup_n f(x_n) \leq f(x_0);$$

(ii) if f is l.s.c. at x_0, then

$$\liminf_n f(x_n) \geq f(x_0).$$

Proof. See for instance pp. 42-43 of Aliprantis & Border [1]. □

Proposition 2.7.17 (Weirstras' Theorem)**.** *Let $f : X \to \mathbb{R}$ and K a compact subset of X. Then*

(i) if f is upper semi-continuous on X, then f assumes its maximum on K; i.e. the problem

$$\max_{x\in K} f(x) = sup$$

has a solution; equivalently, there is $x^* \in K$ such that

$$f(x^*) = \max_{x\in K} f(x) = \sup_{x\in K} f(x)$$

(ii) if f is lower semi-continuous on X, then f assumes its minimum on K; i.e. the problem

$$\min_{x\in K} f(x)$$

has a solution; equivalently, there is $x_* \in K$ such that

$$f(x_*) = \min_{x\in K} f(x) = \inf_{x\in K} f(x).$$

Proof. Cf. pp. 55-56 of Kosmol[17]. □

For metric spaces X and Y, a function $f : X \to Y$ is said to be **bounded on a subset** $S \subset X$ if the image $f(S)$ is a bounded subset of Y. In general, f is called **a bounded function** if f is bounded on X.

Corollary 2.7.18. *Let $f : X \to \mathbb{R}$ and K is a compact subset of X. If f is a continuous function on X, then f assumes both its maximum and minimum values on K; hence, f is a bounded function on K.*

2.7.3 Uniform Continuity

Definition 2.7.19 (uniform continuity)**.** *Let $< X, \rho >$ and $< Y, \sigma >$ be metric spaces and $f : X \to Y$. Then f is said to be uniformly continuous (on X) if given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sigma(f(x), f(z)) < \varepsilon \quad \text{whenever} \quad \rho(x, z) < \delta \text{ and } x, z \in X.$$

Trivially, a uniformly continuous function, is continuous. But, the converse is not always true. For instance, consider the real valued function $f(x) = \frac{1}{x}$.

Proposition 2.7.20. *Let $< X, \rho >$ and $< Y, \sigma >$ be metric spaces and $f : X \to Y$ be uniformly continuous. Then if $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.*

Proof. Let $\{x_n\}$ be a Cauchy Sequence. Suppose an $\varepsilon > 0$ be given. Then, by unform continuity, there is $\delta > 0$ such that

$$\sigma(f(x), f(z)) < \varepsilon, \forall x, z : \rho(x, z) < \delta.$$

Since, $\{x_n\}$ is a Cauchy Sequence, there is $N \in \mathbb{N}$:

$$\rho(x_n, x_m) < \delta, \forall n, m \geq N.$$

From this follows that

$$\sigma(f(x_n), f(x_m)) < \varepsilon, \forall n, m \geq N.$$

□

Proposition 2.7.21. *Let $< X, \rho >$ and $< Y, \sigma >$ be metric spaces and $f : X \to Y$ be a function. Then the following statements are equivalent*

(i) f is uniformly continuous;

(ii) for any pair of sequences $\{x_n\}$ and $\{y_n\}$, if $\rho(x_n, y_n) \to 0$, then $\sigma(f(x_n), f(y_n)) \to 0$.

Proof. (i)" $\Rightarrow$ " (ii) Triviall!! (similar to the proof of Prop. 2.7.20).

(ii) $\Leftarrow$(i) Prove by contradiction.

□

Proposition 2.7.22. *If $f : X \to Y$ is a continuous function and X is a compact set, then f is uniformly continuous on X.*

Proof. Assume that f is not uniformly continuous and arrive at a contradiction.

□

Excercises 2.7.23. *Prove that:*

(a) If $f : X \to Y$ is uniformly continuous and X is totally bounded, then $f(X)$ is totally bounded.

(b) Let $f : X \to X$ be a continuous function. If X is a compact metric space, then there is a non-empty subset $A \subset X$ such that $f(A) = A$. (Hint: use $X_1 = f(X), X_2 = f(X_1)$, and so on. In general, $X_{n+1} = f(X_n)$ and $A := \bigcap_{k=1}^{\infty} X_k$).

2.7.4 Convergence Properties of Sequences of Functions

Let X and Y be metric spaces. Then, for each $n \in \mathbb{N}$, we consider a function $f_n : X \to Y$. Consequently, for each fixed $x \in X$, we have a sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ of elements of Y. We also refer to the sequence $\{f_n\}$ as a sequence of functions.

Definition 2.7.24 (pointwise convergence)**.** *Let X and Y be metric spaces and $S \subset Y$. A sequence of functions $\{f_n\}$ is said to be **pointwise** convergent to a function $f : X \to Y$ on S if, for each fixed $x \in S$, we have*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

*We write $f_n \to f$ **pointwise** on S.*

Even if the functions f_n are continuous, for all $n \in \mathbb{N}$, the pointwise limit function f may not be continuous.

Example 2.7.25. *Let $f_n(x) = x^n, x \in [0, 1]$. Hence,*

$$\lim_{n} f_n(x) = f(x),$$

where

$$f(x) = \begin{cases} 0, x \in [0, 1) \\ 1, x = 1. \end{cases}$$

Hence, for f with $\lim_{n\to} f_n(x) = f(x)$ to be continuous we need a strong convergence property.

Definition 2.7.26 (uniform convergence)**.** *Let $< X, \rho >$ and $< Y, \sigma >$ be metric spaces and $S \subset Y$. A sequence of functions $\{f_n\}$ is said to be **uniformly** convergent to a function $f : X \to Y$ on S if, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that*

$$\sigma(f_n(x), f(x)) < \varepsilon, \forall n \geq N, \forall x \in S.$$

*In this case, we write $f_n \to f$ **uniformly** on S.*

Obviously,

- unform convergence is stronger than pointwise convergence; i.e. uniform convergence implies pointwise convergence.
- if $D \subset S$ and $\{f_n\}$ is uniformly convergent on S, then it is also uniformly convergent on D.

Hence, the following is a direct consequence of Def. 2.7.26.

Proposition 2.7.27. *Let $S \subset X$, $\{f_n\}$ be a sequence such that $f_n \to f$ pointwise on S and, for each $n \in \mathbb{N}, M_n := \sup_{x\in S} \sigma(f_n, f(x))$. Then $f_n \to f$ uniformly on S iff $M_n \to 0$. In short*

$$f_n \to f \ \textit{ uniformly on } S \ \Leftrightarrow \lim_{n\to\infty} \left[\sup_{x\in S} \sigma(f_n(x), f(x)) \right] = 0.$$

Theorem 2.7.28. *Let X and Y be metric spaces, $S \subset X$ and, for each $n \in \mathbb{N}$, $f_n : X \to Y$ be a continuous function. If $f_n \to f$ uniformly on S, then f is a continuous function on S.*

Proof. Suppose we are given an arbitrary point $x_0 \in S$. If $\varepsilon > 0$, then for any $x \in S$

$$\sigma(f(x), f(x_0)) \leq \sigma(f(x), f_n(x)) + \sigma(f_n(x), f(x_0))$$

This implies

$$\sigma(f(x), f(x_0)) \leq \sigma(f(x), f_n(x)) + \sigma(f_n(x), f_n(x_0)) + \sigma(f_n(x_0), f(x_0))$$

By the continuity of f_n, there is a $\delta > 0$ such that

$$\sigma(f_n(x), f_n(x_0)) < \frac{\varepsilon}{3}, \forall x \in B_\delta(x_0).$$

Thus, by the uniform convergence of f_n to f on S, we see that f_n converges uniformly to f on $S \cap B_\delta(x_0) =: B_\delta^S(x_0)$. This implies, there is $N > 0$ such that

$$\sigma(f(x), f_n(x)) < \frac{\varepsilon}{3}, \forall n \geq N, \forall x \in B_\delta^S(x_0).$$

Consequently,

$$\sigma(f(x), f(x_0)) < \varepsilon, \forall x \in B_\delta^S(x_0).$$

Hence, f is continuous at x_0 relative to S. Since, $x_0 \in S$ is arbitrary, we conclude that f is a continuous function relative to S; therefore, f is continuous on S. □

Excercises 2.7.29. *Prove the following statements:*

(a) Prop. 2.7.27.

(b) Which of the following is uniformly convergent

(i) $f_n(x) = xe^{-nx^2}$,

(ii) $f_n(x) = n^2x(1-x)^n$,

(iii) $f_n(x) = \frac{x}{(1+nx)}$,

(iv) $f_n(x) = \frac{x^n}{(1+x^n)}$.

(c) Let f_n be a sequence of real valued functions. If $f_n \to f$ uniformly on S and, for each $n \in \mathbb{N}$, f_n is bounded on D, then

(i) f is bounded on S;

(ii) there is a uniform bound for $\{f_n\}$; i.e. there is $M \in \mathbb{R}$ such that

$$|f_n(x)| \leq M, \forall n \in \mathbb{N}, \forall x \in S.$$

(d) Let $f_n \to f$ uniformly on S. If, for each $n \in \mathbb{N}$, f_n is uniformly continuous on X, then f is also uniformly continuous on X.

2.7.5 Equicontinuiuty and the Ascoli-Arzelá Theorem

In many situations we may need to know if a sequence of functions $\{f_n\}$ has a convergent subsequence.

Definition 2.7.30 (equicontinuiuty). *Let $\mathcal{F}$ be a family of functions from a metric space $< X, \rho >$ to a metric space $< Y, \sigma >$. The family $\mathcal{F}$ is said to be* **equicontinuous** *at $x_0 \in X$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\sigma(f(x), f(x_0)) < \varepsilon, \forall x \in B_\delta(x_0), \forall f \in \mathcal{F}.$$

The family $\mathcal{F}$ is called equicontinuous (on X) if it is equicontinuous at each point x in X.

Lemma 2.7.31. *Let X and Y be metric spaces and $D \subset X$, and $\{f_n\}$ be a sequence of functions from X to Y. If D is a countable set and, for each $x \in S$, the set $\{f_n(x) \mid n \in \mathbb{N}\}$ is compact, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}(x)\}$ converges for each $x \in D$.*

Proof. Let $D = \{x_1, x_2, \ldots\}$. For $x_1 \in D$, there is a convergent subsequence $\{f_{1n}(x_1)\}$ of $\{f_n(x_1)\}$(Observe that $\{f_{1n}(x_1)\} \subset cl\{f_{1n}(x_1)\}$ and $\{f_{1n}(x_1)\}$ is compact)[‖]. For $x_2 \in D$, there is a convergent subsequence $\{f_{2n}(x_2)\}$ of $\{f_{1n}(x_2)\}$, and so on. Proceeding in this manner, for $x_k \in D$, we obtain a convergent subsequence $\{f_{kn}(x_k)\}$ of $\{f_{(k-1)n}(x_{k-1})\}$. Hence

$$\begin{array}{l}
f_{11}(x_1), f_{12}(x_1), \ldots, f_{11}(x_1), \ldots \\
f_{21}(x_2), f_{22}(x_2), \ldots, f_{2n}(x_2), \ldots \\
f_{31}(x_3), f_{32}(x_3), \ldots, f_{3n}(x_3), \ldots \\
\vdots \qquad\quad \vdots \qquad\quad \vdots \\
f_{k1}(x_k), f_{k2}(x_k), \ldots, f_{kn}(x_k), \ldots \\
\vdots \qquad\quad \vdots \qquad\quad \vdots
\end{array}$$

Now, consider the (diagonal) sequence $\{f_{nn}\}$. The sequence $\{f_{nn}\}$ is a subsequence of $\{f_{kn}\}$ for each $k \geq n$. Hence, for each $x_k \in S$, $\{f_{nn}(x_k)\}$ is convergent. ☐

Remark 2.7.32. *In Lem. 2.7.31, it is enough to have $cl\{f_n(x) \mid n \in \mathbb{N}\}$ compact, for each $x \in D$.*

Lemma 2.7.33. *Let $\{f_n\}$ be an equicontinuous sequence of functions from a metric space X to a complete metric space Y. If the sequence $\{f_n(x)\}$ converges for each point x in a dense subset D of X, then*

(i) $\{f_n\}$ converges at each point x in X, and

(ii) the limit function is continuous.

Proof. (i) Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Then there is $\delta > 0$ such that

$$\sigma(f_n(x), f_n(y)) < \varepsilon, \forall y \in B_\delta(x), \forall n \in \mathbb{N}. \tag{2.1}$$

Since D is a dense set, there is $\overline{y} \in D \cap B_\delta(x)$. Thus, by assumption, $\{f_n(\overline{y})\}$ is a convergent sequence. This implies that $\{f_n(\overline{y})\}$ is a Cauchy sequence. Hence, there is a sufficiently large N such that

$$\sigma(f_n(\overline{y}), f_m(\overline{y})) < \frac{\varepsilon}{3}, \forall n, m \geq N. \tag{2.2}$$

[‖]A discrete subset of a compact set is compact.

From (2.1) and (2.2), it follows that

$$\sigma(f_n(x), f_m(x)) \leq \sigma(f_n(x), f_n(\overline{y})) + \sigma(f_n(\overline{y}), f_m(\overline{y})) + \sigma(f_m(x), f_m(\overline{y})) < \varepsilon, \forall n, m \geq N.$$

This implies that $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is a complete metric space, we conclude that $\{f_n(x)\}$ is convergent.

(ii) For $x \in X$, let $f(x) = \lim_{n\to\infty} f_n(x)$. To show f is continuous at x, let $\varepsilon > 0$ be given. Since, $\{f_n\}$ is equicontinuous at x, there is $\delta > 0$ such that

$$\sigma(f_n(x), f_n(y)) < \varepsilon, \forall x \in B_\delta(x), \forall n \in \mathbb{N}.$$

This implies

$$\sigma(f(x), f(y)) = \lim_{n\to\infty} \sigma(f_n(x), f_n(y)) \leq \varepsilon, \forall x \in B_\delta(x).$$

Since $\sigma : Y \times Y \to \mathbb{R}_+$ is continuous(see Cor.2.7.8). Therefore, f is continuous at x. Since, $x \in X$ is arbitrary, we conclude that f is a continuous function.

□

Lemma 2.7.34. *Let X and Y be metric spaces, $K \subset X$ and $\{f_n\}$ be an equicontinuous sequence. If K is compact and $\{f_n(x)\}$ converges to $f(x)$ at each $x \in K$, then $\{f_n\}$ converges uniformly to f on K.*

Proof. Let $\varepsilon > 0$. By the equicontinuiuty of $\{f_n\}$, for each $x \in K$, there is an open ball $B_{\delta(x)}(x)$ such that

$$\sigma(f_n(x), f_n(y)) < \frac{\varepsilon}{3}, \forall y \in B_{\delta(x)}(x), \forall n.$$

This implies that

$$\sigma(f(x), f(y)) < \frac{\varepsilon}{3}, \forall y \in B_{\delta(x)}(x),$$

Hence, by the compactness of K, there is a finite collection $\{x_1, \ldots, x_m\}$ such that

$$K \subset \bigcup_{i=1}^{m} B_{\delta(x_i)}(x_i).$$

Since, $f_n(x_i) \to f(x_i)$, for each $i = 1, \ldots, m$, choose N sufficiently large so that

$$\sigma(f_n(x_i), f(x_i)) < \frac{\varepsilon}{3}, \forall i \in \{1, \ldots, m\}, \forall n \geq N.$$

Then, for any $y \in K$, there is $i_0 \in \{1, \ldots, m\}$ such that $y \in B_{\delta(x_{i_0})}(x_{i_0})$. Hence

$$\sigma(f_n(y), f(y)) \leq \sigma(f_n(y), f_n(x_{i_0})) + \sigma(f_n(x_{i_0}), f(x_{i_0})) + \sigma(f_n(x_{i_0}), f(y)) < \varepsilon, \forall n \geq N.$$

Consequently, $\{f_n\}$ converges uniformly to f on K. □

Using the above three lemmas and the fact that a compact subset of a metric space is compete, one can verify the validity of the following well known theorem:

Theorem 2.7.35 (Ascoli-Arzelá Theorem, Thm. 40, p. 169, Royden[21]). *Let $\mathcal{F}$ be a family of equicontinuous functions from a separable metric space X to a metric space Y. Let $\{f_n\}$ be a sequence in $\mathcal{F}$ such that for each $x \in X$ the closure of the set $\{f_n(x) \mid n \in \mathbb{N}\}$ is compact. Then there is a subsequence $\{f_{n_k}\}$ that converges pointwise to a continuous function f, and the convergence is uniform on each compact subset of X.*

Proof. (see also pp. 189-191 of Shirali & Vasudeva [23])

(a) Since X is separable, there is $D \subset X$ such that D is dense and countable. Then $cl\{f_n(x) \mid n \in \mathbb{N}\}$ is a compact set, for each $x \in D$. Then, by Lem. 2.7.31, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}(x)\}$ converges for each $x \in D$.

(b) Since $\mathcal{F}$ is equicontinuous, then $\{f_{n_k}\}$ is equicontinuous, too. Consequently, by Lem. 2.7.33, $f_{n_k} \to f$ pointwise on X, where f is a continuous function.

(c) Since $\{f_{n_k}\}$ is equicontinuous, Lem. 2.7.34 concludes that $f_{n_k} \to f$ uniformly on X.

□

2.7.6 Homeomorphisms and Isometries in Metric Spaces

Definition 2.7.36 (a homeomorphism)**.** *Let X and Y be metric spaces. A function $f : X \to Y$ is an* ***homeomorphism*** *between X and Y if*

(i) f is a one-to-one and onto function; and

(ii) both f and f^{-1} are continuous functions.

When there is a homeomorphism between two metric spaces X and Y, we say that X and Y are **homeomorphic metric spaces**.

Theorem 2.7.37. *Let $f : X \to Y$ be both one-to-one and onto. Then the following statements are equivalent:*

(i) f is a homeomorphism from X to Y;

(ii) for each subset $A \subset X, f(clA) = cl(f(A))$;

(iii) for each closed set $F \subset X, f(F)$ is closed in X; and for each closed set in $E \subset Y, f^{-1}(E)$ is closed in X;

(iv) for every open set $O \subset X$, $f(O)$ is open in Y; and for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

Consequently, when two metric spaces X and Y are homeomorphic, it follows that: if X is complete, then Y will be complete; if X is separable, then Y will be separable; if X is compact, then Y will be compact, and so on. In general, properties of the metric space X, that could be characterized by open sets, also hold true in Y; vice versa. Such properties are usually known as **topological properties**[**]. Hence, homeomorphic metric spaces have identical topological properties.

However, note that distance is not a topological property; i.e., even if $f : X \to Y$ is a homeomorphism, the distance $\rho(x, y)$, for $x, y \in X$, may not be the same as the distance $\sigma(f(x), f(y))$.

Example 2.7.38. *Let $< X, \rho >$ and $< Y, \sigma >$ with $X = Y = \mathbb{R}^2$ and, for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have*

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ and } \sigma(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

The identity map $\iota : X \to Y$ is a homeomorphism between and X and Y, but $\rho(x, y) \neq \sigma(\iota(x), \iota(y))$.

Definition 2.7.39 (isometry)**.** *Let $< X, \rho >$ and $< Y, \sigma >$ be two metric spaces. A mapping $f : X \to Y$ is an* ***isometric mapping*** *(or simply an* ***isometry****) if, for each $x, y \in X$,*

$$\rho(x, y) = \sigma(f(x), f(y)).$$

Two metric spaces X and Y are called isometric if there is an isometry between them.

[**]In other words, a property that remains true under homeomorphic maps is said to be a **topological property**.

Accordingly, an isometry preserves distance.

Proposition 2.7.40. *Let X and Y be metric spaces and $f : X \to Y$.*

(i) If $f : X \to Y$ is an isometry, then f is is a continuous function.

(ii) If $f : X \to Y$ is an isometry, then f is one-to-one.

*(iii) If $f : X \to Y$ is an isometry, then f is one-to-one (**injective**).*

Hence, an isometry $f : X \to Y$ will be a homeomorphism between X and Y if it maps X **onto** Y; i.e. if it is **surjective**. In fact, an isometric map is a homeomorphism between X and $f(X)$. But, not every homeomorphism is an isometric mapping. Hence, a homeomorphism may not preserve distance.

Excercises 2.7.41. *(i) Give some examples of properties which are not topological.*

(ii) Let X and Y be a metric spaces, $f : X \to X$ be a continuous function and $h : Y \to X$ is a homeomorphism, then $f \circ h^{-1}$ is a continuous function on Y.

2.7.7 Contractive Maps and Fixed Point Properties

Definition 2.7.42 (Lipschitz Continuity)**.** *Let $< X, \rho >$ and $< Y, \sigma >$ be two metric spaces. A function $f : X \to Y$ is said to be **Lipschitz continuous**, if there is a constant $L > 0$ such that, for each $x, z \in X$*

$$\sigma(f(x), f(z)) \leq L\ \rho(x, z). \tag{2.3}$$

*The smallest number L for which (2.3) holds is called the **Lipschitz constant** of the function f.*

It is easy to verify that: an isometry is Lipschitz continuous - with Lipschitz constant $L = 1$.

Example 2.7.43. *Let $< X, \rho >$ and $< Y, \sigma >$ be as given in Example 2.7.38 with the identity map $\iota : X = \mathbb{R}^2 \to Y = \mathbb{R}^2$. Then ι is a Lipschitz continuous function. (But, recall that, ι is not an isometry).*

Corollary 2.7.44. *Every Lipschitz continuous function is uniformly continuous.*

Definition 2.7.45 (contraction and non-expansive maps)**.** *Let $< X, \rho >$ be a metric space and $f : X \to X$. If there is a constant $\gamma \in [0, 1]$ such that*

$$\rho(f(x), f(y)) \leq \gamma\ \rho(x, y), \forall x, y \in X,$$

then

*(i) f is called **contractive** if $0 \leq \gamma < 1$;*

*(ii) f is called **non-expansive** if $0 \leq \gamma \leq 1$.*

Trivially, a non-expansive map is Lipschitz continuous. Hence, $f : X \to X$ is a contractive map if there is $\gamma \in [0, 1)$ such that

$$\rho(f(x), f(y)) \leq \rho(x, y).$$

Definition 2.7.46. *Let $< X, \rho >$ be a metric space and $f : X \to X$ a map. Then we say that $x \in X$ is a* ***fixed point*** *of f if*

$$f(x) = x.$$

The relation $f(x) = x$ is a fixed point equation.

Theorem 2.7.47 (Banach Fixed point Theorem)**.** *Let $< X, \rho >$ and $f : X \to X$. If X is a complete metric space and f is a contractive map (with $\gamma \in [0, 1)$), then f has a unique fixed point in X; i.e. there is $\overline{x} \in X$ such that*

$$f(\overline{x}) = \overline{x}.$$

Proof. Existence: Let $x_0 \in X$ be arbitrary and set $x_1 = f(x_0), x_2 = f(x_1)$, and so on, so that $x_{n+1} = f(x_n), n = 0, 1, 2, \ldots$ We show that the sequence $\{x_n\}$ is convergent. Thus, it is enough to show that $\{x_n\}$ is a Cauchy sequence. Let $n > m$. Note that

$$\begin{aligned}\rho(x_{k+1}, x_k) = \rho(f(x_k), f(x_{k-1})) \quad &\leq \gamma\rho(x_k, x_{k-1}) = \rho(f(x_{k-1}), f(x_{k-2})) \leq \gamma^2\rho(x_{k-1}, x_{k-2}) \\ &\leq \ldots \leq \gamma^k\rho(x_1, x_0).\end{aligned}$$

Hence,

$$\rho(x_n, x_m) \leq \rho(x_n, x_{n-1}) + \rho(x_{n-1}, x_{n-2}) + \ldots + \rho(x_{m+1}, x_m).$$

From this it follows that

$$\rho(x_n, x_m) \leq \left(\gamma^{n-1} + \ldots + \gamma^m\right)\rho(x_1, x_0) = \frac{\rho(x_1, x_0)}{1-\gamma}\left[\gamma^m - \gamma^n\right].$$

The sequence $\{\gamma^n\}$ is a Cauchy sequence. Consequently, $\{x_n\}$ is a Cauchy sequence. Since, X is complete, there is $\overline{x} \in X$ such that $x_n \to \overline{x}$. And f is contractive

$$\rho(f(x_n), f(\overline{x})) \leq \gamma\rho(x_n, \overline{x}).$$

This implies

$$f(\overline{x}) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \overline{x}.$$

Therefore, $f(\overline{x}) = \overline{x}$.

Uniqueness: Let $\overline{x}, \overline{y} \in X$ such that $f(\overline{x}) = \overline{x}$ and $f(\overline{y}) = \overline{y}$. Then

$$\rho(\overline{x}, \overline{y}) = \rho(f(\overline{x}), f(\overline{y})) \leq \gamma\rho(\overline{x}, \overline{y}).$$

$\Rightarrow$

$$(1-\gamma)\rho(\overline{x}, \overline{y}) \leq 0 \Rightarrow \rho(\overline{x}, \overline{y}) \leq 0 \Rightarrow \overline{x} = \overline{y}.$$

□

Corollary 2.7.48. *Let X be a complete metric space, $f : X \to X$ and f is a contractive mapping. If $\overline{x} \in X$ is the fixed point of f, then*

$$\rho(x, \overline{x}) \leq \frac{1}{1-\gamma}\rho(f(x), x)$$

for any $x \in X$.

Proof. In the the proof of Thm. 2.7.47 take $x_0 = x$. □

Cor. 2.7.48 gives an estimate of how far the chosen initial iterate x_0 lies from the fixed point $\overline{x}$ of f.

Excercises 2.7.49. *Prove the following statements:*

(i) Let $S \subset X$ and define function

$$d(x) := \inf_{z \in S} \rho(x,z) =: dist(x,S),$$

for each $x \in X$ - known as the distance function. *Then d is a Lipschitz continuous function from X to $\mathbb{R}$ ($\mathbb{R}$ with the usual metric).*

(ii) Let $< \mathbb{R}^n, d_\infty >$, with $d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|$, $b \in \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping given by $Tx = Ax + b$ for an $n \times n$ matrix $A = (a_{ij})_{1 \le 1, i, j, \le n}$. Show that if there is α such that

$$\sum_{j=1}^{n} |a_{ij}| \le \alpha < 1, i = 1, \ldots, n,$$

then the system of equations

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i, i = 1, \ldots, n$$

has a unique solution. (Hint: Show that T is contractive).

(iii) Suppose X is a complete metric space and $T : X \to X$ such that, form some integer n, $T^{(n)}$ is a contraction, where

$$T^{(n)} = \underbrace{T \circ T \circ \ldots \circ T}_{r \text{ times}},$$

then T has a unique fixed point $\overline{x}$ in X and, for any $x \in X$, the sequence $\{T^n(x)\}$ converges to $\overline{x}$.

3 Topological Spaces

Definition 3.0.50 (topological space)**.** *Let X be a non-empty set. Then a family τ of subsets of X is called a **topology** on X if the following statements (axioms) hold true*

A1: X and $\emptyset$ are elements of τ;

A2: $A, B \in \tau \Rightarrow A \cap B \in \tau$;

A3: for any family $\{A_\alpha : \mid \alpha \in \Omega\} \subset \tau$, $\bigcup_{\alpha \in \Omega} A_\alpha \in \tau$.

*The set X with a topology τ is called a **topological spaces**, denoted by $< X, \tau >$.*

Note that if τ is a topology, then the intersection of any finite number of elements of τ is again an element of τ. In a topological space $< X, \tau >$, the elements of τ are called **open sets**.

Example 3.0.51. *Examples of topological spaces.*

*(i) Let $\tau_1 = \{X, \emptyset\}$, then $< X, \tau_1 >$ is a topological space. The topology τ_1 in known as the **trivial topology**. The only open sets are X and $\emptyset$.*

*(ii) For $X \neq \emptyset$, let $\tau_2 = 2^X$, then $< X, \tau_2 >$ is a topological space. This topology in known as the **discrete topology**. Here, every subset of X is an open set.*

*(iii) If $< X, \rho >$ is a metric spaces and τ_3 is a family of open sets of $< X, \rho >$, then $< X, \tau_3 >$ is **a topological space associated with the metric** ρ. Hence, different metrics may give rise to different topologies on a given set.*. A topological space which could associated with a metric space is called metrizable.*

Observe that, there might be several topologies being defined on a given set. (see Example 3.0.51(i) & (ii)).

Proposition 3.0.52. *Let $\{\tau_\alpha \mid \alpha \in \Omega\}$ be any collection of topologies of a set X. Then the intersection*

$$\bigcap_{\alpha \in \Omega} \tau_\alpha$$

is again a topology of X.

Suppose that τ and σ be two topologies on a set X. If $\tau \subset \sigma$, then we say that σ is a **stronger (finer)** topology than τ; equivalently, τ is a **weaker (coarse)** topology than σ. Consequently, topologies are partially ordered with respect to "$\subset$".

On a given set X, the trivial topology is the weakest topology and the discrete topology is the strongest topology.

Proposition 3.0.53. *Let X be a non-empty set and $\mathcal{C}$ be any collection of subsets of X. Then there is a weakest topology $\mathcal{W}$ that contains $\mathcal{C}$.*

*If two metrics are equivalent, then they give rise to the same topology

Proof. Follows from Prop. 3.0.52. □

Proposition 3.0.54 (relative topology)**.** *Let $< X, \tau >$ be a topological space and $S \subset X$. Then the family of sets*

$$\tau_S := \{S \cap O \mid O \in \tau\}$$

*is a topology on S. The topology τ_S is called the **relative topology** or the **induced topology** on S.*

Definition 3.0.55 (closed set)**.** *A set $F \subset X$ is said to be **closed** if $X \setminus F \in \tau$.*

Proposition 3.0.56. *In a topological space $< X, \tau >$,*

(i) the sets $\emptyset$ and X are both closed and open;

(ii) if $\{F_1, \ldots, F_n\}$ is any finite collection of closed subsets of X, then $\bigcup_{i=1}^{n} F_i$ is closed;

(iii) if $\{F_\alpha \mid \alpha \in \Omega\}$ is any collection of closed subsets of X, then $\bigcap_{\alpha\in\Omega} F_\alpha$ is closed.

Definition 3.0.57 (closure,interior)**.** *Let $< X, \tau >$ be a topological space and $A \subset X$. Then*

(i) the intersection of all closed sets containing A is called the closure of A, denoted by clA; i.e.

$$clA := \bigcap\{F \mid A \subset F \text{ and } X \setminus F \in \tau\}.$$

(ii) the union of all open sets contained in A is called the interior of A, denoted by $intA$; i.e.

$$intA := \bigcup\{O \mid O \subset A \text{ and } O \in \tau\}.$$

Corollary 3.0.58. *For a set A, clA is the smallest closed set containing A and $intA$ is the largest open set contained in A.*

Proposition 3.0.59. *Let $< X, \tau >$ be a topological space and $A, B \subset X$. Then*

(i) $cl(A \cup B) = clA \cup clB$;

(ii) $cl(A \cap B) \subset clA \cap clB$;

(iii) $int(A \cap B) = intA \cap intB$;

(iv) $int(A \cup B) \supset intA \cup intB$;

(v) $int(X \setminus A) = X \setminus clA$.

Definition 3.0.60 (exterior, boundary)**.** *Let $< X, \tau >$ be a topological space and $A \subset X$. Then*

*(i) the **exterior** of the set A, denoted $extA$, is defined as*

$$extA := int(X \setminus A).$$

*(ii) the **boundary** of the set A, denoted ∂A, is defined as*

$$\partial A := X \setminus (extA \cup intA).$$

Definition 3.0.61 (accumulation point)**.** *Let $< X, \tau >$ be a topological space and $A \subset X$.*

*(i) A point $x \in X$ is an **accumulation point** of A iff*

$$O \in \tau, x \in O \Rightarrow O \cap (A \setminus \{x\}) \neq \emptyset.$$

Denote by A' the set of all accumulation points of A.

*(i) A point $x \in A$ is an **interior point** of A iff*

$$\exists O \in \tau, x \in O \subset A.$$

Denote by $\overset{0}{A}$ the set of all interior points of A.

Proposition 3.0.62. *Let $< X, \tau >$ be a topological space. Then*

(i) for any set A, it follows that $A' \subset clA$;

(ii) F is a closed set if and only if $F = clF = F \cup F'$; i.e. if and only if $F' \subset F$;

(iii) for any set A, we have $\overset{0}{A} = intA$;

(iv) if O is an open set, then $O = intO$.

Proof. (ii),(iii) and (iv) follow trivially. Thus it remains to show (i). Let $x \in X$, but $x \notin clA$. Then there is a closed set F such that $A \subset F$ such that $x \notin F$.
$\Rightarrow$

$$x \in X \setminus F =: O.$$

$\Rightarrow$

$$x \in O, \text{ but } O \cap (A \setminus \{x\}) = \emptyset \Rightarrow x \notin A'.$$

Consequently, $A' \subset clA$.

□

Definition 3.0.63 (dense set). *Let $< X, \tau >$ be a topological space. A subset D of X is **dense** in X iff $clD = X$.*

Thus if a set D is dense in X, then any element $x \in X$ is an accumulation point of D. Hence, we have

Proposition 3.0.64. *If D is dense in X, then*

$$\forall O \in \tau : O \cap D \neq \emptyset.$$

Definition 3.0.65 (separable topological space). *A topological space X is separable if it has a countable dense subset D.*

3.1 Neighborhood and Neighborhood Systems

Definition 3.1.1 (neighborhood). *Let $< X, \tau >$ be a topological space and $x \in X$. A set $U, U \subset X$, is called a neighborhood of x if there is an open set $O \in \tau$ such that*

$$x \in O \subset U.$$

*If a neighborhood U is an open set, then it is called an **open neighborhood**.*

Definition 3.1.2. *[neighborhood system] Let $x \in X$ and $\mathcal{N}_x \subset 2^X$. Then $\mathcal{N}_x$ is said to be a* ***neighborhood system*** *of x if the following (neighborhood axioms) are satisfied:*

N1: $N \in \mathcal{N}_x \Rightarrow x \in N$;

N2: $N \in \mathcal{N}_x$ *and* $N \subset A \Rightarrow N \in \mathcal{N}_x$;

N3: $N_1, N_2 \in \mathcal{N}_x \Rightarrow N_1 \cap N_2 \in \mathcal{N}_x$;

N4: $N_1 \in \mathcal{N}_x \Rightarrow \exists N_2 \in \mathcal{N}_x : N_2 \subset N_1$ *and* $N_2 \in \mathcal{N}_y, \forall y \in N_2$.

According to Def. 3.1.2, the set of all neighborhoods of a point x, satisfies the axioms N1 - N4.

Proposition 3.1.3 (relative neighborhood)**.** *Let $< X, \tau >$ be a topological space, $S \subset X$ with the relative topology τ_S and $x \in S$. If $V \subset S$ is a neighborhood of x w.r.t. the relative topology τ_A, there is a neighborhood U of x w.r.t. τ such that*

$$V = S \cap U.$$

The neighborhood V is called the ***relative neighborhood*** *of x; conversely, if U is a neighborhood of x in X, then $U \cap S$ is a neighborhood of x in S.*

Excercises 3.1.4. *Suppose that $< X, \tau >$ is a topological space and prove the following*

1. *for any set A, $\partial A = clA \cup cl(X \setminus A)$;*
2. *if $U \in \tau$, then, for each $x \in U$, U is a neighborhood of x;*
3. *if x is an accumulation point of A, then x is also an accumulation point of $A \setminus \{x\}$;*
4. *let $S \subset X$ and $G \subset S$, then G is a closed set in S if there is closed set F in X such that $G = F \cap S$;*
5. *given $A \subset X$, what condition should be satisfied, so that a point $x \in X$ is not an accumulation point of A;*
6. *if τ is the discrete topology and $A \subset X$, then the set of accumulation points $A' = \emptyset$;*
7. *if $A \subset B$, then $clA \subset clB$ and $intA \subset intB$;*
8. *if D is dense in X and $D \subset D_2 \subset X$, then D_2 is also dense in X;*
9. *for any set A, $\partial A = extA \cup intA$ is cannot be a dense set in X;*
10. *for any set A, $clA = intA \cup \partial A$; i.e. $\partial A \subset clA$;*
11. *if $\mathcal{N}_x$ is a neighborhood system of x, then $\cap\{N \mid N \in \mathcal{N}_x\} \in \mathcal{N}_x$.*

3.2 Bases and Subbases

Definition 3.2.1 (a base). *Let $\mathcal{B}$ be any collection of open sets in a topological space $< X, \tau >$; i.e. $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ be is said to be a* ***base*** *for the topology τ iff for each open set O and each $x \in O$, there is a set $B \in \mathcal{B}$ such that $x \in \mathcal{B} \subset O$.*

Proposition 3.2.2. *(base) Let $< X, \tau >$ be a topological space. A collection $\mathcal{B} \subset \tau$ is a base for τ iff every $O \in \tau$ is a union of sets from $\mathcal{B}$.*

Example 3.2.3.

(i) If $\mathcal{B} := \{(a,b) \mid a, b \in \mathbb{R}\}$ is a collection of all open intervals, then $\mathcal{B}$ is a base for the usual topology on $\mathbb{R}$ (i.e. for the topology generated by the absolute value metric).

(ii) Let $< X, \tau >$ be the discrete topological space. Then the collection $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a base for τ.

Suppose we have some collection $\mathcal{B}$ of sets, is there a topology for which $\mathcal{B}$ is a base? In other words, given a collection $\mathcal{B}$ can we generate a topology for which $\mathcal{B}$ is a base? The following statement gives a necessary and sufficient condition that a collection $\mathcal{B}$ in order to be a base for some topology τ.

Proposition 3.2.4. *Let $\mathcal{B}$ be a collection of subsets of X, $X \neq \emptyset$. Then $\mathcal{B}$ is a base for some topology on X if and only*

(i) $X = \bigcup\{B \mid B \in \mathcal{B}\}$;

(ii) for any two sets $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Proof. $\Rightarrow$: Suppose $\mathcal{B}$ is a base for some topology τ on X. Since X is open and, for $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is open. Hence, the claim follows by Def. 3.2.1.

$\Leftarrow$: Suppose given $\mathcal{B}$ that (i) and (ii) are satisfied. New define the collection

$$\tau := \{U \mid x \in U \Rightarrow \exists B \in \mathcal{B} : x \in B \subset U\}.$$

Claim: (a) τ is a topology and (b) $\mathcal{B}$ is a base for τ.

(a) Obviously, $\mathcal{B} \subset \tau$, $X \in \tau$, $\emptyset \in \tau$. Moreover, for any family $\{U_\alpha \mid \alpha \in \Omega\}$, we have $\cup_{\alpha \in \Omega} U_\alpha \in \tau$.

Now, let $U_1, U_2 \in \tau$ and $x \in U_1 \cap U_2$. Since $x \in U_1$ and $x \in U_2$, there is $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By (ii), there is B_3 such that $x \in B_3 \subset B_1 \cap B_2$. Consequently, $x \in B_3 \subset U_1 \cap U_2$. This implies $U_1 \cap U_2 \in \tau$. Hence, τ is a topology.

(b) Suppose $O \in \tau$ and $x \in O$ be any. Then by definition of τ, $\exists B \in \mathcal{B}$ such that $x \in B \subset O$. Thus, $\mathcal{B}$ is a base for τ.

□

Corollary 3.2.5. *Let $\mathcal{S}$ be an arbitrary collection of set. If $X = \bigcup\{S \mid S \in \mathcal{S}\}$ and $\mathcal{B}$ is the collection of all finite intersections of elements of $\mathcal{S}$, then $\mathcal{B}$ is a base for some topology on X.*

Proof. We verify (i) and (ii) of Prop. 3.2.4. In fact, (i) is obvious, since $X = \bigcup\{S \mid S \in \mathcal{S}\} = \bigcup\{B \mid B \in \mathcal{B}\}$. Then, let $B_1, B_2 \in \mathcal{B}$. Then there are $S_1^1, \ldots, S_{n_1}^1 \in \mathcal{S}$ and $S_1^2, \ldots, S_{n_2}^2 \in \mathcal{S}$ such that

$$B_1 = \bigcap_{k=1}^{n_1} S_k^1 \text{ and } B_2 = \bigcap_{k=1}^{n_2} S_k^2.$$

Hence, if $x \in B_1 \cap B_2$, then

$$x \in B_3 := \bigcap_{k=1}^{n_1} S_k^1 \cap \bigcap_{k=1}^{n_2} S_k^2.$$

Consequently, $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Therefore, by Prop. 3.2.4, $\mathcal{B}$ is a base for some topology on X.

□

Definition 3.2.6. *(subbase) A family $\mathcal{S}$ of sets is a **subbase** if the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a base for some topology τ on $X = \bigcup\{S \mid S \in \mathcal{S}\}$.*

Definition 3.2.7 (neighborhood base). *Let X be a topological space and $x \in X$. Then the collection $\mathcal{N}_x$ of open set that contain x is said to be a **neighborhood (local)** base at x if for any set U with $x \in U$, there is $N_x \in \mathcal{N}_x$ such that $x \in N_x \subset U$.*

Proposition 3.2.8. *If $\mathcal{B}$ is a base for some topology τ on X and $x \in X$, then*

$$\mathcal{N}_x := \{B \in \mathcal{B} \mid x \in B\}$$

forms a neighborhood base at x.

Proposition 3.2.9. *Let $< X, \tau >$ be a topological space and $A \subset X$. Then $x \in X$ is an accumulation point of A; i.e. $x \in clA$, if and only if,*

$$\forall N \in \mathcal{N}_x, \exists y \in N \cap A : y \neq x.$$

Definition 3.2.10 (first countability). *A topological space X is said to satisfy the **first axiom of countability** (or X is **first countable**) if there exists a countable neighbourhood base $\mathcal{N}_x$, for each $x \in X$.*

Proposition 3.2.11. *Every metric space is first countable.*

Remark 3.2.12. *If X is first countable, then every $x \in X$ has a countable neighborhood base, say $\mathcal{N}_x = \{B_n \mid n \in \mathbb{N}\}$, we can also assume, w.l.o.g, that*

$$B_1 \supset B_2 \supset B_3 \supset \ldots$$

Definition 3.2.13 (second countability). *A topological space $< X, \tau >$ is said to satisfy the **second axiom of countability** (or $< X, \tau >$ is **second countable**) if there is a countable base for τ.*

Example 3.2.14. *For each, $n \in \mathbb{N}$, the space $X = \mathbb{R}^n$ with the usual topology is second countable. For instance, the collection $\mathcal{B} := \{(a, b) \mid a, b \in \mathbb{Q}\}$ is a countable base for the space $X = \mathbb{R}$.*

Proposition 3.2.15. *Every second countable space is first countable.*

Proposition 3.2.16. *If a topological space is second countable, then it is separable.*

Proof. Let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$. Define the set

$$D := \{x_n \in B_n\}.$$

To show D is a dense subset of X, for $\overline{x} \in X \setminus D$, we show that $\overline{x}$ is an accumulation point of D. Let U be any open set such that $\overline{x} \in U$. Since $\mathcal{B}$ is a base, there is some $B_{n_0} \in \mathcal{B}$ such that

$$\overline{x} \subset B_{n_0} \subset U,$$

$x_{n_0} \in B_{n_0} \subset U$ and $\overline{x} \neq x_{n_0}$. Consequently, U contains an element of D other that $\overline{x}$. Since U is arbitrary, we conclude that $\overline{x}$ is an accumulation point of D. Hence, $clD = X$ and D is a countable dense subset of X. Therefore, X is separable. □

Corollary 3.2.17. *For each $n \in \mathbb{N}$, then space $X = \mathbb{R}^n$, with the usual topology, is separable.*

Definition 3.2.18 (a covering)**.** *Let $< X, \tau >$ be a topological space. A family of sets $\{U_\alpha \mid \alpha \in \Omega\}$ is said to be a **covering** of X if*

$$X \subset \bigcup_{\alpha \in \Omega} U_\alpha.$$

*A covering $\{U_\alpha \mid \alpha \in \Omega\}$ is an **open covering** if each of the sets U_α is an open set; i.e. $U_\alpha \in \tau, \forall \alpha \in \Omega$.*

Theorem 3.2.19 (Lindelöf)**.** *Let $< X, \tau >$ be second countable and A be any subset of X. Then every open covering of A has a countable subcover.*

Proof. Since X is second countable, τ has a countable basis $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$. Suppose that $\{O_\alpha \mid \alpha \in \Omega\}$ is an open covering of A; i.e.

$$A \subset \bigcup_{\alpha \in \Omega} O_\alpha.$$

This implies, for each $x \in A$, there is $\alpha \in \Omega$ such that $x \in O_\alpha$. Since $\mathcal{B}$ is a base for τ, there is $B_n \in \mathcal{B}$ such that $x \in B_n \subset O_\alpha$. If we now let $O_{\alpha_n} = O_\alpha$ whenever $B_n \subset O_\alpha$, then there is $N \subset \mathbb{N}$ such that

$$A \subset \bigcup_{n \in N} B_n \subset \bigcup O_{\alpha_n}.$$

Consequently, $\{O_{\alpha_n} \mid n \in N\}$ is a countable subcovering of A. □

Definition 3.2.20 (a Lindelöf topological space)**.** *A topological space X is said to be **Lindelöf** if every open covering of X has a countable subcover.*

Hence, the space $\mathbb{R}^n$ is Lindelöf.

Theorem 3.2.21. *Let $< X, \rho >$ be a metric space. Then the following statements are equivalent.*

(i) $< X, \rho >$ is separable;

(ii) $< X, \rho >$ satisfies the second axiom of coutability;

(i) $< X, \rho >$ is Lindelöf;

Proof. Exercise! □

3.3 Sequences, Continuity and Homeomorphism

Definition 3.3.1 (convergent sequence)**.** *Let $< X, \tau >$ be a topological space. A sequence $\{x_n\} \subset X$ is said to* ***converge*** *to an element $x_0 \in X$ if for every neighborhood U of x_0 (i.e. $\forall U \in \mathcal{N}_{x_0}$) there is $N \in \mathbb{N}$ such that*

$$x_n \in U, \forall n \geq N.$$

In this case we write $x_n \to x_0$ and we call x_0 the ***limit*** *of $\{x_n\}$. When such an x_0 exists the sequence $\{x_n\}$ is called* ***a convergent sequence****.*

Definition 3.3.2 (continuity)**.** *Let $< X, \tau >$ and $< Y, \sigma >$ be topological spaces. Then $f : X \to Y$ is a* ***continuous function*** *if, for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.*

Proposition 3.3.3. *A function $f : X \to Y$ is continuous if and only if the inverse image of any closed set is closed.*

Proof. "⇒": Suppose $f : X \to Y$ be continuous and $G \subset Y$ is a closed set. Then

$$Y \setminus G \text{ open } \Rightarrow f^{-1}(Y \setminus G) \text{ is an open set in X }.$$

Since $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$. Hence, $f^{-1}(G)$ is a closed set.

"⇐": Let $U \subset Y$ be any open set. Then $Y \setminus U$ is a closed set. Then, by assumption, $f^{-1}(Y \setminus U)$ is a closed set. Since, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$, we have that $f^{-1}(U)$ is an open set in X. Hence, f is a continuous function.

□

Proposition 3.3.4. *Let $< X, \tau >$ and $< Y, \sigma >$ be topological spaces, $f : X \to Y$ be a function and $\{x_n\}$ be a sequence in X. If $x_n \to \overline{x}$, then $f(x_n) \to f(\overline{x})$.*

Definition 3.3.5 (continuity at a point)**.** *A function $f : X \to Y$ is* ***continuous at a point*** *$x \in X$ if for any open neighborhood U of $f(x)$ in Y, there is an open neighborhood O of x such that*

$$\forall x \in O : f(x) \in U; \text{ i.e. } O \subset f^{-1}(U).$$

Proposition 3.3.6. *A function $f : X \to Y$ is continuous iff f is continuous at each point in X.*

Definition 3.3.7 (sequential continuity)**.** *A function $f : X \to Y$ is* ***sequentially continuous*** *if for every convergent sequence $\{x_n\}$ with $x_n \to x_0$, $\{f(x_n)\}$ is a convergent sequence and $f(x_n) \to f(x_0)$.*

Proposition 3.3.8. *If $f : X \to Y$ is a continuous function, then f is sequentially continuous.*

Proof. Follows directly from Prop. 3.3.4. □

Remark 3.3.9. *The converse of Prop. 3.3.8 is not always true.*

Proposition 3.3.10. *If $f : X \to Y$ is a sequentially continuous, then, for every $A \subset X$,*

$$f(clA) \subset clf(A).$$

Definition 3.3.11 (open, closed functions)**.** *Let $f : X \to Y$ be a function. Then*

(i) f is said to be an ***open function (an open map)*** *if, for every open set $O \subset X$, $f(O)$ is an open set in X;*

(ii) f is said to be a ***closed function (a closed map)*** *if, for every closed set $F \subset X$, $f(F)$ is a closed set in X.*

Example 3.3.12. *The function $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) = x$. Then $\pi(\cdot)$ is an open mapping, but not a closed one.*

Definition 3.3.13 (homeomorphism)**.** *Two topological spaces X and Y are said to be homeomorphic if there is a one-to-one and onto function $f : X \to Y$ with both f and f^{-1} are continuous.*

Proposition 3.3.14. *Let X and Y be topological spaces and $f : X \to Y$. Then f^{-1} is a continuous function if and only if f is an open map; equivalently, if and only if f is a closed map.*

Corollary 3.3.15. *Let X and Y be topological spaces. A function $f : X \to Y$ is a homeomorphism if and only if f is a non-to-one open or closed map.*

Excercises 3.3.16. *Prove the following statements:*

1. *Prop. 3.3.4.*
2. *Prop. 3.3.10.*

3.4 Classification of Topological Space: Separation Axioms

If two elements x, y of a metric space X are not equal, then is it possible to put these elements in two separate open sets. But, this is not always possible in a general topological space(unless the topological space is metrizable). Hence, we have the following major classification of topological spaces based on separation axioms.

(i) A topological space X is a T_1-space (Fréchet-Riesz) if whenever $x, z \in X$ and $x \neq z$, there exist open (not necessarily disjoint) sets O_1 and O_2 of X such that

$$x \in O_1, z \notin O_1 \text{ and } z \in O_2, x \notin O_2;$$

(ii) A topological space X is a T_2-space (Hausdorff) if whenever $x, z \in X$ and $x \neq z$, there exist disjoint open sets O_1 and O_2 of X such that

$$x \in O_1 \text{ and } z \in O_2;$$

(iii) A topological space X is called **regular** (Vietoris) if for each $x \in X$ and each closed set F of X with $x \notin F$, there exist disjoint open sets O_1 and O_2 such that

$$x \in O_1 \text{ and } F \subset O_2;$$

(iv) A topological space is a T_3-space if it is regular and T_1;

(v) A topological space X is called **normal** (Tietze) if whenever $F_1, F_2 \subset X$ are disjoint closed sets, there exist disjoint open sets O_1 and O_2 such that

$$F_1 \subset O_1 \text{ and } F_2 \subset O_2;$$

(vi) A topological space is called T_4 if it is normal and T_1;

Proposition 3.4.1. *Let $< X, \tau >$ be a topological space. Then*

$$X \ \ is \ \ T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1.$$

Proposition 3.4.2. *A topological space X is T_1 iff, for every $x \in X$, $\{x\}$ is a closed set; i.e. every singleton is closed.*

Proof. ”⇒” : Suppose a topological space X is T_1. For $x \in X$, we show that $\{x\}$ is a closed set. Consider the set $X \setminus \{x\}$ and let $z \in X \setminus \{x\}$. Then $x \neq z$. Since X is T_1, there is an open set O_z such that

$$z \in O_z \text{ but } x \notin O_z.$$

⇒

$$X \setminus \{x\} = \bigcup_{z \in X \setminus \{x\}} O_z.$$

Consequently, $X \setminus \{x\}$ is open. Hence, $\{x\}$ is closed.

”⇐” : Suppose for each $x \in X, \{x\}$ is closed. Let $x, z \in X$ and $x \neq y$. If $O_1 := X \setminus \{x\}$ and $O_2 := X \setminus \{z\}$, then O_1 and O_2 are open sets, $z \in O_1, x \notin O_1$ and $x \in O_2, z \notin O_2$. Therefore, X is T_1.

□

Corollary 3.4.3. *Every finite subset of a T_1 space is closed.*

Proposition 3.4.4. *If $< X, \tau >$ is a Hausdorff topological space and $S \subset X$, then $< S, \tau_S >$ is also a Hausdorff topological space.*

Proposition 3.4.5. *In a Hausdorff topological space, every convergent sequence has a unique limit.*

Proof. Let $\{x_n\}$ be a sequence such that $x_n \to x$ and $x_n \to z$. Assume that $x \neq z$. Since, X is Hausdorff, there are disjoint open sets O_1 and O_2 such that

$$x \in O_1 \ \text{ and } \ z \in O_2.$$

Since $x_n \to x$, there is N such that

$$x_n \in O_1, \forall n \geq N.$$

From this follows that, there are only a finite number of elements of $\{x_n\}$ that can belong to O_2. This implies that x_n cannot converge to z. But this is a contradiction. Hence, the assumption is false and $x = z$. □

Proposition 3.4.6. *Suppose X is a topological space that is first countable. Then the following statements are equivalent:*

(i) X is Hausdorff;

(ii) every convergent sequence in X has a unique limit.

Proof. ”(i)⇒ (ii)”: See Prop. 3.4.5.

"(ii)$\Rightarrow$ (i)": Assume that X is not Hausdorff. Hence, there are elements $x, z \in X$, $x \neq z$, such that every open neighborhood O_x of x and U_z of z have a non-empty intersection; i.e. $O_x \cap U_z \neq \emptyset$.

By first countability of X, let $\{O_n\}$ and $\{U_n\}$ be a decreasing sequence of open neighborhoods of x and z (see Rem.3.2.12), respectively. Then, there is a sequence $\{x_n\}$ such that

$$x_n \in O_n \cap U_n \neq \emptyset.$$

Then $x_n \to x$ and $x_n \to z$. By assumption $x = z$. But this is a contradiction. Therefore, X is Hausdorff.

□

Theorem 3.4.7. *Let X be a topological space. Then the the following statements are equivalent:*

(i) X is normal;

(ii) whenever F is a closed and O an open sets, with $F \subset O$, there is an open set U such that

$$F \subset U \subset clU \subset O.$$

Proof. (a) Let F be a closed and O be an open sets with $F \subset U$. Then $F_2 := X \setminus O$ is a closed set and $F \cap F_2 = \emptyset$. Since X is a normal space, there exist disjoint opens set U and U_1 such that

$$F \subset U \text{ and } F_2 \subset U_1$$

Hence

$$U \cap U_1 = \emptyset \Rightarrow U \subset X \setminus U_1 \text{ and } F_2 = X \setminus O \subset U_1 \Rightarrow X \setminus U_1 \subset O$$

$\Rightarrow$

$$F \subset U \subset X \setminus U_1 \subset O.$$

Since $X \setminus U_1$ is a closed set, we further have $clU \subset X \setminus U_1$. Consequently,

$$F \subset U \subset clU \subset O.$$

(b) Suppose F_1 and F_2 are disjoint closed sets. Then $F_1 \subset X \setminus F_2 =: O$ and O is open. By assumption, there is an open set U such that

$$F_1 \subset U \subset clU \subset X \setminus F_2.$$

It follows that $F_1 \subset U$ and $F_2 \subset X \setminus clU$. Setting $U_1 := X \setminus clU$, we see that $U \cap U_1 = \emptyset$.

□

3.5 Uryson's Lemma, Tietze's Extension Theorem and Metrizability

Consider the interval $[0, 1]$, we construct the following set of real numbers

$$D_0 = \{0, 1\}, D_1 = \{0, \frac{1}{2}, 1\}, D_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}, D_3 = D_2 = \{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\}, \ldots$$

Define D to be the union of all these set as

$$D := \bigcup_{n=0}^{\infty} D_n.$$

This set is known as the set of all **dyadic** rational numbers obtained by dividing $[0, 1]$.

Remark 3.5.1. *Note that the sequence* $\{\frac{1}{2^n}\} \subset D$. *Moreover, for any* k *with* $1 \le k \le n$, $\frac{2^k}{2^n} = \frac{1}{2^{n-k}} \in D$.

Lemma 3.5.2. *The set* D *of dyadic rational numbers of* $[0,1]$ *is dense in* $[0,1]$*; i.e.* $clD = [0,1]$.

Proof. We show that every open interval of $[0,1]$ contains an element of D. Let $x \in [0,1]$ be any and $(x-\delta, x+\delta)$, for an arbitrary $\delta > 0$.
Since $\frac{1}{2^n} \to 0$, for the given $\delta > 0$, there is $n_0 \in \mathbb{N}$ such that

$$0 < \frac{1}{2^{n_0}} < \delta.$$

Now set $q := 2^{n_0}$. Then it follows that $0 < \frac{1}{q} < \delta$ and

$$x \in [0,1] = \bigcup_{k=0}^{n_0} \left[\frac{2^k - 1}{2^{n_0}}, \frac{2^k}{2^{n_0}}\right].$$

Hence, there is $m, 1 \le m \le q = 2^{n_0}$ (i.e., $m \in \{2^k \mid k = 0, \ldots, n_0\}$), such that

$$x \in \left[\frac{m-1}{q}, \frac{m}{q}\right]$$

$\Rightarrow \frac{m-1}{q} \le x \le \frac{m}{q}$. Since $0 < \frac{1}{q} < \delta$, we have

$$x - \delta < x - \frac{1}{q} \le \frac{m}{q} - \frac{1}{q} = \frac{m-1}{q} \le x < x + \delta$$

Consequently,

$$\frac{m-1}{q} \in (x-\delta, x+\delta).$$

Note that $\{\frac{i-1}{q} \mid i = 1, \ldots q\} \subset D$. Hence, $(x-\delta, x+\delta) \cap D \neq \emptyset$. Since, $x \in [0,1]$ and $\delta > 0$ are arbitrary, we conclude that D is dense in $[0,1]$. □

Theorem 3.5.3 (Uryson's Lemma)**.** *Let* X *be a topological space. Then the following statements are equivalent:*

(i) X *is a normal space;*

(ii) for any two disjoint subsets F_1 *and* F_2 *there is a continuous function* $f : X \to [0,1]$ *such that*

$$f(F_1) = \{0\}, f(F_2) = \{1\}.$$

That is

$$f(x) = \begin{cases} 0, & \text{if } x \in F_1 \\ 1, & \text{if } x \in F_2. \end{cases}$$

Proof. $(ii) \Rightarrow (i)$: Let F_1 and F_2 be any two closed sets in X. By assumption there is a continuous function $f : X \to [0,1]$ such that $f(F_1) = \{0\}$ and $f(F_2) = \{1\}$. The sets $[0, \frac{1}{3})$ and $(\frac{1}{3}, 1]$ are open sets in $[0,1]$. Consequently, by the continuity of f, it follows that

$$U_1 := f^{-1}([0, \frac{1}{3})) \text{ and } U_2 := f^{-1}((\frac{1}{3}, 1])$$

are open sets in X and $U_1 \cap U_2 = \emptyset$; moreover

$$F_1 \subset U_1 \text{ and } F_2 \subset U_2.$$

Therefore, X is a normal space.

$(i) \Rightarrow (ii)$: Let X be a normal space and F_1 and F_2 are any two disjoint closed sets. Then $F_1 \subset X \setminus F_2 =: O_2$ and O_2 is an open set. By Thm. , there is an open sets $U_{\frac{1}{2}}$ such that

$$F_1 \subset U_{\frac{1}{2}} \subset clU_{\frac{1}{2}} \subset O_2$$

Using Thm. again we obtain open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that

$$F_1 \subset U_{\frac{1}{4}} \subset clU_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset clU_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset clU_{\frac{3}{4}} \subset O_2$$

Proceeding in this manner we construct a sequence of sets $\{U_t \mid t \in D\}$, corresponding to the set D of dyadic rational numbers of $[0,1]$. Then the following hold true

- for each $t \in D, clU_t \subset X \setminus F_2 \Rightarrow clU_t \cap F_2 = \emptyset, \forall t \in D$;
- for $t_1, t_2 \in D$ and $t_1 < t_2$, it follows that $F_1 \subset clU_{t_1} \subset U_{t_2} \subset X \setminus F_2$.
- for each $t \in D, F_1 \subset U_t$; in particular

$$F_1 \subset U_{\frac{1}{2^n}}, \forall n \in \mathbb{N}.$$

Now define the function $f : X \to [0,1]$ such that for $x \in X$

$$f(x) := inf\{t \mid x \in U_t\}$$

Then f is a well defined function.

(a) Now let $x \in F_1$ be any. Since $x \in F_1 \subset U_{\frac{1}{2^n}}, \forall n \in \mathbb{N}$, it follows that

$$0 \le f(x) \le \frac{1}{2^n}, \forall n \in \mathbb{N} \Rightarrow f(x) = 0.$$

Since $x \in F_1$ is arbitrary, we conclude that $f(F_1) = \{0\}$.

(b) If $x \in F_2$, then $x \notin U_t, \forall t \in D$. In particular, given $n \in \mathbb{N}$

$$x \notin clU_t, \forall t \in D, t \le \frac{n}{n+1}$$

$\Rightarrow$

$$f(x) \ge \frac{n}{n+1}.$$

But this holds true for any given $n \in \mathbb{N}$. Hence,

$$1 \ge f(x) \ge \frac{n}{n+1}, \forall n \in \mathbb{N}.$$

$\Rightarrow f(x) = 1$. Moreover, since $x \in F_2$ is arbitrary, we conclude that

$$f(x) = 1, \forall x \in F_2 \Rightarrow f(F_2) = \{1\}.$$

(c) It remains, now to show that f is a continuous function. For any $\alpha \in [0,1]$, if we show that $f^{-1}([0,\alpha))$ and $f^{-1}((\alpha,1])$ are open sets, then we are done.

(i) First, we claim that

$$f^{-1}([0,\alpha)) = \cup\{U_t \mid t \in D, t < \alpha\}.$$

To show this, let $x \in f^{-1}([0,\alpha)) \Rightarrow 0 \le f(x) < \alpha$. Hence, by the definition of f, there is $t \in D$ such that $f(x) < t < \alpha$ and $x \in U_t$. This implies that $f^{-1}([0,\alpha)) \subset \cup\{U_t \mid t \in D, t < \alpha\}$.

Conversely, let $x \in \cup\{U_t \mid t \in D, t < \alpha\}$. Then $x \in U_{t_0}$, for some $t_0 \in D$ and $t_0 < \alpha \Rightarrow f(x) \le t_0 < \alpha \Rightarrow x \in f^{-1}([0,\alpha))$. Consequently, $\cup\{U_t \mid t \in D, t < \alpha\} \subset f^{-1}([0,\alpha))$.

(ii) Next, we claim that

$$f^{-1}((\alpha,1]) = \cup\{X \setminus clU_t \mid t \in D, \alpha < t\}.$$

Let $x \in f^{-1}((\alpha,1])$, then $\alpha < f(x) \leq 1$. Since D is dense in $[0,1]$, there are $t_1, t_2 \in D$ such that $\alpha < t_1 < t_2 < f(x)$. This implies that $x \notin U_{t_2}$. Furthermore, from $t_1 < t_2$, we have $clU_{t_1} \subset U_{t_2}$. Consequently, $x \notin clU_{t_1} \Rightarrow x \in X \setminus clU_{t_1}$. Hence,

$$f^{-1}((\alpha,1]) \subset \cup\{X \setminus clU_t \mid t \in D, \alpha < t\}.$$

Conversely, let $x \in \cup\{X \setminus clU_t \mid t \in D, \alpha < t\}$. Then $x \in X \setminus clU_{t_0}$, for some $t_0 > \alpha \Rightarrow x \notin clU_{t_0}$. Moreover, for any $t \in D, t < t_0$, we have $U_t \subset U_{t_0} \subset clU_{t_0}$. Consequently, $x \notin U_t, \forall t < t_0$. From this follows that

$$f(x) = inf\{t \mid x \in U_t\} \geq t_0 > \alpha.$$

$\Rightarrow \alpha < f(x) \leq 1 \Rightarrow x \in f^{-1}((\alpha,1])$. Hence,

$$\{X \setminus clU_t \mid t \in D, \alpha < t\} \subset f^{-1}((\alpha,1]).$$

From (a),(b) and (c), the claim of the theorem follows.

□

Corollary 3.5.4 (a generalization of Uryson's Lemma). *Let X be a topological space and a, b are any two real numbers, with $a < b$. Then the following statements are equivalent:*

(i) X is a normal space;

(ii) for any two disjoint subsets F_1 and F_2 there is a continuous function $f : X \to [a,b]$ such that

$$f(F_1) = \{a\}, f(F_2) = \{b\}.$$

That is

$$f(x) = \begin{cases} a, & \text{if } x \in F_1 \\ b, & \text{if } x \in F_2. \end{cases}$$

Definition 3.5.5 (Tychonoff or completely regular spaces). *A topological space X is* ***completely regular*** *or* ***Tychonoff*** *iff for any closed set F of X and $p \in X$, with $p \notin F$, there exists a continuous function $f : X \to [a,b]$ such that $f(p) = a, f(F) = \{b\}$.*

Proposition 3.5.6. *If a topological space X is completely regular, then X is regular.*

Definition 3.5.7 (a $T_{3\frac{1}{2}}$ or Tychonoff space). *A T_1 topological space which is also completely regular is called $T_{3\frac{1}{2}}$ or a Tychonoff space.*

3.5.1 Tietze's Extension Theorem

Theorem 3.5.8 (Tietze's Extension Theorem). *Let X be a topological space and F be a closed subset of X. If X is normal and f is a continuous real valued function such that $f : F \to [0,1]$. Then there is a continuous real valued function g with $g : X \to [0,1]$ such that $g|_F = f$; i.e. $g(x) = f(x)$ for $x \in F$*[†].

[†] $g|_F$ is the restriction of f to the set F.

Proof. Define the following sets

$$A_1 := \{x \in F \mid f(x) \leq \frac{1}{3}\}$$
$$B_1 := \{x \in F \mid f(x) \geq \frac{2}{3}\}.$$

Then both A_1 and B_1 are closed sets and $A_1 \cap B_1 = \emptyset$. Then, by Uryson's Lemma (see Cor. 3.5.4), there is a continuous funciton $f_1 : X \to [\frac{1}{3}, \frac{2}{3}]$ such that $f(A_1) = \frac{1}{3}$ and $f(B_1) = \frac{2}{3}$.

Hence, for $x \in F$,

$$|f(x) - f_1(x)| \leq \begin{cases} \frac{1}{3} - 0 = \frac{1}{3}, & \text{if } f(x) < \frac{1}{3} \\ 1 - \frac{2}{3} = \frac{1}{3}, & \text{if } f(x) > \frac{2}{3} \\ \frac{2}{3} - \frac{1}{3} = \frac{1}{3}, & \text{if } \frac{1}{3} \leq f(x) \leq \frac{2}{3}, \text{ since } \frac{1}{3} \leq f(x) \leq \frac{2}{3}. \end{cases}$$

$\Rightarrow |f(x) - f_1(x)| \leq \frac{1}{3}$, for $x \in F$. Then the function $h_1 := f - f_1$ maps F to $[0, \frac{1}{3}]$.

Repeating the above process, let

$$A_2 := \{x \in F \mid h_1(x) \leq \frac{1}{9}\} = \{x \in F \mid h_1(x) \leq \frac{1}{3^2}\}$$
$$B_2 := \{x \in F \mid h_1(x) \geq \frac{2}{9}\} = \{x \in F \mid h_1(x) \geq \frac{2}{3^2}\}.$$

Thus A_2 and B_2 are closed and disjoint sets. Hence, there is a continuous function $f_2 : X \to [\frac{1}{9}, \frac{2}{9}]$ such that $f(A_2) = \frac{1}{9}$ and $f(B_2) = \frac{2}{9}$. Furthermore, we have

$$|f(x) - (f_1(x) + f_2(x))\,| = |\,(f(x) - f_1(x)) - f_2(x)| = |h_1(x) - f_2(x)| \leq \frac{1}{3^2}.$$

Proceeding inductively, we construct a sequence of closed sets $\{A_n\}$ and $\{B_n\}$, with $A_n \cap B_n = \emptyset$ and a sequence of continuous functions

$$f_n : X \to \left[\frac{1}{3^n}, \frac{2}{3^n}\right]$$

such that $f_n(A_n) = \frac{1}{3^n}$ and $f_n(B_n) = \frac{2}{3^n}$ and

$$|f(x) - \sum_{k=1}^{n} f_k(x)| \leq \frac{1}{3^n}, \forall x \in F.$$

Now let $s_n(x) := \sum_{k=1}^{n} f_k(x)$. Then, for each n, $s_n(\cdot)$ is a continuous function on X. Moreover,

$$\lim_{n\to\infty} \sup_{x\in F} |f(x) - s_n(x)| \leq \lim_{n\to\infty} \frac{1}{3^n} = 0.$$

This implies, $s_n(x) \to f(x)$ uniformly on F. Thus

$$f(x) = \sum_{k=1}^{\infty} f_k(x), x \in F. \tag{3.1}$$

Moreover, for each $x \in X$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \leq \sum_{k=1}^{\infty} \frac{2}{3^k} = 2\sum_{k=1}^{\infty} \frac{1}{3^k} = 2\left(\sum_{k=0}^{\infty} \frac{1}{3^k} - 1\right) = 2\left(\frac{3}{2} - 1\right) = 1. \tag{3.2}$$

This implies $\sum_{k=1}^{\infty} f_k(x)$ is summable for each $x \in X$. Now, define

$$g(x) := \sum_{k=1}^{\infty} f_k(x), x \in X.$$

Claim:

(i) if $x \in F$, then $g(x) = f(x)$;

(ii) $0 \leq g(x) \leq 1, \forall x \in X$;

(iii) g is a continuous function on X.

Claim (i) follows from (3.1) and by the definition of g. Claim (ii) is has been shown in (3.2). Thus it remains to show (iii).

Let $x \in X$ be any fixed element. Then for any $z \in X$ we have

$$|g(x) - g(z)| = \left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{\infty} f_k(z) \right| \leq \sum_{k=1}^{\infty} |f_k(x) - f_k(z)| . \tag{3.3}$$

We know that $\sum_{k=1}^{n} \frac{1}{3^n} \to \sum_{k=1}^{\infty} \frac{1}{3^k}$. Hence, for any given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{\infty} \frac{1}{3^k} - \sum_{k=1}^{n} \frac{1}{3^k} \right| < \frac{\varepsilon}{2}, \forall n \geq n_0.$$

$\Rightarrow$

$$\sum_{k=n_0+1}^{\infty} \frac{1}{3^k} < \frac{\varepsilon}{2}.$$

Thus, from 3.3, it follows that

$$\begin{aligned} |g(x) - g(z)| \leq \sum_{k=1}^{\infty} |f_k(x) - f_k(z)| &= \sum_{k=1}^{n_0} |f_k(x) - f_k(z)| + \sum_{k=n_0+1}^{\infty} |f_k(x) - f_k(z)| \\ &\leq \sum_{k=1}^{n_0} |f_k(x) - f_k(z)| + \sum_{k=n_0+1}^{\infty} \frac{1}{3^k} \\ &\leq \sum_{k=1}^{n_0} |f_k(x) - f_k(z)| + \frac{\varepsilon}{2}. \end{aligned}$$

Since, for each $k = 1, 2, \ldots, n_0$, the function f_k is continuous at x, there is an open neighborhood $U_k(x)$ such that

$$|f_k(x) - f_k(z)| < \frac{\varepsilon}{2n_0}, \forall z \in U_k(x).$$

Set $U(x) := \cap_{k=1}^{n_0} U_k(x)$. Then $U(x)$ is an open neighborhood of x and

$$|g(x) - g(z)| < n_0 \left(\frac{\varepsilon}{2n_0} \right) + \frac{\varepsilon}{2} = \varepsilon, \forall z \in U(x).$$

Consequently, g is continuous at x. Since $x \in X$ is arbitrary, we conclude that g is a continuous function.

□

Corollary 3.5.9. *Let X be a topological space and F be a closed subset of X. If X is normal and f is a continuous real valued function such that $f : F \to [a, b]$, $a, b \in \mathbb{R}$ and $a < b$. Then there is a continuous real valued function g with $g : X \to [a, b]$ such that $g|_F = f$; i.e. $g(x) = f(x)$ for $x \in F$.*

Theorem 3.5.10 (generalized Tietze's extension theorem)**.** *Let X be a topological space and F be a closed subset of X. If X is normal and f is a continuous real valued function such on F. Then there is a continuous real valued function g on X such that $g|_F = f$; i.e. $g(x) = f(x)$ for $x \in F$.*

3.5.2 Urysohn's Metrizability

Definition 3.5.11 (metrizabllity)**.** *A topological space is said to be **metrizable** if it is homeomorphic to a metric space.*

A metrizable topological space inherits all the properties of the metric space associated with it. But not all topological spaces are metrizable; i.e. there are metric spaces which are not metrizable.

Definition 3.5.12 (Hilbert Cube)**.** *The cartesian product $\mathcal{H} = \prod_{k=1}^{\infty}[0, 1] =: [0, 1]^{\aleph_0}$ is known as the **Hilbert cube**.*

Lemma 3.5.13. *Let $\mathcal{H}_s$ be the set of all sequences of the form*

$$\mathcal{H}_s := \{\{x_n\} \mid 0 \leq x_n \leq \frac{1}{n}\}$$

and, for $x = \{x_n\}, y = \{y_n\} \in \mathcal{H}_s$, let

$$\rho(x, y) := \left[\sum_{k=1}^{\infty}(x_k - y_k)^2\right]^{\frac{1}{2}}.$$

Then

(i) ρ is a metric on $\mathcal{H}_s$;

(ii) $< \mathcal{H}_s, \rho >$ is a metric space; and

(iii) $< \mathcal{H}_s, \rho >$ is homeomorphic to a subspace of the Hilbert cube $\mathcal{H}$.

Lemma 3.5.14. *Let X be a T_4 topological space and $\mathcal{B}$ be a basis for X. If $U \in \mathcal{B}$, then, for every $x \in U$, there exists $U_x \in \mathcal{B}$ such that*

$$x \in clU_x \subset U.$$

Proof. Let $U \in \mathcal{B}$ and $x \in U$ be any. Since X is a T_1 space, the set $\{x\}$ is closed. Hence, $\{x\} \subset U$. Then, by Thm. 3.4.7, there exists an open set G such that

$$\{x\} \subset G \subset clG \subset U.$$

$\Rightarrow x \in G$ and G is an open set. Consequently, there is $U_x \in \mathcal{B}$ such that $x \in U_x \subset G$. This implies

$$x \in U_x \subset clU_x \subset clG \subset U.$$

$\Rightarrow$

$$x \in clU_x \subset U.$$

□

Theorem 3.5.15 (Urysohn's Metrizability). *Every second countable T_4 space is metrizable.*

Proof. Let X be a second countable T_4 space. If X is finite, the the claim follows trivially. Hence, assume w.l.o.g. that X is infinite. In this case we show that X is homeomorphic to a subspace of $\mathcal{H}_s$.

Since X is second countable, then X has a countable base $\{U_n\}$. Then, by Lem. 3.5.14, for each $U_k \in \mathcal{B}$, there is $U_i \in \mathcal{B}$ such that

$$clU_i \subset U_k.$$

Then the system

$$\{(U_i, U_k) \mid clU_i \subset U_k; \;\; U_i, U_k \in \mathcal{B}\}$$

is countable. Consequently, we can use the representation $P_n := (U_{i_n}, U_{k_n})$, where $clU_{i_n} \subset U_{k_n}, n \in \mathbb{N}$.

Hence, for each $n \in \mathbb{N}$ and $P_n = (U_{i_n}, U_{k_n})$, the sets clU_{i_n} and $X \setminus U_{k_n}$ are disjoint closed sets. Then, by Urysohn's Lemma, there is a continuous function $f_n : X \to [0,1]$ such that

$$f_n(clU_{i_n}) = 0 \;\text{ and }\; f_n(X \setminus U_{k_n}) = 1.$$

Hence, for each $x \in X$ and each $n \in \mathbb{N}$,

$$\left|\frac{f_n(x)}{2^n}\right| \leq \frac{1}{2^n} \leq \frac{1}{n}$$

Now, for $x \in X$, if we define

$$f(x) = \left\{\frac{f_n(x)}{2^n}\right\}_{n\in\mathbb{N}},$$

then f is a function from X to $\mathcal{H}_s$; i.e. $f : X \to \mathcal{H}_s$. Furthermore, we claim that

- f is a one-to-one;
- f is continuous;
- f^{-1} is continuous on a subspace of $\mathcal{H}_s$.

(i) Let $x, z \in X$ such that $x \neq z$. Then there exists $U_k \in \mathcal{B}$ such that $x \in U_k$ and $z \notin U_k$. Then, by Lem. 3.5.14, there is U_i with $x \in clU_i \subset U_k$ and $P_m = (U_i, U_k)$. Since $x \in clU_i$ and $z \in X \setminus U_k$, it follows that

$$f_m(x) = 0 \text{ and } f_m(z) = 1.$$

$\Rightarrow$

$$\left\{\frac{f_n(x)}{2^n}\right\}_{n\in\mathbb{N}} \neq \left\{\frac{f_n(z)}{2^n}\right\}_{n\in\mathbb{N}}.$$

$\Rightarrow f(x) \neq f(z)$. Hence, f is one-to-one.

(ii) Since $f_n \in C[0,1]$, for $x, z \in X$, it follows that

$$\left|\frac{f_n(x) - f_n(z)}{2^{2n}}\right| \leq \frac{1}{2^{2n}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{2^{2n}}$ converges, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \frac{1}{2^{2n}} < \frac{\varepsilon^2}{2}.$$

Moreover, $f(x), f(z) \in H_s$ yields

$$\rho(f(x), f(z))^2 = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(z)|}{2^{2n}} = \sum_{n=n_0+1}^{\infty} \frac{|f_n(x) - f_n(z)|}{2^{2n}} + \sum_{n=1}^{n_0} \frac{|f_n(x) - f_n(z)|}{2^{2n}}$$

$\Rightarrow$

$$\rho(f(x), f(z))^2 \leq \sum_{n=n_0+1}^{\infty} \frac{1}{2^{2n}} + \sum_{n=1}^{n_0} \frac{|f_n(x) - f_n(z)|}{2^{2n}} < \frac{\varepsilon^2}{2} + \sum_{n=1}^{n_0} \frac{|f_n(x) - f_n(z)|}{2^{2n}}$$

For each $l = 1, \ldots, n_0$, there is an open neighborhood U_l of x such that

$$|f_k(x) - f_k(z)| < \frac{\varepsilon^2}{2n_0}, \forall z \in U_l.$$

Then the set $O := \bigcap_{l=1}^{n_0} U_l$ is open and

$$\rho(f(x), f(z))^2 < \frac{\varepsilon^2}{2} + n_0 \left(\frac{\varepsilon^2}{2n_0} \right) = \varepsilon^2, \forall z \in O.$$

Consequently, f is a continuous function.

(iii) Let $Y := f(X) \subset H_s$. Next we show that $f^{-1} : Y \to X$ is continuous. Assume that there is $\overline{y} \in Y$, f^{-1} is not continuous at $\overline{y}$. This implies , there is a sequence $\{y_n\}$

$$y_n \to \overline{y} \quad \text{but} \quad x_n = f^{-1}(y_n) \nrightarrow f^{-1}(\overline{y}) = \overline{x}.$$

Hence, there is a neighborhood U of $\overline{x}$ that contains only a finite number of elements of $\{x_n\}$. This implies, there is $N \in \mathbb{N}$ such that $\{x_n \mid n \geq N\} \subset X \setminus U$.

Then there exists $U_k \in \mathcal{B}$ such that $\overline{x} \in U_k \subset U$. By Lem. 3.5.14, there $U_i \in \mathcal{B}$ such that

$$\overline{x} \in U_i \subset\in clU_i \subset U_k \subset U.$$

Hence, for some fixed $m \in \mathbb{N}, P_m = (H_i, H_k)$ and it follows that

$$\overline{x} \in U_i \;\text{ and }\; x_n \in X \setminus U_k, \forall n \geq N.$$

$\Rightarrow$

$$f_m(\overline{x}) = 0 \;\text{ and }\; f_m(x_n) = 1, \forall n \geq N.$$

$\Rightarrow$ for each $n \geq N, |f_m(x_n) - f_m(\overline{x})|^2 = 1$ and

$$\rho(f(x_n), f(\overline{x}))^2 = \sum_{k=1}^{\infty} \frac{|f_k(x_n) - f_k(\overline{x})|}{2^{2n}} \geq \frac{|f_m(x_n) - f_m(\overline{x})|^2}{2^{2m}} = \frac{1}{2^{2m}}$$

$\Rightarrow$

$$\rho(f(x_n), f(\overline{x})) \geq \frac{1}{2^{2m}}, \forall n \geq N.$$

$\Rightarrow$

$$y_n = f(x_n) \nrightarrow f(\overline{x}) = \overline{y}.$$

But this a contradiction. Hence, the assumption is false and f^{-1} must be continuous.

Consequently, f is a homeomorphism between X and $Y = f(X) \subset H_s$. Therefore, X is metrizable. □

The converse of the above statement is not always true.

Excercises 3.5.16. *Prove the following statements:*

1. *Let* $< x, \tau >$ *and* $< Y, \sigma >$ *be topological spaces,* $f : X \to Y$ *and* $\mathcal{S}$ *is a subbase for the topology* σ *on* Y*. Then the function* f *is continuous iff*

$$\forall S \in \mathcal{S} : f^{-1}(S) \in \tau.$$

2. *Let* $f : X \to Y$ *and* $\mathcal{B}$ *be a basis for the topological space* X*. If, for every* $B \in \mathcal{B}$*,* $f(B)$ *is open in* Y*, then* f *is an open map.*

3. *Let* $< X, \tau >$ *and* $< Y, \sigma >$ *be topological space and* $f : X \to Y$ *be a function. Then*

   (a) f *is a closed function if and only if, for any set* $A \subset X$*,* $clf(A) \subset f(clA)$*;*

   (b) f *is an open function if and only if, for any set* $A \subset X$*,* $f(intA) \subset intf(A)$*.*

4. *The topological spaces* $X = (0,1)$ *and* $Y = \mathbb{R}$*, with the usual topology, are homeomorphic;*

5. *Let* X *and* Y *be topological spaces and* $f : X \to Y$*. If* f *is a continuous function and* Y *is a Hausdorff space, then*

$$Graph(f) := \{(x, y) \in X \times Y \mid y = f(x)\} \quad \text{(graph } \textit{of } f\text{)}$$

   is a closed set in $X \times Y$*.*

6. *Let* X *and* Y *be topological spaces and* $f, g : X \to Y$ *are continuous functions. If* Y *is a Hausdorff space, then the set*

$$\{x \in X \mid f(x) = g(x)\}$$

   is a closed set in X*.*

7. *If* $f : X \to X$ *is a continuous function and* X *is Hausdorff space, then the* ***set of fixed points*** *of* f*, given by*

$$\{x \in X \mid x = f(x)\}$$

   is a closed set in X*.*

8. *Let* $< X, \rho >$ *be a metric space. For* $A \subset X, A \neq \emptyset$ *and* $x \in X$ *we set the (distance) function as*

$$f_A(x) = dist(x, A) = \inf_{z \in A} \rho(x, z).$$

   Then the map $f_A : X \to \mathbb{R}$ *is continuous. Moreover, for* A *and* B *be disjoint closed sets,* $g : X \to \mathbb{R}$ *and* $g := f_A - f_B$*, we have* $g^{-1}(0, \infty) \cap g^{-1}(-\infty, 0) = \emptyset$ *and* $g^{-1}(0, \infty) \subset A$ *and* $g^{-1}(-\infty, 0) \subset B$*. This implies that every metric space is a normal topological space.*

3.6 Compact Topological Spaces

3.6.1 Definitions

Definition 3.6.1 (a refinement). *A covering* $\{V_\lambda \mid \lambda \in \Lambda\}$ *of* X *is a refinement of a covering* $\{U_\alpha \mid \alpha \in \Omega\}$ *of* X *if*

$$\forall \lambda \in \Lambda, \exists \alpha \in \Omega : V_\lambda \subset U_\alpha.$$

Definition 3.6.2 (finite subcovering). *Let* $< X, \tau >$ *be a topological space and* $\{U_\alpha \mid \alpha \in \Omega\}$ *be a covering of* X*. If there is a finite index* $\{\alpha_1, \ldots, \alpha_n\} \subset \Omega$ *such that*

$$X \subset \bigcup_{k=1}^{n} U_{\alpha_k},$$

then the collection $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ *is called a* ***finite subcovering*** *of* X*; i.e.* $\{U_\alpha \mid \alpha \in \Omega\}$ *has a finite subcovering of* X*.*

Definition 3.6.3 (a compact set). *Let* $< X, \tau >$ *be a topological space and* $K \subset X$*. Then the set* K *is said to be a compact set if every open covering* $\{O_\alpha \mid \alpha \in \Omega\}$ *of* K *has a finite subcovering* $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$*; i.e.*

$$K \subset \bigcup_{\alpha \in \Omega} O_\alpha \Rightarrow K \subset \bigcup_{k=1}^{n} O_{\alpha_k}.$$

If X *itself is a compact set, then* $< X, \tau >$ *is called a* ***compact topological space****.*

Proposition 3.6.4. *Let* $< X, \tau >$ *and* $< Y, \sigma >$ *be topological spaces and* $f : X \to Y$ *be a continuous function. If* K *is a compact set in* X*, then* $f(K)$ *is a compact set in* Y*.*

Proof. Let $\{O_\alpha \mid \alpha \in \Omega\}$ be an open covering of $f(K)$;i.e

$$f(K) \subset \bigcup_{\alpha \in \Omega} O_\alpha$$

$\Rightarrow$

$$K \subset \bigcup_{\alpha \in \Omega} f^{-1}(O_\alpha)$$

Since f is a continuous function, the collection $\{f^{-1}(O_\alpha) \mid \alpha \in \Omega\}$ is an open covering of the compact set K. Consequently, there exists $\{O_{\alpha_1}, \ldots, O_{\alpha_n}\}$ such that

$$K \subset \bigcup_{k=1}^{n} f^{-1}(O_{\alpha_k})$$

$\Rightarrow$

$$f(K) \subset \bigcup_{k=1}^{n} O_{\alpha_k}.$$

Hence, $\{O_{\alpha_1}, \ldots, O_{\alpha_n}\}$ is a finite subcovering of $f(K)$. Consequently, $f(K)$ is a compact set. □

Proposition 3.6.5.

(a) A closed subset of a compact topological space is compact;

(b) A compact subset of a Hausdorff topological space is closed.

Proof. (a) Let $< X, \tau >$ be a compact topological space and F be a closed set. Suppose $\{O_\alpha \mid \alpha \in \Omega\}$ is an open covering of F. Then the collection

$$\{O_\alpha, X \setminus F \mid \alpha \in \Omega\}$$

is an open covering of X. Hence, there is $\{O_{\alpha_1}, \ldots, O_{\alpha_n}\} \subset \{O_\alpha \mid \alpha \in \Omega\}$ such that

$$X \subset \bigcup_{k=1}^{n} O_{\alpha_k} \cup (X \setminus F).$$

$\Rightarrow$

$$F \subset \bigcup_{k=1}^{n} O_{\alpha_k} \cup (X \setminus F).$$

But, $F \cap (X \setminus F) = \emptyset$. Hence,

$$F \subset \bigcup_{k=1}^{n} O_{\alpha_k}.$$

Consequently, F is a compact set.

(b) Let $< X, \tau >$ be a Hausdorff topological space and K be a compact subset of X. To show K is a closed set, we show $X \setminus K$ is an open set.

Let $z \in X \setminus K$ be any fixed element, then $z \notin K$. Since X is a Hausdorff space,

$$\forall x \in K, \exists U_x \in \tau, \exists O_x \in \tau : z \in U_x, x \in O_x \text{ and } U_x \cap O_x = \emptyset.$$

Hence, the family $\{O_x \mid x \in K\}$ is an open covering of K. This implies there is a finite subcovering $\{O_{x_1}, \ldots, O_{x_n}\}$ such that

$$K \subset \bigcup_{k=1}^{n} O_{x_k}.$$

Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be the corresponding collection of open sets such that $z \in U_{x_k}, k = 1, \ldots, n$. Define

$$U := \bigcap_{k=1}^{n} U_{x_k}.$$

Hence, U is an open set, $z \in U$ and $U \cap O_{x_k} = \emptyset$ for each $k = 1, \ldots, n$. This implies

$$U \cap \bigcup_{k=1}^{n} O_{x_k} = \emptyset \Rightarrow U \cap K = \emptyset \Rightarrow z \in U \subset X \setminus K.$$

Consequently, z is an interior point of $X \setminus K$. Since $z \in X \setminus K$ is arbitrary, we conclude that $X \setminus K$ is an open set. Therefore, K is a closed set.

□

3.6.2 The Finite Intersection Property

Definition 3.6.6 (the finite intersection property). *Let $< X, \tau >$ be a topological space and $\{A_\alpha \mid \alpha \in \Omega\}$ be a collection in X. Then $\{A_\alpha \mid \alpha \in \Omega\}$ is said to have* ***the finite intersection property*** *if for every finite collection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$, the intersection*

$$\bigcap_{k=1}^{n} A_{\alpha_k}$$

is non-empty.

Proposition 3.6.7. *A topological space X is compact if and only if every collection $\{F_\alpha \mid \alpha \in \Omega\}$ of closed sets with the finite intersection property has a non-empty intersection.*

Proof. "$\Rightarrow$" : Assume that

$$\bigcap_\alpha F_\alpha = \emptyset.$$

$\Rightarrow$

$$X = X \setminus \bigcap_\alpha F_\alpha = \bigcup_{\alpha\in\Omega} (X \setminus F_\alpha).$$

Since X is a compact set, there exist $F_{\alpha_1}, \dots, F_{\alpha_n}$ such that

$$X \subset \bigcup_{k=1}^{n} (X \setminus F_{\alpha_k})$$

$\Rightarrow$

$$\bigcap_{k=1}^{n} F_{\alpha_k} = \emptyset.$$

But this contradicts the finite intersection property. Hence, $\bigcap_\alpha F_\alpha \neq \emptyset$.

"$\Leftarrow$": Assume that X is not a compact set. Then there is an open covering $\{O_\alpha \mid \alpha \in \Omega\}$ of X with no finite subcovering. This implies, for every finite subcollection $\{O_{\alpha_1}, \dots, O_{\alpha_n}\}$

$$X \setminus \bigcup_{k=1}^{n} O_{\alpha_k} \neq \emptyset.$$

$\Rightarrow$ the family of closed sets $\{X \setminus O_\alpha \mid \alpha \in \Omega\}$ satisfies the finite intersection property. Then, by assumption,

$$\bigcap_{\alpha\in\Omega} (X \setminus O_\alpha) \neq \emptyset.$$

$\Rightarrow$

$$X \setminus \bigcup_{\alpha\in\Omega} O_\alpha \neq \emptyset.$$

This implies the collection $\{O_\alpha \mid \alpha \in \Omega\}$ does not cover X. But this is a contradiction. Consequently, X should be a compact set.

□

Remark 3.6.8. *From the proof of Prop. 3.6.7 we can easily verify that the following two statements are equivalent*

- *X is a compact topological space;*
- *for every family of closed subsets $\{F_\alpha \mid \alpha \in \Omega\}$ of X with the property that $\bigcap_{\alpha\in\Omega} F_\alpha = \emptyset$, there is a finite subcollection $\{O_{\alpha_1}, \dots, O_{\alpha_n}\}$ such that*

$$\bigcap_{k=1}^{m} O_{\alpha_k} = \emptyset.$$

3.6.3 Compact Hausdorff Spaces

Compact Hausdorff topological spaces exhibit very interesting properties, which are very important from practical point of view.

Corollary 3.6.9. *Let $< X, \tau >$ be a Hausdorff topological space. If K is a compact subset of X, $z \in X$ and $z \notin K$, then there exist disjoint open sets O and U in X such that*

$$K \subset U \text{ and } z \in O.$$

Proof. See the proof of part (b) of Prop. 3.6.5. □

Proposition 3.6.10. *Let $< X, \tau >$ be a Hausdorff topological space. If K_1 and K_2 are disjoint compact sets, then there exists disjoint open sets O_1 and O_2 of X such that*

$$K_1 \subset O_1, K_2 \subset O_2.$$

Proof. Each $z \in K_1$ is such that $z \notin K_3$. Using Cor. 3.6.9, there are disjoint open sets O_z and U_z such that

$$z \in O_z \text{ and } K_2 \subset U_z.$$

Hence, the family $\{O_z \mid z \in K_1\}$ is an open covering of K_1. By the compactness of K_1, there is a finite open covering $\{O_{z_1}, \ldots, O_{z_m}\}$, i.e.

$$K_1 \subset \bigcup_{k=1}^{m} O_{z_k} =: O_1$$

and there is a corresponding finite collection $\{U_{z_1}, \ldots, U_{z_m}\}$ with $K_2 \subset U_{z_k}$ for each $k = 1, \ldots, m$. Then

$$K_2 \subset \bigcap_{k=1}^{m} U_{z_k} =: O_2.$$

Then O_1 and O_2 are the required open sets. □

Corollary 3.6.11. *If $< X, \tau >$ is a compact Hausdorff topological space, then X is a normal topological space.*

Proof. Follows from Prop. 3.6.5(a) and Prop. 3.6.10. □

Corollary 3.6.12. *A compact Hausdorff second countable topological space is metrizable.*

Proof. Follows from Cor. 3.6.11 and Thm. 3.5.15 . □

Excercises 3.6.13. *Show that*

1. *The finite union of compact sets is again compact.*
2. *Let X be a compact topological space, U be an open subset of X and $\{K_n\}_{n\in\mathbb{N}}$ be a family of compact subsets of X. If $\bigcap_{n\in\mathbb{N}} K_n \subset U$, then there exists a finite index $I \subset \mathbb{N}$ such that*

$$\bigcap_{n\in I} K_n \subset U.$$

3. *Let X be a compact topological space and let $\{K_n\}_{n\in\mathbb{N}}$ be a family of non-empty closed subsets of X with $K_{n+1} \subset K_n$ for each $n \geq 1$. Then*

$$\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset.$$

4. (a) *It is necessary and sufficient for a topological space X to be compact that: if $\{V_\alpha | \alpha \in \Omega\}$ is any family of closed subsets of X such that $\cap_{\alpha\in\Omega'} V_\alpha \neq \emptyset$, for any subset $\Omega' \subset \Omega$, then $\cap_{\alpha\in\Omega} V_\alpha \neq \emptyset$*

   (b) *Let X be a compact topological space and for each $n \in \mathbb{N}$, V_n is a non-empty closed subset of X such that*

   $$V_n \supset V_{n+1}.$$

   Then

   $$\bigcap_n V_n \neq \emptyset.$$

   (c) *Suppose $f : X \to X$ is continuous, where X is a compact metric space. Then there exists a non-empty subset $A \subset X$ such that $f(A) = A$. (Hint: put $X_1 = f(X)$, $X_{n+1} = f(X_n)$ and $A = \cap_{n=1}^{\infty} X_n$ and use (b)),*

5. *Let X be a compact* ***metric space****, F be a closed subset of X and U be an open subset of X. If $F \subset U$, then there exists $\varepsilon > 0$ such that*

$$\bigcup_{x\in F} \mathbf{B}_\varepsilon(x) \subset,$$

where $\mathbf{B}_\varepsilon(x)$ represents the open ball centered at x and with radius ε.

6. *Let X be Hausdorff and Y be a compact topological spaces. If $f : X \to Y$ is a continuous one-to-one function from X on to Y, then f is a Homeomorphism.*

7. *Let X and Y be metric spaces. If X is compact and $f : X \to Y$ be a continuous function, then $f(X) = \{f(x) \mid x \in X\}$ is a bounded subset of Y.*

8. *If K_1 and K_2 be compact subsets of a metric space $< X, \rho >$, then there exist $x \in K_1$ and $z \in K_2$ such that*

$$\rho(x, z) = dist(K_1, K_2).$$

3.7 Locally Compact Spaces

Locally compact Hausdorff space are among the most important topological spaces, say in non-linear analysis and abstract measure theory. For instance, measure and integral theory on topological space usually assume the underlying space to be locally compact Hausdorff topological space (eg. Borel, Radon measure on topological spaces, etc).

Definition 3.7.1 (locally compact spaces)**.** *A topological space $< X, \tau >$ is said to be* ***locally compact*** *if*

$$\forall x \in X, \exists O \in \tau : x \in O \text{ and } clO \text{ is a compact set}.$$

Proposition 3.7.2. *Every compact topological space is locally compact.*

The converse of Prop. 3.7.2 is not always true. For instance, the Euclidean space $\mathbb{R}^n$ is locally compact, but it is not a compact space.

Lemma 3.7.3. *If X is a compact T_2 space, then X is T_4.*

Proof. Let F_1 and F_2 be two closed sets in X such that $F_1 \cap F_2 = \emptyset$. For each $x \in F_1$ and $z \in F_2$, there are disjoint open sets U_x and V_z such that $x \in U_x$ and $z \in V_z$. This implies

$$F_1 \subset \bigcup_{x \in F_1} U_x \text{ and } F_2 \subset \bigcup_{z \in F_2} V_z.$$

(Note that: each U_x is so that $U_x \cap V_{z_1} = \emptyset$ and $U_x \cap V_{z_2} = \emptyset$ for $z_1 \neq z_2$ and vice-versa). By the compactness of F_1 and F_2, there are $x_1, \ldots, x_n \in F_1$ and $z_1, \ldots, z_m \in F_2$ such that

$$F_1 \subset \bigcup_{k=1}^{n} U_{x_k} =: U \text{ and } F_2 \subset \bigcup_{k=1}^{m} V_{z_k} =: V.$$

$\Rightarrow F_1 \subset U, F_2 \subset V$ and $U \cap V = \emptyset$. Therefore, X is T_4. ☐

Lemma 3.7.4. *Let X be a T_2 space. If $U \subset X$ is an open set such that clU is compact, then, for every $x \in U$, there is a compact neighborhood V_x such that*

$$x \in V_x \subset U.$$

Proof. A closed subset of a T_2 space is T_2 (cf. Prop. 3.4.4). Hence, clU is a T_2 and compact subspace of X. By, Lem. 3.7.3, clU is a T_4 subspace of X. Since

$$\{x\} \subset U \subset clU \text{ and } U \text{ is again an open set w.r.t. the topology of } clU,$$

by Thm. 3.4.7, there is an open set O in X such that

$$\{x\} \subset O \cap clU \subset cl\,(O \cap clU) = cl(O \cap U) \subset U \subset clU.$$

(Note that $O \cap clU$ is an open set w.r.t. the relative topology on clU). Now set, $V_x = cl(O \cap U)$ and the claim follows. ☐

Proposition 3.7.5. *Let X be a locally compact Hausdorff space. If K is a compact subset of X, then*

(i) there is an open set O such that $K \subset O$, clO is compact; and

(ii) given such a set O there is non-negative function $f : X \to [0, 1]$; i.e. $0 \leq f(x) \leq 1$ such that

$$f(X \setminus O) = 0 \text{ and } f(K) = 1.$$

Hence, $\{x \in X \mid f(x) \neq 0\} \subset O$.

Proof. (i) For each $x \in K$, there is an open sets O_x such that $x \in O_x$ and clO_x is compact. Then

$$K \subset \bigcup_{x \in K} O_x.$$

$\Rightarrow$ there are $x_1, \ldots, x_n \in K$ such that

$$K \subset \bigcup_{k=1}^{n} O_{x_k}.$$

If we let $O := \bigcup_{k=1}^{n} O_{x_k}$, it follows that $K \subset O$ and clO is a compact set (note that clO is a finite union of compact sets, according to Ex. 3.6.13(1)).

(ii) Now for each $x \in K$, (by Lem. 3.7.4) choose a compact neighborhood V_x such that $V_x \subset O$. By the compactness of K, there are $x_1, \dots, x_n \in K$ such that

$$A = \bigcup_{k=1}^{n} V_{x_k}, \text{ A is compact and } K \subset intA \subset A \subset O.$$

By Lem 3.7.4 A is a T_4 subspace of X. Hence, by Uryson's Lemma(Lem. 3.5.3), there is a continuous function $g : A \to [0,1]$ such that $g(K) = 1$ and $g(A \setminus intA) = 0$. Now define $f : X \to [0,1]$ as

$$f(x) = \begin{cases} g(x), & \text{if } x \in A \\ 0, & \text{if } x \in X \setminus A. \end{cases}$$

It remains now to show that f is continuous on X. For this it is enough to show that f continuous at $x_0 \in \partial A \Rightarrow x_0 \notin intA \Rightarrow x_0 \in A \setminus K$. Thus,

(a) there is a neighborhood V_{x_0} such that $V_{x_0} \cap A \subset (A \setminus K) \Rightarrow f(x) = g(x) = 0, \forall x \in V_{x_0} \cap A$; and

(b) by the definition of f, $f(x) = 0, \forall x \in V_{x_0} \cap (X \setminus A)$.(Note that $A \setminus K$ is an open set relative to A.)

From (a) and (b) it follows that $f(x) = 0, \forall x \in V_{x_0}$. Hence, f is continuous at x_0. Moreover, $X \setminus O \subset X \setminus A$ implies that $f(X \setminus O) = 0$. Hence, $cl\{x \in X \mid f(x) \neq 0\} \subset A \subset O$.[‡] □

Excercises 3.7.6. *Prove the following statements:*

1. *If X is a locally compact T_3 topological space, then each point $x \in X$ has a neighborhood base of the form*
$$\mathcal{N}_x = \{x \in O \in \tau \mid clO \text{ compact }\}.$$
2. *Let X be a locally compact space and $\{D_n \mid n \in \mathbb{N}\}$ be a countable collection of dense set. Then $\bigcap_{n \in \mathbb{N}} D_n$ is dense in X.*
3. *Let X be a locally compact space. A subset F of X is closed if and only if $F \cap K$ is closed for closed compact set K.*
4. *A closed subset of a locally compact space is locally compact.*
5. *An open subset of a locally compact space is locally compact.*

3.8 Sigma-Compact Topological Spaces

Definition 3.8.1 (σ-compact spaces). *A topological space $< X, \tau >$ is said to be σ-compact if it is the union of a countable number of compact sets.*

Spaces which are σ-compact are also known as **countably compact** spaces. Thus, a compact space is a σ-compact.

[‡] The set $cl\{x \in X \mid f(x) \neq 0\}$ is known as the support of the function f on the set X, denoted by $sup\ f$,

Theorem 3.8.2. *Let $< X, \tau >$ be a locally compact Hausdorff topological space. Then (i) X is Lindelöf $\Leftrightarrow$ (ii) X is σ-compact $\Leftrightarrow$ (iii) There is a sequence $\{O_n\}$ of open sets with clO_n compact, $clO_n \subset O_{n+1}$ and $X \subset \bigcup_{n\in\mathbb{N}} O_n$.*

Proof. (i)$\Rightarrow$(ii): X is locally compact implies that for each $x \in X$, there is a neighborhood U_x such that $x \in U_x$ and clU is compact. Hence, $X \subset \bigcup_{x\in X} U_x$. Since X is Lindelöf, there is a countable subcover $\{U_n \mid n \in \mathbb{N}\}$ such that

$$X = \bigcup_{n\in\mathbb{N}} U_n \Rightarrow X = \bigcup_{n\in\mathbb{N}} clU_n.$$

Consequently, X is σ-compact.

(ii)$\Rightarrow$ (iii): X is σ-compact $\Rightarrow X = \bigcup_{n\in\mathbb{N}} K_n$, where for each $n \in \mathbb{N}$, K_n is a compact set. Now, since K_1 is compact, by Prop. 3.7.5(i), there is an open set O_1 such that $K_1 \subset O_1$ and clO_1 is compact. Proceeding inductively, for $n = 2, \ldots$, the set $K_n \cup clO_{n-1}$ is compact. Hence, there is O_n such that $K_n \cup clO_{n-1} \subset O_n$ and clO_n is compact. Consequently, we have

$$clO_n \subset O_{n+1}, n = 1, \ldots \text{ and } X = \bigcup_{n\in\mathbb{N}} O_n.$$

(iii)$\Rightarrow$ (i): Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open covering of X. Then, by (iii) there is a sequence $\{O_n \mid n \in \mathbb{N}\}$ such that $clO_n \subset O_n, clO_n$ compact and $X = \bigcup_{n\in\mathbb{N}} O_n = \bigcup_{n\in\mathbb{N}} clO_n$. This implies, for each $n \in \mathbb{N}$,

$$clO_n \subset \bigcup_{\alpha\in\Omega} U_\alpha \Rightarrow \exists\{O_{\alpha_1}, \ldots, O_{\alpha_{m_n}}\} \subset \mathcal{C} : clO_n \subset \bigcup_{k=1}^{m_n} U_{\alpha_k}.$$

Set

$$\mathcal{F} := \bigcup_{n\in\mathbb{N}} \{U_{\alpha_1}, \ldots, U_{\alpha_{m_n}} \mid clO_n \subset \bigcup_{k=1}^{m_n} U_{\alpha_k}, n \in \mathbb{N}\}.$$

It follows that $\mathcal{F}$ is countable, $\mathcal{F} \subset \mathcal{C}$ and $\bigcup \mathcal{F} = X$. Consequently, X is Lindelöf.

□

3.9 Paracompact Topological Spaces

Definition 3.9.1 (refinement). *Let $< X, \tau >$ be a topological space and $\mathcal{C}$ be a collection of subsets of X. A collection $\mathcal{F}$ of subsets of X is said to be a refinement of $\mathcal{C}$ if*

(i) $\bigcup \mathcal{F} = \bigcup \mathcal{C}$; and

(ii) for each $V \in \mathcal{F}$, there is $U \in \mathcal{C}$ such that $V \subset U$.

Accordingly, if $\mathcal{F}$ is a refinement of $\mathcal{C}$ and $\mathcal{C}$ is a covering of X, then $\mathcal{F}$ should also be a covering of X.

Definition 3.9.2 (locally finite). *Let $< X, \tau >$ be a topological space. A family $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ of subsets of X is said to be **locally finite** if for each $x \in X$, there is a neighborhood O of x such that O intersects only a finite number of elements of $\mathcal{C}$.*

Definition 3.9.3 (σ-locally finite). *Let $< X, \tau >$ be a topological space. A family $\mathcal{C}$ of subsets of X is said to be σ-**locally finite** if*

$$\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n;$$

where, for each $n \in \mathbb{N}$, $\mathcal{C}_n$ locally finite. That, is $\mathcal{C}$ is a countable union of locally finite families.

Remark 3.9.4. *Observe that,*

- *any finite collection $\mathcal{F}$ of subsets of X is locally finite;*
- *if $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ is locally finite, then $\widetilde{\mathcal{C}} := \{clU_\alpha \mid \alpha \in \Omega\}$ is also locally finite;*
- *any refinement of a locally finite collection is again locally finite;*
- *every locally finite collection is also σ-locally finite.*

Lemma 3.9.5. *Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ a locally finite family. If, for each $\alpha \in \Omega, V_\alpha \subset U_\alpha$, then the family $\mathcal{F} := \{V_\alpha \mid \alpha \in \Omega\}$ is also locally finite.*

Proof. Trivial! □

In Lem 3.9.5, the elements of $\mathcal{F}$ can be any type of sets; i.e. it is not necessarily that they are closed or open sets. Recall also that, for a collection $\{U_\alpha \mid \alpha \in \Omega\}$ we have

$$cl\left(\bigcup_{\alpha \in \Omega} U_\alpha\right) \subset \bigcup_{\alpha \in \Omega} clU_\alpha$$

but equality does not hold always. However,

Lemma 3.9.6. *Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be a locally finite family, then*

$$cl\left(\bigcup_{\alpha \in \Omega} U_\alpha\right) = \bigcup_{\alpha \in \Omega} clU_\alpha$$

Proof. (a)

$$\bigcup_{\alpha \in \Omega} clU_\alpha \subset cl\left(\bigcup_{\alpha \in \Omega} U_\alpha\right) \quad \text{is obvious.}$$

(b) Now, let $x \in cl\left(\bigcup_{\alpha \in \Omega} U_\alpha\right)$. Then there is a neighborhood N of x such that N intersects only a finite number of elements of $\mathcal{C}$. Let $U_{\alpha_1}, \ldots, U_{\alpha_n} \in \mathcal{C}$ such that $N \cap U_{\alpha_k} \neq \emptyset, k = 1, \ldots, n$. Hence,

$$x \in N, N \cap \bigcup_{k=1}^{n} U_{\alpha_k} \neq \emptyset, \text{ and } N \cap \left(\bigcup_{\alpha \in \Omega \setminus \{\alpha_1, \ldots, \alpha_n\}} U_\alpha\right) = \emptyset.$$

This implies that $x \in cl(\bigcup_{k=1}^{n} U_{\alpha_k}) = \bigcup_{k=1}^{n} clU_{\alpha_k} \subset \bigcup_{\alpha \in \Omega} clU_\alpha$[§]. Consequently,

$$cl\left(\bigcup_{\alpha \in \Omega} U_\alpha\right) \subset \bigcup_{\alpha \in \Omega} clU_\alpha.$$

□

[§]Observe that: if $x \in cl(A \cup B)$ and $x \notin clB$, then $x \in clA$.

Lemma 3.9.7. *Let $< X, \tau >$ be a topological space. If $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ is locally finite family of subsets of X and K is a compact subset of X, then K intersects only a finite number of elements of $\mathcal{C}$.*

Proof. Take any element $x \in K$. Then there is an open neighborhood O_x such that O_x intersects only a finite number of elements of $\mathcal{C}$. Hence,

$$K \subset \bigcup_{x \in K} O_x.$$

Since K is compact, there are $x_1, \ldots, x_n \in K$, for some $n \in \mathbb{N}$, such that

$$K \subset \bigcup_{k=1}^{n} O_{x_k}.$$

But, for each $k \in \{1, \ldots, n\}$, O_{x_k} can intersect only a finite number of elements of $\mathcal{C}$, say about m_k of them. Hence, K can only intersect about $\sum_{k=1}^{n} m_k$ elements of $\mathcal{C}$.

□

Definition 3.9.8 (paracompact topological space)**.** *A Hausdorff topological space $< X, \tau >$ is said to be paracompact if every open cover of X has a locally open refinement.*

Proposition 3.9.9. *A closed subset of a paracompact space is paracompact.*

Proposition 3.9.10. *If $< X, \tau >$ is a σ-compact locally compact Hausdorff topological space, then $< X, \tau >$ is paracompact.*

Proof. Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open covering of X. Then

- Thm. 3.8.2(i) $\Rightarrow X$ is Lindelöf. Hence, $\mathcal{C}$ has a countable sub-cover, say $\{U_n \mid n \in \mathbb{N}\}$; and
- Thm. 3.8.2(iii) $\Rightarrow$ there is a sequence of sets $\{O_n \mid n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}, clO_n$ is compact, $clO_n \subset O_{n+1}$ and $X = \bigcup_{n \in \mathbb{N}} O_n$.

Now, for each $n \in \mathbb{N}$, define the set

$$\widetilde{U}_n := U_n \cap (O_n \setminus clO_{n-2}),$$

where, w.l.o.g, we assume that $O_{-1} = O_0 = \emptyset$.

(i) <u>The collection $\{\widetilde{U}_n \mid n \in \mathbb{N}\}$ is an open refinement of $\mathcal{C}$.</u>

For each $n \in \mathbb{N}, \widetilde{U}_n \subset U_n \in \mathcal{C}$, and

$$\bigcup_n \widetilde{U}_n = \bigcup_n U_n \cap \bigcup_n (O_n \setminus clO_{n-2}) = X \cap \bigcup_n (O_n \setminus clO_{n-2}) = \bigcup_n (O_n \setminus clO_{n-2}). \tag{3.4}$$

We show next that $\bigcup_n (O_n \setminus clO_{n-2}) = X$. Since $X = \bigcup_n O_n$, for $x \in X$, there is a smallest $n(x) \in \mathbb{N}$ such that $x \in O_{n(x)}$.

(a) If $n(x) = 1$, then

$$x \in O_1 \setminus clO_{-1} \Rightarrow x \in \bigcup_n (O_n \setminus clO_{n-2}).$$

(b) If $n(x) \geq 2$, then

$$x \notin O_{n(x)-1} \Rightarrow x \notin clO_{n(x)-2} \Rightarrow x \in O_{n(x)} \setminus clO_{n(x)-2} \Rightarrow x \in \bigcup_n (O_n \setminus clO_{n-2}).$$

From (a) and (b), we conclude that $X = \bigcup_n (O_n \setminus clO_{n-2})$.

Consequently, from eqn. (3.4), we conclude that $\bigcup_n \widetilde{U}_n = X$. That is, $\{\widetilde{U}_n \mid n \in \mathbb{N}\}$ is an open refinement of $\mathcal{C}$.

(ii) $\{\widetilde{U}_n \mid n \in \mathbb{N}\}$ is locally finite. To see this, let $x \in X$ be any. Since $X = \bigcup_{n\in\mathbb{N}} O_n = \bigcup_{n\in\mathbb{N}} clO_n$, there is a smallest $n_0 \in \mathbb{N}$ such that

$$x \in O_{n_0}.$$

This implies that

$$O_{n_0} \cap [U_k \cap (O_k \setminus clO_{k-2})] = \emptyset, \forall k \geq n_0 + 2 \Rightarrow O_{n_0} \cap \widetilde{U}_k = \emptyset, \forall k \geq n_0 + 2.$$

Hence, O_{n_0} can only intersect a finite number of elements of $\{\widetilde{U}_n \mid n \in \mathbb{N}\}$. Consequently, $\{\widetilde{U}_n \mid n \in \mathbb{N}\}$ is locally finite.

Hence, form (i) & (ii) $\{\widetilde{U}_n \mid n \in \mathbb{N}\}$ is a locally finite open refinement of $\mathcal{C}$. Therefore, X is paracompact. □

Corollary 3.9.11. *Every compact Hausdorff space is paracompact.*

Proposition 3.9.12. *Let $< X, \tau >$ be a topological space. If an open cover $\mathcal{C}$ of X has a σ-locally finite refinement, then $\mathcal{C}$ has a locally finite refinement.*

Proof. Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open covering of X. Then by assumption $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$, where for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is a locally finite family.

Thus, for each $\alpha \in \Omega$,

$$\exists n \in \mathbb{N} : U_\alpha \in \mathcal{C}_n.$$

Let $n(\alpha)$ be the smallest natural number for which $U_\alpha \in \mathcal{C}_{n(\alpha)}$. Define now

$$W_\alpha := U_\alpha \cap \bigcap \{X \setminus U_\beta \mid U_\beta \in \mathcal{C}_m \text{ for some } m < n(\alpha)\}.$$

It follows that:

(i) for each $\alpha \in \Omega : W_\alpha \subset U_\alpha$;

(ii) the family $\{W_\alpha \mid \alpha \in \Omega\}$ is a covering of X. To see this let $x \in X$ be any. This implies

$$\exists \alpha_0 \in \Omega : x \in U_{\alpha_0}.$$

For the corresponding smallest natural number $n(\alpha_0)$ it follows that

$$m < n(\alpha_0) \text{ and } U_\beta \in \mathcal{C}_m \Rightarrow x \notin U_\beta \Rightarrow x \in X \setminus U_\beta.$$

Hence, $x \in W_{\alpha_0}$; i.e. $X = \bigcup_{\alpha\in\Omega} W_\alpha$.

(iii) The family $\{W_\alpha \mid \alpha \in \Omega\}$ is locally finite. Let $x \in X$, α_0 and $n(\alpha_0)$ be as in (ii).

Case a: If $\alpha \in \Omega$ such that $n(\alpha) > n(\alpha_0)$, then $x \in U_{\alpha_0}$ and $U_{\alpha_0} \cap W_\alpha = \emptyset$.

Case b: If $\alpha \in \Omega$ such that $n(\alpha) \leq n(\alpha_0)$; i.e. $n(\alpha) \in \{1, \ldots, n(\alpha_0)\}$; say $n(\alpha) = m \leq n(\alpha_0)$. Then the family

$$\mathcal{B}_m := \{U_\alpha \in \mathcal{C} \mid n(\alpha) = m\} \subset \mathcal{C}_m.$$

is locally finite. Hence, there exists a neighborhood of O_m of x such that O_m intersects only a finite number of elements of $\mathcal{B}_m$. Since $W_\alpha \subset U_\alpha$, we also have that

$$\mathcal{W}_m = \{W_\alpha \in \mathcal{C} \mid n(\alpha) = m\}$$

is also locally finite; i.e. that O_m intersects only a finite number of elements of $\mathcal{W}_m$. Now, define the set

$$O_x := U_{\alpha_0} \cap O_1 \cap O_2 \cap \ldots \cap O_{n(\alpha_0)}.$$

Then O_x is a neighborhood of x which intersects only a finite number of elements of $\{W_\alpha \mid \alpha \in \Omega\}$.

□

Proposition 3.9.13. *Let $< X, \tau >$ be a T_3 topological space. If every open cover of $\mathcal{C}$ of X has a locally finite refinement, then $\mathcal{C}$ has a closed locally finite refinement.*

Proof. Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open cover of X. Then for each $x \in X$, there is $U_{\alpha(x)} \in \mathcal{C}$ such that $x \in U_{\alpha(x)}$. Since X is T_3, there is an open set O_x such that

$$x \in clO_x \subset U_{\alpha(x)}.$$

Then the collection $\mathcal{O} =: \{O_x \mid x \in X\}$ is an open refinement of $\mathcal{C}$ (note that $X = \bigcup_{x\in X} O_x$) such that $\widetilde{\mathcal{O}} := \{clO_x \mid x \in X\}$ is a closed refinement of $\mathcal{C}$. By assumption, there is a locally finite refinement $\mathcal{F}$ of $\mathcal{O}$. Then, the family

$$\widetilde{\mathcal{F}} := \{clV \mid V \in \mathcal{F}\}$$

is a closed locally finite refinement of $\widetilde{\mathcal{O}}$. Consequently, $\widetilde{\mathcal{F}}$ is a closed locally finite refinement of $\mathcal{C}$. (Observe that: $V \in \mathcal{F} \Rightarrow V \subset O_x$, for some $x \in X \Rightarrow clV \subset clO_x \subset U_{\alpha(x)} \in \mathcal{C}$). □

Proposition 3.9.14. *Let $< X, \tau >$ be a Hausdorff topological space. If every open cover $\mathcal{C}$ of X has a closed locally finite refinement, then X is paracompact.*

Theorem 3.9.15. *Every paracompact space is normal.*

Proof. Let X be a paracompact topological space. First we show that X is regular.

(i) Let $x \in X$, $F \subset X$ closed and $x \notin F$. Since X is Hausdorff, for each $z \in F$ there exists a neighborhoods U_z such that

$$x \notin clU_z.$$

Define

$$\mathcal{C} := \{U_z \mid z \in F\} \cup \{X \setminus F\}$$

Then $\mathcal{C}$ is an open covering of X. Hence,

X is paracompact $\Rightarrow \exists \mathcal{F}$ which is a locally finite open refinement of $\mathcal{C}$. (Hence, $X = \bigcup \mathcal{F}$).

Define now $\mathcal{A} := \{V \in \mathcal{F} \mid V \subset U_z,$ for some $U_z \in \mathcal{C}\}$¶.

Then

(a) Lem. 3.9.5 implies that $\mathcal{A}$ is locally finite. And, from Lem. 3.9.6, it follows that

$$cl\left(\bigcup_{V\in\mathcal{A}} V\right) = \bigcup_{V\in\mathcal{A}} clV;$$

(b) $F \subset \bigcup \mathcal{A} =: O$ and O is an open set;

¶Indeed $\mathcal{A} \neq \emptyset$. If $z \in F$, then $x \in V$ for some $V \in \mathcal{F}$. But then $V \nsubseteq X \setminus F$. Hence, there is U_z such that $V \subset U_z$; i.e. $z \in V \subset U_z$ and $V \in \mathcal{A}$. So, $\mathcal{A} \neq \emptyset$.

(c) since $x \notin clU_z$ for each $z \in F$, $x \notin clV$ for each $V \in \mathcal{A}$. This implies that

$$x \notin \bigcup_{V\in\mathcal{A}} clV \Rightarrow x \notin cl\left(\bigcup_{V\in\mathcal{A}} V\right) \Rightarrow x \notin clO.$$

$\Rightarrow$

$$x \in X \setminus clO \text{ and } F \subset O.$$

Consequently, X is regular; i.e. T_3.

(ii) Let now F be a closed and U and an open subsets of X such that

$$F \subset U.$$

Since X is regular, for each $x \in F$ there is an open neighborhood U_x such that

$$x \in U_x \subset clU_x \subset U. \qquad (\star)$$

Define the collection $\mathcal{C} := \{U_x \mid x \in F\} \cup \{X \setminus F\}$ and let $\mathcal{F}, \mathcal{A}$ and O be as in part (i). Then, under the condition $(\star)$, it is easy to see that

$$F \subset O \subset clO \subset U.$$

Which implies that X is normal; hence, T_4.

□

Corollary 3.9.16. *A second countable paracompact space is metrizable.*

Proof. A direct consequence of Thm. 3.9.15 and Uryson's metrizability (Thm. 3.5.15). □

Lemma 3.9.17. *Let $< X, \rho >$ be a metric space. If $\{U_n \mid n \in \mathbb{N}\}$ is a countable open cover of X, then there is a locally finite open refinement $\{V_n \mid n \in \mathbb{N}\}$ such that $V_n \subset U_n$, for each $n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$, define the function

$$\varphi_n : X \to [0,1] \text{ such that } \varphi_n(x) = \min\{1, dist(x, X \setminus U_n)\}.$$

Now let

- $V_1 := U_1$; and
- for $n = 2, \ldots$

$$V_n := U_n \cap \bigcap_{k=1}^{n-1} \left\{ x \in X \;\middle|\; \varphi_k(x) < \frac{1}{n} \right\}.$$

Then

(i) for each $n \in \mathbb{N}$, $V_n \subset U_n$ and V_n is an open set;

(ii) the family $\{V_n\}$ covers X. To see this, let $x \in X$. Since $\{U_n\}$ covers X, we have $x \in U_n$ for some $n \in \mathbb{N}$. If $x \in U_1$, then we are done. Otherwise, let $n_0 \in \mathbb{N}$ be the smallest index such that $x \in U_{n_0}$. This implies

$$x \notin U_k, k = 1, \ldots, n_0-1 \Rightarrow x \in X\setminus U_k, k = 1, \ldots, n_0-1 \Rightarrow dist(x, X\setminus U_k) = 0, k = 1, \ldots, n_0-1.$$

From this follows that

$$\varphi_k(x) = 0, k = 1, \dots, n_0 - 1 \Rightarrow x \in U_{n_0} \cap \bigcap_{k=1}^{n_0-1} \left\{ x \in X \;\middle|\; \varphi_k(x) < \frac{1}{n} \right\} = V_{n_0}.$$

Since $x \in X$ is arbitrary, we conclude that

$$X = \bigcup_{n \in \mathbb{N}} V_n;$$

(iii) the family $\{V_n \mid n \in \mathbb{N}\}$ is locally finite. By part (ii), $\{V_n \mid n \in \mathbb{N}\}$ covers X implies for $x_0 \in X$ there is $n_0 \in \mathbb{N}$ such that $x \in V_{n_0}$. But

$$V_{n_0} = U_{n_0} \cap \bigcap_{k=1}^{n_0-1} \left\{ x \in X \;\middle|\; \varphi_k(x) < \frac{1}{n_0} \right\} = U_{n_0} \cap \bigcap_{k=1}^{n_0-1} \left\{ x \in X \;\middle|\; min\{1, dist(x, X \setminus U_k)\} < \frac{1}{n_0} \right\}.$$

But note that

$$\bigcap_{k=1}^{n_0-1} X \setminus U_k \subset \bigcap_{k=1}^{n_0-1} \left\{ x \in X \;\middle|\; min\{1, dist(x, X \setminus U_k)\} < \frac{1}{n_0} \right\}$$

$\Rightarrow$

$$U_{n_0} \cap \bigcap_{k=1}^{n_0-1} X \setminus U_k \subset V_{n_0} \Rightarrow U_{n_0} \setminus \bigcup_{k=1}^{n_0-1} U_k \subset V_{n_0}.$$

Moreover,

$$U_{n_0+1} \setminus \bigcup_{k=1}^{n_0} U_k \subset V_{n_0+1}.$$

Since, $V_{n_0} \subset U_{n_0}$, we observe that $V_{n_0} \cap V_{n_0+1} = \emptyset$. In fact, for $n > n_0, V_{n_0} \cap V_n = \emptyset$. Consequently, $x_0 \in V_{n_0}$ and V_{n_0} can intersect only a finite number of elements of $\{V_n \mid n \in \mathbb{N}\}$. In particular, V_{n_0} can only intersect $V_1, \dots, V_{n_0-1}$. Therefore, $\{V_n \mid n \in \mathbb{N}\}$ is a locally finite family.

□

In the following, for $\varepsilon > 0$, we use $\mathbf{B}_\varepsilon(A) := \{x \in X \mid dist(x, A) < \varepsilon\}$. If $A = \{z\}$, then $\mathbf{B}_\varepsilon(\{z\}) = \mathbf{B}_\varepsilon(z)$.

Theorem 3.9.18. *Every metric space is paracompact.*

Proof. Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open covering of X and let "$\preccurlyeq$" be a well ordering on Ω with $\alpha_1 \in \Omega$ being the first element. Now for each $(n, \alpha) \in \mathbb{N} \times \Omega$ define

$$H_{n,\alpha_1} := \left\{ x \in X \;\middle|\; dist(x, X \setminus U_{\alpha_1}) \geq \frac{1}{n} \right\}$$

$$H_{n,\alpha} := \left\{ x \in X \;\middle|\; dist(x, X \setminus U_\alpha) \geq \frac{1}{n} \right\} \cap \left\{ x \in X \;\middle|\; dist\left(x, \bigcup_{\lambda \precneqq \alpha} H_{n,\lambda} \right) \geq \frac{1}{n} \right\}$$

Obviously, we have that

$$\left\{ x \in X \;\middle|\; dist(x, X \setminus U_\alpha) \geq \frac{1}{n} \right\} \setminus \bigcup_{\lambda \precneqq \alpha} H_{n,\lambda} \subset H_{n,\alpha}.$$

Moreover, the following hold true:

(a) <u>for each $(n,\alpha)\in\mathbb{N}\times\Omega$, $\mathbf{B}_{\frac{1}{2n}}(H_{n,\alpha})\subset U_\alpha$</u>. To see this, let $z\in H_{n,\alpha}$ and $x\in B_{\frac{1}{2n}}(z)$ and $w\in X$ be any. Then

$$\rho(w,z)\le\rho(w,x)+\rho(x,z)\Rightarrow\rho(w,z)-\rho(x,z)\le\rho(w,x).$$

In particular, for any $w\in X\setminus U_\alpha$ we have

$$\frac{1}{n}-\frac{1}{2n}<\rho(w,x)\Rightarrow\rho(w,x)>\frac{1}{2n},\forall w\in X\setminus U_\alpha\Rightarrow dist(x,X\setminus U_\alpha)\ge\frac{1}{2n}.$$

$\Rightarrow x\notin X\setminus U_\alpha\Rightarrow x\in U_\alpha$. Consequently, $\mathbf{B}_{\frac{1}{2n}}(H_{n,\alpha})\subset U_\alpha$.

(b) <u>If $\lambda\precneqq\alpha$, then $dist(H_{n,\lambda},H_{n,\alpha})\ge\frac{1}{2n}$</u>. Let $z\in H_{n,\alpha}$ be arbitrary and consider $B_{\frac{1}{n}}(z)=\{x\in X\mid\rho(z,x)<\frac{1}{n}\}$. By definition of $H_{n,\alpha}$ we have that

$$dist\left(z,\bigcup_{\lambda\precneqq\alpha}H_{n,\lambda}\right)\ge\frac{1}{n}\Rightarrow B_{\frac{1}{n}}(z)\cap\bigcup_{\lambda\precneqq\alpha}H_{n,\lambda}=\emptyset\Rightarrow B_{\frac{1}{n}}(z)\cap H_{n,\lambda}=\emptyset\text{ for each }\lambda\text{ with }\lambda\precneqq\alpha.$$

Since, $B_{\frac{1}{2n}}(z)\subset B_{\frac{1}{n}}(z)$ it follows that

$$B_{\frac{1}{2n}}(z)\cap H_{n,\lambda}=\emptyset\text{ for each }\lambda\text{ with }\lambda\precneqq\alpha.$$

Note that $z\in H_{n,\alpha}$ was chose arbitrarily. Consequently,

$$dist(H_{n,\lambda},H_{n,\alpha})\ge\frac{1}{2n},\text{ for each }\lambda\text{ with }\lambda\precneqq\alpha.$$

(c) <u>$\bigcup_{n\in\mathbb{N}}\bigcup_{\alpha\in\Omega}H_{n,\alpha}=X$</u>. To verify this, let $x\in X$. Then

(i) if $x\in U_{\alpha_1}$, then we are done. Since $x\notin X\setminus U_{\alpha_1}$, it follows that

$$dist(x,X\setminus U_{\alpha_1})>0\Rightarrow\exists m\in\mathbb{N}:dist(x,X\setminus U_{\alpha_1})\ge\frac{1}{m}\Rightarrow x\in H_{m,\alpha_1}.$$

(ii) Otherwise, by the well ordering principle, the set $\Omega_x=\{\alpha\in\Omega\mid x\in U_\alpha\}$ has a least element, say α_x. Hence, $x\in U_{\alpha_x}$ and $x\notin U_\lambda$ for $\lambda\precneqq\alpha_x$. The set U_{α_x} is open implies

$$\exists n_x\in\mathbb{N}:\mathbf{B}_{\frac{1}{n_x}}(x)\subset U_{\alpha_x}\Rightarrow dist(x,X\setminus U_{\alpha_x})\ge\frac{1}{n_x}.\tag{**}$$

Furthermore,

$$\mathbf{B}_{\frac{1}{n_x}}(x)\cap\left(\bigcup_{\lambda\precneqq\alpha_x}H_{n_x,\lambda}\right)=\emptyset.\left(\Rightarrow dist\left(x,\bigcup_{\lambda\precneqq\alpha_x}H_{n_x,\lambda}\right)\ge\frac{1}{n_x}.\right)\tag{***}$$

If we assume that (***) does not hold, then there is $z\in\mathbf{B}_{\frac{1}{n_x}}(x)$ such that $z\in H_{n_x,\lambda}$, for some $\lambda\precneqq\alpha_x$. But then

$$\rho(x,z)<\frac{1}{n_x}\Rightarrow dist(x,X\setminus\{z\})\ge\frac{1}{n_x}.$$

and $z\in H_{n_x,\lambda},\lambda\precneqq\alpha_x$, implies $z\in U_\lambda$. From this follows that

$$X\backslash U_\lambda\subset X\backslash\{z\}\Rightarrow dist(x,X\backslash U_\lambda)\ge dist(x,X\backslash\{z\})\ge\frac{1}{n_x}.\text{(That is, }x\notin X\setminus U_\lambda\text{, so that }x\in U_\lambda.\text{)}$$

Hence, $x \in U_\lambda$. But this is a contradiction to the definition of α_x. Consequently, (***) holds.
From (**) and (***) we conclude now that $x \in H_{n_x,\alpha}$. Therefore,

$$X = \bigcup_{n\in\mathbb{N}} \bigcup_{\lambda\in\Omega} H_{n,\alpha}.$$

Now define

$$V_{n,\alpha} = B_{\frac{1}{6n}}(H_{n,\alpha}). \text{ Then it follows that } X = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\in\Omega} V_{n,\alpha}; \text{ since } H_{n,\alpha} \subset V_{n,\alpha}. \qquad (****)$$

That is, the collection $\{V_{n,\alpha} \mid (n,\alpha) \in \mathbb{N}\times\Omega\}$ is an open cover of X. Moreover, if

$$A_n := \bigcup_{\alpha\in\Omega} V_{n,\alpha},$$

then $\mathcal{A} := \{A_n \mid n \in \mathbb{N}\}$ is a countable open cover for X. By Lem. 3.9.17, there is a locally finite open refinement $\mathcal{F} = \{A'_n \mid n \in \mathbb{N}\}$ of $\mathcal{A}$ such that $A'_n \subset A_n = \bigcup_{\alpha\in\Omega} V_{n,\alpha}$ for each $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} A'_n$.
Let next

$$O_{n,\alpha} := A'_n \cap V_{n,\alpha}, n \in \mathbb{N}, \alpha \in \Omega.$$

Then we claim

(d) $\underline{\mathcal{O} := \{O_{n,\alpha} \mid n \in \mathbb{N}, \alpha \in \Omega\} \text{ is a cover for } X.}$
Let $x \in X$. Since $X = \bigcup_{n\in\mathbb{N}} A'_n$, there is $n \in N$ such that

$$x \in A'_n \subset A_n = \bigcup_{\alpha\in\Omega} V_{n,\alpha} \Rightarrow \exists n \in \mathbb{N}, \alpha \in \Omega : x \in A'_n \text{ and } x \in V_{n,\alpha} \Rightarrow \exists n \in \mathbb{N}, \alpha \in \Omega : x \in O_{n,\alpha}.$$

Hence,

$$X = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\in\Omega} O_{n,\alpha}.$$

(e) $\underline{\mathcal{O} := \{O_{n,\alpha} \mid n \in \mathbb{N}, \alpha \in \Omega\} \text{ is a locally finite open refinement of } \mathcal{C}.}$
Note that

$$O_{n,\alpha} \subset V_{n,\alpha} = \mathbf{B}_{\frac{1}{6n}}(H_{n,\alpha}) \subset \mathbf{B}_{\frac{1}{2n}}(H_{n,\alpha}) \underbrace{\subset}_{\text{part }(a)} U_\alpha \Rightarrow O_{n,\alpha} \subset U_\alpha.$$

Furthermore, for $x \in X$, since $\mathcal{F}$ is locally finite, there is a neighborhood $N(x)$ of x such that $N(x)$ intersects only a finite number of elements of $\mathcal{F} \Rightarrow$

$$\exists n_0 \in \mathbb{N} : N(x) \cap A'_n \neq \emptyset, 1 \leq n \leq n_0 \text{ and } N(x) \cap A'_n = \emptyset, n \geq n_0 + 1.$$

Moreover, for $\lambda \precneqq \alpha$ it follows that $dist(V_{n,\alpha}, V_{n,\lambda}) \geq \frac{1}{6n}$. To verify this, let $w \in H_{n,\alpha}, x \in V_{n,\alpha}, y \in H_{n,\lambda}, z \in V_{n,\lambda}$. We estimate $\rho(x,z)$.
Repeatdly, using the triangle inequality, we obtain

$$\rho(w,y) \leq \rho(w,x) + \rho(x,y) \leq \rho(w,x) + \rho(x,z) + \rho(z,y)$$

$\Rightarrow$ (using part (b), and the definitions of $V_{n,\alpha}$ and $V_{n,\lambda}$)

$$\rho(w,y) - \rho(w,x) - \rho(z,y) \leq \rho(x,z) \Rightarrow \frac{1}{2n} - \frac{1}{6n} - \frac{1}{6n} \leq \rho(w,y) - \rho(w,x) - \rho(z,y) \leq \rho(x,z)$$

$\Rightarrow$

$$\rho(x,z) \geq \frac{1}{6n}.$$

Hence, $dist(V_{n,\alpha}, V_{n,\lambda}) \geq \frac{1}{6n}$. This implies the open ball $\mathbf{B}_{\frac{1}{8n}}(x)$ can intersect only one $V_{n,\alpha}$ whenever $\lambda \not\preceq \alpha$. Consequently, the neighborhood $N(x) \cap \mathbf{B}_{\frac{1}{8n}}(x)$ intersects only at most n_0 elements of $\mathcal{O}$; i.e. $\mathcal{O}$ is locally finite.

□

Remark 3.9.19. *Thms. 3.9.18 and 3.9.15 imply that every metric space is normal; hence, Hausdorff.*

Excercises 3.9.20. *Verify the statements of Rem. 3.9.4.*

3.10 Partition of Unity

Definition 3.10.1 (support of a function)**.** *Let $< X, \tau >$ be a topological space and $f : X \to \mathbb{R}$, i.e. f is a real valued function. Then the support of the function f is defined as*

$$supp\ f := cl\{x \in X \mid f(x) \neq 0\}.$$

Definition 3.10.2. *Let $< X, \tau >$ be a topological space and $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ is a covering of X. Then a collection of real-valued functions $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ is said to be* ***subordinate*** *to the covering $\mathcal{C}$ on X if*

$$\forall \lambda \in \Lambda, \exists \alpha \in \Omega : supp\ \varphi_\lambda \subset U_\alpha.$$

Excercises 3.10.3. *If K is a compact subset of Hausdorff space, then there is an open set O such that $K \subset O$ and clO is compact.*

Proposition 3.10.4. *Let $< X, \tau >$ be a locally compact Hausdorff space and K be a compact subset of X. If $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ is an open covering of K, then there is a collection $\{\varphi_1, \ldots, \varphi_n\}$ of continuous real-valued non-negative functions subordinate to the collection $\mathcal{C}$ on K.*

Proof. By Prop. 3.7.5(i), there is O open such that $K \subset O$ and clO is compact. Hence,

$$K \subset O \cap \bigcup_{\alpha \in \Omega} U_\alpha.$$

(i) If $x_0 \in K$, then there is $\alpha(x_0) \in \Omega$ such that $x_0 \in O \cap U_{\alpha(x_0)}$. Thus $\{x_0\}$ is compact, $O \cap U_{\alpha(x_0)}$ open and $cl\left(O \cap U_{\alpha(x_0)}\right)$ is compact. Using Prop. 3.7.5(ii), there is a continuous function $f_{x_0} : X \to [0,1]$ such that

$$f_{x_0}(x_0) = 1 \text{ and } f(X \setminus cl(O \cap U_{\alpha(x_0)})) = 0.$$

This implies,

- $suppf_{x_0} \subset O \cap U_{\alpha(x_0)}$; i.e. $suppf_{x_0} \subset U_{\alpha(x_0)}$; and
- $f_{x_0}(x) > 0, \forall x \in cl(O \cap U_{\alpha(x_0)})$.

(ii) If $x_0 \in clO \setminus K$, there is an open neighborhood $N(x_0)$ of x_0 such that $x_0 \in N(x_0)$ and $clN(x_0)$ is compact (since X is locally compact). Hence, $N(x_0) \cap (X \setminus K)$ is open and $cl\,(N(x_0) \cap (X \setminus K))$ is compact. Once more by Prop. 3.7.5(ii), there is a continuous function $g_{x_0} : X \to [0,1]$ such that

$$g_{x_0}(x_0) = 1, \text{ and } g_{x_0}(X \setminus cl\,(N(x_0) \cap (X \setminus K))) = 0.$$

This implies

$$supp\ g_{x_0} \subset cl\,(N(x_0) \cap (X \setminus K)) = clN(x_0) \cap X \setminus K \Rightarrow supp\ g_{x_0} \subset X \setminus K.$$

Consequently,

$$supp\ g_{x_0} \cap K = \emptyset \Rightarrow g_{x_0} = 0, \forall x \in K \Rightarrow g_{x_0} \equiv 0 \text{ on } K.$$

(iii) Now define the sets

$$V_{x_0} := \begin{cases} \{x \in O \cap U_{\alpha(x_0)} \mid f_{x_0}(x) > 0\}, & \text{if } x_0 \in K; \\ \{x \in O_{x_0} \cap (X \setminus K) \mid g_{x_0}(x) > 0\}, & \text{if } x_0 \in clO \setminus K. \end{cases}$$

Thus, for each $x_0 \in clO$, the sets V_{x_0} are open and $x_0 \in V_{x_0}$. Consequently,

$$clO \subset \bigcup \{V_{x_0} \mid x_0 \in clO\}.$$

Since, clO is compact, there is a finite covering of clO from $\{V_{x_0} \mid x_0 \in clO\}$. Correspondingly, there are functions $f_{x_0^1}, \ldots, f_{x_0^n}, g_{x_0^{n+1}}, \ldots, g_{x_0^{n+m}}$.

Set now

$$f = \sum_{k=1}^{n} f_{x_0^k} \quad \text{and } g = \sum_{k=n+1}^{m} g_{x_0^k}.$$

Hence,

(a) Note that for each $k \in \{n+1, \ldots, m\}$, $supp\ g_{x^k} \subset X \setminus K \Rightarrow K \cap supp\ g_{x^k} = \emptyset \Rightarrow g_{x^k}(x) = 0, \forall x \in K$. Hence, $g \equiv 0$ on K.

(b) Let the corresponding finite covering be $\{V_{x_0^1}, \ldots, V_{x_0^n}, V_{x_0^{n+1}}, \ldots, V_{x_0^{n+m}}\}$, such that

$$K \subset clO \subset \bigcup_{k=1}^{n} V_{x_0^k} \cup \bigcup_{k=n+1}^{m} V_{x_0^k}$$

By part (a), we have $K \cap \left(\bigcup_{k=n+1}^{m} V_{x_0^k}\right) = \emptyset \Rightarrow K \subset \bigcup_{k=1}^{n} V_{x_0^k} \Rightarrow f_{x^k}(x) > 0, \forall x \in K$. Hence, $f > 0$ on K.

From this follows that

$$\frac{f}{f+g} \equiv 1 \text{ on } K.\text{(Note that } g \equiv 0 \text{ on K).}$$

Now define

$$\varphi_k := \frac{f_{x_0^k}}{f+g}, k = 1, \ldots, n.$$

It follows that $\varphi_1 + \ldots + \varphi_n = 1$, for each $k \in \{1, \ldots, n\}$

$$supp\ \varphi_k \subset U_{\alpha(x_0^k)} \text{ and } \varphi_k \geq 0, \text{ on } X.$$

□

Definition 3.10.5 (partition of unity). *Let $< X, \tau >$ be a topological space. A family of functions $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ is called a* **partition of unity** *on X iff*

(i) $\{supp\ \varphi_\lambda \mid \lambda \in \Lambda\}$ is a closed locally finite covering of X;

(ii) for each $\lambda \in \Lambda, \varphi_\lambda \geq 0$ on X;

(iii) for each $x \in X$

$$\sum_{\lambda \in \Lambda} \varphi_\lambda(x) = 1.$$

If φ_λ is continuous (or Lipschitz continuous or differentiable), for each $\lambda \in \Lambda$, then the partition is said to be a continuous (or Lipschitz continuous or differentiable) partition of unity.

If a partition $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ is subordinate to a family $\{U_\alpha \mid \alpha \in \Omega\}$, then it is called a **partition of unity subordinate** to $\{U_\alpha \mid \alpha \in \Omega\}$.

Lemma 3.10.6. *If X is a paracompact space, then every open covering $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ of X has a locally finite open refinement $\{V_\alpha \mid \alpha \in \Omega\}$ such that $\emptyset \neq clV_\alpha \subset U_\alpha$.*

Proof. (i) For each $x \in X$, there is $\alpha \in \Omega$ such that $x \in U_\alpha$.

(ii) Since X is paracompact, X is normal by Thm. 3.9.15. Then, by Thm.3.4.7, there is an open set W_x such that

$$x \in W_x \subset clW_x \subset U_\alpha.$$

Hence, $\mathcal{W} := \{W_x \mid x \in X\}$ is an open cover of X which refines $\mathcal{C}$.

(iii) By paracompactness of X, there is a locally finite open refinement $\mathcal{O}$ of $\mathcal{W}$.

(iv) Now define, for each $\alpha \in \Omega$

$$\mathcal{A}_\alpha := \{O \in \mathcal{O} \mid O \subset U_\alpha\}.$$

(v) Next define

$$V_\alpha := \begin{cases} \bigcup_{O \in A_\alpha} O, & \text{if } A_\alpha \neq \emptyset \\ \text{Any open set with } \emptyset \neq V_\alpha \subset clV_\alpha \subset U_\alpha, & \text{if } A_\alpha = \emptyset. \end{cases}$$

(vi) Let $x \in X$, then $x \in O$, for some $O \in \mathcal{O}$. Thus, there is $W_x \in \mathcal{W}$ such that $x \in \mathcal{O} \subset W_x$. Correspondingly, there is $\alpha \in \Omega$ such that $W_x \subset clW_x \subset U_\alpha$. Hence,

$$\exists \alpha \in \Omega : x \in \mathcal{O} \subset U_\alpha \Rightarrow x \in V_\alpha.$$

From this follows that $\{V_\alpha \mid \alpha \in \Omega\}$ is a covering of X. Furthermore, $\{V_\alpha \mid \alpha \in \Omega\}$ is locally finite and, for each $\alpha \in \Omega$, $V_\alpha \subset clV_\alpha \subset U_\alpha$ (cf. Lem. 3.9.6).

□

Theorem 3.10.7. *Let $< X, \tau >$ be a topological space. If X is paracompact, then every open cover of X has a partition of unity subordinate to it.*

Proof. Let $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open cover of X. Then, by Lem. 3.10.6, there is a locally finite open refinement of $\mathcal{C}$ such that $clO_\alpha \subset U_\alpha$ for each $\alpha \in \Omega$. Then, by Thm. 3.5.3 (since X is normal) , there is a continuous function $f_\alpha : X \to [0, 1]$ such that

$$f_\alpha(clO_\alpha) = 1 \text{ and } f_\alpha(X \setminus U_\alpha) = 0, \text{ for each } \alpha \in \Omega. \text{ (That is, } supp\ f_\alpha \subset U_\alpha.\text{)}$$

Now, for $x \in X$, define

$$\varphi_\alpha(x) := \frac{f_\alpha(x)}{\sum_{\lambda\in\Omega} f_\alpha(x)}.$$

Claim $\{\varphi_\alpha \mid \alpha \in \Omega\}$ is a partition of unity subordinate to $\mathcal{C}$.

(i) Obviously, for each $\alpha \in \Omega, \varphi \geq 0$ on X;

(ii) To show continuity, let $x_0 \in X$, there is a neighborhood $U(x_0)$ such that $U(x_0)$ intersects only a finite number of elements of $\{clO_\alpha \mid \alpha \in \Omega\}$. That is there is $\{\alpha_1, \ldots, \alpha_{n(x_0)}\}$ such that

$$U(x_0) \cap O_\alpha = \emptyset, \forall \alpha \in \Omega \setminus \{\alpha_1, \ldots, \alpha_{n(x_0)}\}$$

[‖] This implies

$$U(x_0) \subset X \setminus O_\alpha, \forall \alpha \in \Omega \setminus \{\alpha_1, \ldots, \alpha_{n(x_0)}\} \Rightarrow f_\alpha(U(x_0)) = 0, \forall \alpha \in \Omega \setminus \{\alpha_1, \ldots, \alpha_{n(x_0)}\} \quad (3.5)$$

and

$$U(x_0) \cap supp\ f_\alpha = \emptyset, \forall \alpha \in \Omega \setminus \{\alpha_1, \ldots, \alpha_{n(x_0)}\}. \quad (3.6)$$

Thus

$$\sum_{\lambda\in\Omega} f_\alpha(x) = \sum_{k=1}^{n(x_0)} f_{\alpha_k}(x) + \sum_{\lambda\in\Omega\setminus\{\alpha_1,\ldots,\alpha_{n(x_0)}\}} f_\alpha(x) \Rightarrow \sum_{\lambda\in\Omega} f_\alpha(x) = \sum_{k=1}^{n(x_0)} f_{\alpha_k}(x), \text{ for } x \in U(x_0).$$

Consequently, for $x \in U(x_0)$:

$$\varphi_\alpha(x) = \frac{f_\alpha(x)}{\sum_{k=1}^{n(x_0)} f_{\alpha_k}(x)}.$$

This implies, φ_α is a continuous function at x_0. Note also that $\sum_{k=1}^{n(x_0)} f_{\alpha_k}(x_0) \neq 0$, since $x_0 \in clO_{\alpha_{k_0}}$, for some $k_0 \in \{1, \ldots, n(x_0)\}$. Hence, φ_α is a continuous on X.

(iii) For each $\alpha \in \Omega$, $clO_\alpha \subset supp\ \varphi_\alpha \subset U_\alpha$ and , from eqn. (3.6), $\{supp\ \varphi_\alpha \mid \alpha \in \Omega\}$ is locally finite. Hence, $\{supp\ \varphi_\alpha \mid \alpha \in \Omega\}$ is a closed locally finite covering of X.

(iv) Moreover, $\sum_{\alpha\in\Omega} \varphi_\alpha(x) = 1$ for each $x \in X$.

Consequently, by Def. 3.10.5, $\{\varphi_\alpha \mid \alpha \in \Omega\}$ is a partition of unity of subordinate to $\mathcal{C}$.

□

Remark 3.10.8. *The converse of Thm. 3.10.7 also holds true. (Proof is left as an exercise!).*

Corollary 3.10.9 (Thm. 2, p. 10, Aubin/Cellina)**.** *Let $< X, \rho >$ is a metric space. Then to any locally finite open covering $\mathcal{C} := \{U_\alpha \mid \alpha \in \Omega\}$ of X, there is locally Lipschitzean partition of unity subordinate to it.*

[‖] In fact we have here $U(x_0) \cap clO_\alpha = \emptyset, \forall \alpha \in \Omega \setminus \{\alpha_1, \ldots, \alpha_{n(x_0)}\}$.

Proof. Let $\{U_\alpha \mid \alpha \in \Omega\}$ be a locally finite open covering of X. Then, by Lem. 3.10.6, there is a locally finite open covering $\{O_\alpha \mid \alpha \in \Omega\}$ such that $clO_\alpha \subset U_\alpha$. Now, define

$$f_\alpha(x) := dist(x, X \setminus O_\alpha) \text{ and } \varphi_\alpha(x) := \frac{f_\alpha(x)}{\sum_{\lambda\in\Omega} f_\lambda(x)}.$$

Then the following hold true :

(i) $f_\alpha : X \to \mathbb{R}_+$ Lipschitz continuous and

$$supp\ f_\alpha = supp\ \varphi_\alpha = cl\{x \in X \mid \varphi_\alpha(x) \neq 0\} = clO_\alpha;$$

(ii) $\{\varphi_\alpha \mid \alpha \in \Omega\}$ is a partition of unity subordinate to $\mathcal{C}$ (as in the proof of Thm. 3.10.7);

(iii) for each $\alpha \in \Omega$, φ_α is locally Lipschitz continuous on X.

□

Excercises 3.10.10. *Prove the following statements:*

1. *A closed subspace of a paracompact topological space is normal.*
2. *A closed subspace of a paracompact topological space is paracompact.*
3. *If $< X, \tau >$ is a regular Lindlöf space, then $< X, \tau >$ is paracompact.*
4. *If $< X, \tau >$ is paracompact and sparable, then $< X, \tau >$ is Lindelöf.*
5. *The following statements are equivalent for a T_1 space*
   (i) X is normal;
   (ii) every locally finite open cover $\mathcal{C} = \{U_\alpha \mid \alpha \in \Omega\}$ of X has an open refinement $\mathcal{O} = \{O_\alpha \mid \alpha \in \Omega\}$ with the property that $\emptyset \neq clO_\alpha \subset U_\alpha$.

---

4 The Hausdorff Metric and Convergence of Sequences of Sets

4.1 The Hausdorff Metrics

In this this section we consider solely metric spaces and normed linear spaces. Given a metric or a normed space X, we also let that $\mathcal{P}(X) := \{S \mid S \subset X\}$ to represent the *power set* of X.

Definition 4.1.1 (distance from a point to a set)**.** *Let $< X, \rho >$ be a metric space and $A \subset X$, $A \neq \emptyset$ and $y \in X$. Then the* distance *between y to A with respect to ρ is given by*

$$dist(b, A) := \inf \{\rho(b, a) \mid a \in A\} .$$

Proposition 4.1.2. *Let X be a metric space, $b \in X$ and $A \subset X$. Then*

(i) $dist(b, A) \geq 0$;

(ii) if $b \in A$, then $dist(b, A) = 0$;

(iii) if $dist(b, A) = 0$, then $b \in clA$.

Remark 4.1.3. *In Def. 4.1.1, we can in fact allow $A = \emptyset$. But in this case we define*

$$dist(b, \emptyset) = \infty,$$

for any $b \in X$.

Proposition 4.1.4. *Given $A \subset X$ and $b \in X$ it follows that*

$$dist(b, A) \leq \rho(b, c) + dist(c, A)$$

for any $c \in X$.

Recall also that the metric distance between two sets A and B is given by

$$dist(A, B) = \inf \{\rho(x, y) \mid x \in A, y \in B\} .$$

Definition 4.1.5 (The Hausdorff distance)**.** *Let A and B be subsets of a metric space X. Then the Hausdorff distance between A and B is defined as*

$$h(A, B) := \max \left\{ \sup_{a \in A} dist(a, B), \sup_{b \in B} dist(b, A) \right\}$$

Observe that: $dist(A, B) \leq h(A, B)$.

Remark 4.1.6. *The quantity*

$$h^*(A, B) := \sup_{a \in A} dist(a, B).$$

is sometimes termed as the semi-(Hausdorff)distance *from the set A to the set B. Thus, obviously,*

- *$h^*(A, B) \neq h^*(B, A)$ and*
- *$h(A, B) = \max \{h^*(A, B), h^*(B, A)\}$.*

Lemma 4.1.7. *For any three sets A, B and C we have*

$$h^*(A,B) \leq h^*(A,C) + h^*(C,B).$$

Proof. Take $a \in A$ arbitrarily. Then, by Prop. 4.1.4, for any $c \in C$, we have that

$$dist(a,B) \leq \rho(a,c) + dist(c,B).$$

$\Rightarrow$

$$dist(a,B) \leq \inf_{c\in C}\{\rho(a,c) + dist(c,B)\} \leq \inf_{c\in C}\rho(a,c) + \sup_{c\in C} dist(c,B).$$

$\Rightarrow$

$$dist(a,B) \leq dist(a,C) + h^*(C,B).$$

This last inequality holds true for every $a \in A$. Hence

$$\sup_{a\in A} dist(a,B) \leq \sup_{a\in A} dist(a,C) + h^*(C,B)$$

Therefore,

$$h^*(A,B) \leq h^*(A,C) + h^*(C,B).$$

□

That fact that $h^*(A,B) \neq h^*(B,A)$ and the satisfaction of the triangle inequality make h^* a quasi metric (see sec. 1.1 of chap. 1).

Lemma 4.1.8. *For any four real numbers a, b, c, d, the following holds*

$$\max\{a+b, d+e\} \leq \max\{a,d\} + \max\{b,e\}$$

The Hausdorff distance defines some sort of metric on the set $\mathcal{P}(X)$ as given by

Theorem 4.1.9. *The following statements hold true for the Hausdorff distance h:*

(i) for each $A, B \in \mathcal{P}(X)$, $h(A,B) \geq 0$;

(ii) for each $A \in \mathcal{P}(X)$, $h(A,A) = 0$;

(iii) for $A, B \in \mathcal{P}(X)$, $h(A,B) = h(B,A)$;

(iv) for $A, B, C \in \mathcal{P}(X)$, $h(A,B) \leq h(A,C) + h(C,B)$ (triangle inequality).

Thus (i)-(iii) imply that the Hausdorff distance h defines a ***pseudo-metric*** *on $\mathcal{P}(X)$ (sec. 1.1. of chap. 1).*

Proof. (i) - (iii) follow directly from Defs. 4.1.1 and 4.1.5.

(iv) Using Lem. 4.1.7 twice, we find that

$$h^*(A,B) \leq h^*(A,C) + h^*(C,B)$$
$$h^*(B,A) \leq h^*(B,C) + h^*(C,A).$$

$\Rightarrow$

$$\begin{aligned} h(A,B) = \max\{h^*(A,B), h^*(B,A)\} &\leq \max\{h^*(A,C) + h^*(C,B), h^*(B,C) + h^*(C,A)\} \\ &= \max\{h^*(A,C) + h^*(C,B), h^*(C,A) + h^*(B,C)\}. \end{aligned}$$

Next, applying Lem. 4.1.8, we obtain that

$$h(A,B) \leq \max\{h^*(A,C), h^*(C,A)\} + \max\{h^*(C,B), h^*(B,C)\} = h(A,C) + h(C,B).$$

Hence,

$$h(A,B) \leq h(A,C) + h(C,B).$$

□

Remark 4.1.10. *In general, for $A, B \subset X$, $h(A,B) = 0$ does not imply that $A = B$; i.e. h is **not a metric** on $\mathcal{P}(X)$.*

Lemma 4.1.11. *Let A and B be subsets of a metric space. If $h^*(A,B) = 0$, then $A \subset clB$.*

Proof. By definition

$$h^*(A,B) = \sup_{a\in A} dist(a,B) = 0 \Rightarrow \forall a \in A : dist(a,B) = 0, \text{ (since } dist(a,B) \geq 0 \text{ for any point } a \text{ and set } B \text{).}$$

Hence, for each $a \in A$, $a \in clB$ (cf. Prop. 4.1.2); i.e. $A \subset clB$. □

Theorem 4.1.12. *Let X be a metric space and $\mathcal{P}_{cl}(X)$ is the set of all closed subsets of X. Then the Hausdorff distance h defines a metric on $\mathcal{P}_{cl}(X)$; $< \mathcal{P}_{cl}(X), h >$ is a metric space.*

Proof. According to Thm.4.1.9, it remains to show that for $A, B \in \mathcal{P}_{cl}(X)$, $h(A,B) = 0$ implies $A = B$. But then

$$h(A,B) = 0 \Rightarrow h^*(A,B) = 0 \text{ and } h^*(B,A) = 0.$$

Then, using Lem. 4.1.11, we conclude

$$A \subset clB \text{ and } B \subset clA \Rightarrow clA = clB \Rightarrow A = B. \text{ (Since both } A \text{ and } B \text{ are closed sets).}$$

□

Suppose, for a set $A \subset X, b \in X$ and $\varepsilon > 0$, we have

$$\mathcal{U}_\varepsilon(A) := \{x \in X \mid dist(x,A) < \varepsilon\}, \quad \mathbf{B}_\varepsilon(b) := \{x \in X \mid \rho(x,b) < \varepsilon\},$$

$\widetilde{\mathcal{U}_\varepsilon}(A) := \{x \in X \mid dist(x,A) \leq \varepsilon\}$ and $cl\mathbf{B}_\varepsilon(b) := \{x \in X \mid \rho(x,b) \leq \varepsilon\}$

Lemma 4.1.13. *For any two subsets A and B of a metric space X we have*

$$h^*(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\} \tag{4.1}$$

$$h^*(B,A) = \inf\{\varepsilon > 0 \mid B \subset \mathcal{U}_\varepsilon(A)\}. \tag{4.2}$$

Proof. We need only to show that $h^*(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}$.

- If $h^*(A,B) = 0$, then the equality follows by using Lem. 4.1.11.

(a) Let $\varepsilon_0 := \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}$. Then, using Prop. 4.1.4, for $a \in A$ and $c \in \mathcal{U}_{\varepsilon_0}(B)$ follows

$$dist(a,B) \leq \rho(a,c) + dist(c,B) \leq \rho(a,c) + \varepsilon_0.$$

$\Rightarrow dist(a,B) \leq \rho(a,c) + \varepsilon_0$. Since $c \in \mathcal{U}_{\varepsilon_0}(B)$ is arbitrary, we obtain

$$dist(a,B) \leq \inf_{c\in\mathcal{U}_\varepsilon(B)} \rho(a,c) + \varepsilon_0 = dist(a, \mathcal{U}_\varepsilon(B)) + \varepsilon_0 = 0 + \varepsilon_0 = \varepsilon_0. \text{ (Note that } A \subset \mathcal{U}_{\varepsilon_0}(B)).$$

Consequently,

$$dist(a,B) \leq \varepsilon_0, \text{ for each } a \in A. \Rightarrow \sup_{a\in A} dist(a,B) \leq \varepsilon_0$$

Implying that

$$h^*(A,B) \leq \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}.$$

(b) Conversely, suppose $h^*(A,B) = r$. Then $\sup_{a\in A} dist(a,B) = r$. Hence, for $a \in A$, we obtain

$$r = h^*(A,B) \geq \rho(a,B).$$

$\Rightarrow$

$$\exists b \in B : \rho(a,b) \leq r \Rightarrow a \in cl\mathbf{B}_r(b) := \{x \in X \mid \rho(x,b) \leq r\}.$$

$\Rightarrow$

$$A \subset \bigcup_{b\in B} cl\mathbf{B}_r(b). \tag{4.3}$$

Now, for any $b \in B$, let $z \in \mathbf{B}_r(b)$. Then

$$dist(z,B) \leq \rho(z,b) + dist(b,B) \Rightarrow dist(z,B) \leq r \Rightarrow z \in \widetilde{\mathcal{U}}_r(B).$$

Since, $z \in \mathbf{B}_r(b)$ is arbitrary, we conclude that $cl\mathbf{B}_r(b) \subset \widetilde{\mathcal{U}}_r(B)$. It then follows that,

$$\bigcup_{b\in B} cl\mathbf{B}_r(b) \subset \widetilde{\mathcal{U}}_r(B).$$

Hence, along with (4.3) we find that $A \subset \widetilde{\mathcal{U}}_r(B)$. Furthermore, if we take an arbitrary $n \in \mathbb{N}$, we obtain

$$A \subset \widetilde{\mathcal{U}}_r(B) \subset \mathcal{U}_{r+\frac{1}{n}}(B).$$

$\Rightarrow$

$$r + \frac{1}{n} \geq \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}, \forall n \in \mathbb{N}$$

Consequently,

$$h^*(A,B) = r \geq \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}.$$

Therefore, from (a) and (b), we conclude that

$$h^*(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}.$$

□

Proposition 4.1.14. *Let X be a metric space and $A, B \subset X$. Then*

$$h(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B) \text{ and } B \subset \mathcal{U}_\varepsilon(A)\}$$

Proof. Use Lem. 4.1.13 □

Remark 4.1.15. *Given a normed linear space X and $A, B \subset X$ and $\gamma \in \mathbb{R}$ we recall that*

$A + B = \{a + b \mid a \in A, b \in B\}$ *- the sum of two sets*
$\gamma A = \{\gamma a \mid a \in A\}$ *- scalar product.*

*The sum of sets $A + B$ is commonly known as **Kuratowski sum**. Thus, for $B = \{b\}$, we write*

$$B + A = b + A = \{b + a \mid a \in A\}.$$

Consequently, for a set $A \subset X$ and $\varepsilon > 0$ we can now write

$$\mathcal{U}_\varepsilon(A) = A + \varepsilon\mathbf{B},$$

where $\mathbf{B}$ represents the unit ball *centered at the zero element of X. Thus, the Hausdorff metric in a normed linear space takes the form*

$$\boxed{h(A,B) = \inf\{\varepsilon > 0 \mid A \subset B + \varepsilon\mathbf{B} \text{ and } B \subset A + \varepsilon\mathbf{B}\}.}$$

Definition 4.1.16 (Pompeiu-Hausdorff distance, see [22]). *Let $< X, \rho >$ be a metric space, $A, B \in \mathcal{P}(X) \setminus \{\emptyset\}$. Then the Pompeiu-Hausdorff distance between A and B is given by*

$$\mathbb{D}_\infty(A, B) = \sup\{|dist(x, A) - dist(x, B)| \ : \ x \in X\} = \sup_{x \in X} |dist(x, A) - dist(x, B)|.$$

Note that, if either $A = \emptyset$ or $B = \emptyset$, then $\mathbb{D}_\infty$ is undefined.

Proposition 4.1.17. *Let $< X, \rho >$ be a metric space and $A, B \in \mathcal{P}(X) \setminus \{\emptyset\}$. Then*

$$h(A, B) = \mathbb{D}_\infty(A, B).$$

Proof. (a) Using Prop. 4.1.4, for any $x \in X$ and $b \in B$, we have

$$dist(x, A) \leq \rho(x, b) + dist(b, A).$$

Since $b \in B$ is arbitrary, this implies that

$$dist(x, A) \leq \inf_{b \in B} \rho(x, b) + dist(b, A) \leq \inf_{b \in B} \rho(x, b) + \sup_{b \in B} dist(b, A) \leq dist(x, B) + h^*(B, A).$$

$\Rightarrow$

$$dist(x, A) - dist(x, B) \leq h^*(B, A).$$

Similarly, we have

$$dist(x, B) - dist(x, A) \leq h^*(A, B)$$

Hence,

$$\max\{dist(x, A) - dist(x, B), dist(x, B) - dist(x, A)\} \leq \max\{h^*(B, A), h^*(A, B)\}.$$

$\Rightarrow$

$$|dist(x, A) - dist(x, B)| \leq h(A, B); \text{ i.e. } \mathbb{D}_\infty \leq h(A, B).$$

(b) Conversely

$$\begin{aligned} h^*(B, A) = \sup_{b \in B} dist(b, A) \quad &= \sup_{b \in B} [dist(b, A) - dist(b, B)] \\ &\leq \sup_{x \in X} [dist(x, A) - dist(x, B)] \\ &\leq \sup_{x \in X} |dist(x, A) - dist(x, B)|. \end{aligned}$$

Analogously

$$h^*(A, B) \leq \sup_{x \in X} |dist(x, A) - dist(x, B)|$$

Consequently, we have

$$\max\{h^*(B, A), h^*(A, B)\} \leq \sup_{x \in X} |dist(x, A) - dist(x, B)|$$

$\Rightarrow$

$$h(A, B) \leq \mathbb{D}_\infty(A, B).$$

Which concludes the proof.

□

Excercises 4.1.18. *Prove the following statements:*

1. *Show that, if $dist(b, A) = 0$, then $b \in clA$.*
2. *Show that if $B_1 \subset B_2$, then $dist(A, B_1) \geq dist(A, B_2)$.*
3. *Prove Prop. 4.1.4.*
4. *Verify the validity of Lem. 4.1.8.*
5. *Show that if $h^*(A, B) = 0$, then $h^*(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{U}_\varepsilon(B)\}$.*
6. *Let X be a normed linear space, $A, B, C, D \in \mathcal{P}(X)$ and $\lambda \in [0, 1]$. Then prove the following*

   (i) $h(rA, rB) \leq |r|h(A, B), \forall r \in \mathbb{R}$;
   (ii) $h(A + B, C + D) \leq h(A, C) + h(B, D)$
   (iii) $h(\lambda A + (1 - \lambda)B, \lambda C + (1 - \lambda)D) \leq \lambda h(A, C) + (1 - \lambda)h(B, D), \forall \lambda \in [0, 1]$.
   (iv) $h(cl\ convA, cl\ convB) \leq h(A, B)$;
   (v) $h(U_\varepsilon(A), U_\varepsilon(A)) \leq h(A, B), \forall \varepsilon > 0$;

   *where $convA$ represents the **convex Hull** of A.*
7. *Considering $h : \mathcal{P}(X) \times \mathcal{P}(X) \to R_+$, show that $h(\cdot, \cdot)$ is a Lipschitz continuous function with Lipschitz constant 1; i.e. $h(\cdot, \cdot)$ is a non-expansive map.*

4.2 Convergence of Sequences of Sets

Definition 4.2.1 (Kuratowski Convergence)**.** *Let $< X, \rho >$ be a metric space and $\{A_n\}_{n\in N}$ be a sequence of subsets of X. Then*

*(i) the **upper limit** or **outer limit** of the sequence $\{A_n\}_{n\in N}$ is a subset of X given by*

$$\limsup_{n\to\infty} A_n = \{x \in X \mid \liminf_{n\to\infty} dist(x, A_n) = 0\};$$

*(ii) the **lower limit*** or **inner limit**† of the sequence $\{A_n\}_{n\in N}$ is a subset of X given by*

$$\liminf_{n\to\infty} A_n = \{x \in X \mid \lim_{n\to\infty} dist(x, A_n) = 0\}.$$

*If $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, then we say that the **limit of** $\{A_n\}_{n\in N}$ **exists** and*

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

*$\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are sometimes called *Kuratowski limit inferior and limit superior*, respectively.(see Papagorgeu)

†The notions *outer limit* and *inner limit* are introduced by Rockafellar and Wets (see)

Remark 4.2.2. *For a fixed set $A \subset X$, the distance function $dist(\cdot, A) : X \to R$ is a **Lipschitz continuous** function. This follows from the fact that*

$$dist(x, A) \leq \rho(x, \overline{x}) + dist(\overline{x}, A) \Rightarrow dist(x, A) - dist(\overline{x}, A) \leq \rho(x, \overline{x}).$$

for any $x, \overline{x} \in X$. Hence,

$$|dist(x, A) - dist(\overline{x}, A)| \leq \rho(x, \overline{x}).$$

Consequently, if $\{x_n\}_{n\in N}$ is sequence in X such that $x_n \to \overline{x}$, then

$$\lim_{n\to\infty} dist(x_n, A) = dist(\overline{x}, A).$$

With this remark the following proposition follows trivially.

Proposition 4.2.3. *Let $\{A_n\}_n$ be a sequence of subsets of a metric space X. Then*

(i) $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$*;*

(ii) the sets $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are closed in X .

Proof. (i) is trivial. To show (ii), let $\overline{x} \in cl\ (\limsup_{n\to\infty} A_n)$. This implies

$$\exists \{x_k\}_{k\in N} \subset \limsup_{n\to\infty} A_n \text{ such that } x_k \to \overline{x}.$$

Consequently, for each k, we have

$$dist(\overline{x}, A_n) \leq \rho(\overline{x}, x_k) + dist(x_k, A_n).$$

$\Rightarrow$

$$\liminf_{n\to\infty} dist(\overline{x}, A_n) \leq \rho(\overline{x}, x_k) + \liminf_{n\to\infty} dist(x_k, A_n) \Rightarrow \liminf_{n\to\infty} dist(\overline{x}, A_n) \leq \rho(\overline{x}, x_k), \forall k.$$

$\Rightarrow$

$$\liminf_{n\to\infty} dist(\overline{x}, A_n) \leq \inf_k \rho(\overline{x}, x_k) = 0.$$

Hence, $\liminf_{n\to\infty} dist(\overline{x}, A_n) = 0 \Rightarrow \overline{x} \in \limsup_{n\to\infty} A_n$. It follows that $cl\,(\limsup_{n\to\infty} A_n) \subset \limsup_{n\to\infty} A_n$. Thus, $\limsup_{n\to\infty} A_n$ is a closed set. The rest is left as an exercise.

The closure of $\limsup_{n\to\infty} A_n$ also follows similarly. □

Proposition 4.2.4. *If $\{A_n\}_{n\in N}$ is a sequence in a metric space X, then*

(i) $\limsup_{n\to\infty} A_n = \{x \in X \mid x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$.

(ii) $\liminf_{n\to\infty} A_n = \{x \in X \mid x_n \in A_n : x_n \to x\}$.

*That is, $\liminf_{n\to\infty} A_n$ is a **collection of limits of sequences** $\{x_n\}_{n\in\mathbb{N}}$, where $x_n \in A_n$; whereas $\limsup_{n\to\infty} A_n$ is **a collection of cluster points of sequences** $\{x_n\}_{n\in\mathbb{N}}$, where $x_n \in A_n$.*

Proof. (i) Obviously, $\limsup_{n\to\infty} A_n \supset \{x \in X \mid x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$. Thus, let $\overline{x} \in \limsup_{n\to\infty} A_n$. Then

$$\liminf_{n\to\infty} dist(\overline{x}, A_n) = 0 \Rightarrow \sup_n \inf_{i\geq n} dist(\overline{x}, A_i) = 0.$$

Let $k \in \mathbb{N}$ be arbitrary, we have

$$\sup_n \inf_{i\geq n} dist(\overline{x}, A_i) \leq \frac{1}{k}$$

Hence,

$$\exists n_k \in \mathbb{N} : \inf_{i \geq n_k} dist(\overline{x}, A_i) \leq \frac{1}{k}.$$

$\Rightarrow$

$$\exists i_k \geq n_k, \exists x_{i_k} \in A_{i_k} : \rho(\overline{x}, x_{i_k}) \leq \frac{1}{k}.$$

But this is true for each $k \in \mathbb{N}$. Hence, there is $\{x_{i_k}\}_{i_k \in \mathbb{N}}$ such that $x_{i_k} \in A_{i_k}$ and $x_{i_k} \to \overline{x}$. Consequently,

$$\overline{x} \in \{x \in X \mid x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$$

Therefore, $\limsup_{n\to\infty} A_n \subset \{x \in X \mid x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$.

(ii) follows trivially.

□

Proposition 4.2.5. *Let $\{A_n\}$ be a sequence in a metric space $< X, \rho >$. Then*

(i) $\limsup_{n\to\infty} A_n = \{x \in X \mid \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N : \mathbf{B}_\varepsilon(x) \cap A_n \neq \emptyset\}$.

(ii) $\liminf_{n\to\infty} A_n = \{x \in X \mid \forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : \mathbf{B}_\varepsilon(x) \cap A_n \neq \emptyset, \forall n \geq N(\varepsilon)\}$

Proof. (i) Suppose $\overline{x} \in \limsup_{n\to\infty} A_n$. Assume that

$$\exists \varepsilon > 0, \exists N \in \mathbb{N} : \mathbf{B}_\varepsilon(\overline{x}) \cap A_n = \emptyset, \forall n \geq N.$$

$\Rightarrow$

$$dist(\overline{x}, A_n) \geq \varepsilon, \forall n \geq N \Rightarrow \liminf_{n\to\infty} dist(\overline{x}, A_n) = \sup_n \inf_{k \geq n} dist(\overline{x}, A_k) \neq 0.$$

$\Rightarrow \overline{x} \notin \limsup_{n\to\infty} A_n$. But this is a contradiction. Consequently,

$$\overline{x} \in \{x \in X \mid \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N : \mathbf{B}_\varepsilon(x) \cap A_n \neq \emptyset\}$$

That is

$$\limsup_{n\to\infty} A_n \subset \{x \in X \mid \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N : \mathbf{B}_\varepsilon(x) \cap A_n \neq \emptyset\}$$

To show the reverse inclusion. By assumption, we have

$$\forall k \in \mathbb{N}, \exists n_k \geq k : B_{\frac{1}{k}}(\overline{x}) \cap A_{n_k} \neq \emptyset.$$

$\Rightarrow \exists x_{n_k} \in A_{n_k} : \rho(\overline{x}, x_{n_k}) \leq \frac{1}{k}$. Consequently, the sequence $\{x_{n_k}\}_{k \in \mathbb{N}}$, with $x_{n_k} \in A_{n_k}$ converges to $\overline{x}$. Hence, by Prop. 4.2.4,

$$\overline{x} \in \limsup_{n\to\infty} A_n$$

Hence, $\limsup_{n\to\infty} A_n = \{x \in X \mid \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N : \mathbf{B}_\varepsilon(x) \cap A_n \neq \emptyset\}$.

(ii) Use a similar argument as in (i). (Exercise!)

□

Remark 4.2.6. *The statements in Prop. 4.2.4 and Prop. 4.2.5 can be used as alternative definitions of inferior and superior limits of a sequence of sets as defined above. In particular, from Prop. 4.2.5, it follows that*

$$(i) \qquad \limsup_{n\to\infty} A_n = \bigcap_{\varepsilon>0}\bigcap_{N\geq 1}\bigcup_{n\geq N}\mathcal{U}_\varepsilon(A_n) \tag{4.4}$$

$$(ii) \qquad \bigcap_{n\geq 1} cl\left(\bigcup_{n\geq m} A_m\right) \subset \limsup_{n\to\infty} A_m \tag{4.5}$$

$$(iii) \qquad \liminf_{n\to\infty} A_n = \bigcap_{\varepsilon>0}\bigcup_{N\geq 1}\bigcap_{n\geq N}\mathcal{U}_\varepsilon(A_n). \tag{4.6}$$

Proposition 4.2.7. *If $\{A_n\}_{n\in N}$ is a sequence such that $A_n \supset A_{n+1}, n \in N$ (a* ***decreasing sequence****), then $\lim_{n\to\infty} A_n$ exists and*

$$\lim_{n\to\infty} A_n = \bigcap_{n\in N} cl\ A_n$$

Proof. Since $dist(\overline{x}, clA_n) = dist(\overline{x}, A_n)$, it follows that,

$$\bigcap_{n\in N} cl\ A_n \subset \liminf_{n\to\infty} A_n. \tag{4.7}$$

If we now show that $\limsup_{n\to\infty} A_n \subset \bigcap_{n\in\mathbb{N}} cl\ A_n$, then we are done!

- Let $\overline{x} \in \limsup_{n\to\infty} A_n$. Hence,

$$\sup_n \inf_{k\geq n} dist(\overline{x}, A_k) = 0.$$

  Since, for $k \geq n$, $A_n \supset A_k$, we have $dist(\overline{x}, A_n) \leq dist(\overline{x}, A_k)$. This implies that

$$\sup_n \inf_{k\geq n} dist(\overline{x}, A_k) = \sup_n dist(\overline{x}, A_n) = 0 \Rightarrow dist(\overline{x}, A_n) = 0, \forall n.$$

  $\Rightarrow$ (by Prop. 4.1.2(iii)) $\overline{x} \in clA_n, \forall n \in \mathbb{N}$. Hence

$$\limsup_{n\to\infty} A_n \subset \bigcap_{n\in N} cl\ A_n.$$

- Moreover, by Prop. 4.2.3, we have $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$. Hence

$$\liminf_{n\to\infty} A_n \subset \bigcap_{n\in N} cl\ A_n.$$

Finally, using (4.7), $\lim_{n\to\infty} A_n$ exists and

$$\lim_{n\to\infty} A_n = \bigcap_{n\in N} cl\ A_n.$$

□

4.2.1 Calculus of Limits of Sequences of Sets

Theorem 4.2.8. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be two sequences of sets, $K \subset X$ be a compact set. If*

for every neighborhood U of $K, \exists N \in \mathbb{N} : A_n \subset U, \forall n \geq N,$

then

for every neighborhood V of $K \cap \left(\limsup_{n\to\infty} B_n\right), \exists N \in \mathbb{N} : A_n \cap B_n \subset V, \forall n \geq N.$

Proof. Let V be any neighborhood of $K \cap (\limsup_{n\to\infty} B_n)$; i.e. $K \cap (\limsup_{n\to\infty} B_n) \subset V$.

(i) If $K \subset V$, then taking $U = V$, we have, by assumption, that

$$\exists N : A_n \subset V, \forall n \geq N \Rightarrow A_n \cap B_n \subset A_n \subset V, \forall n \geq N.$$

(ii) If K is not a subset of V, then $M := K \setminus V \neq \emptyset$, M is compact and $M \cap (\limsup_{n\to\infty} B_n) = \emptyset$. Hence,

$$z \in M \Rightarrow z \notin \limsup_{n\to\infty} B_n \overset{\text{Prop. 4.2.5}}{\Rightarrow} \exists \varepsilon(z) > 0, \exists N(z) \in \mathbb{N} : \mathbf{B}_{\varepsilon(z)}(z) \cap B_n = \emptyset, \forall n \geq N.$$

Hence,

$$M \subset \bigcup_{z\in M} \mathbf{B}_{\varepsilon(z)}(z).$$

Since M is compact, there are $z_k \in M, k = 1, \ldots, m$, such that

$$M \subset \bigcup_{k=1}^{m} \mathbf{B}_{\varepsilon(z_k)}(z_k) =: W.$$

Now define, $N_0 := \max\{N(z_k) \mid k = 1, \ldots, m\}$. It follows that

$$W \cap B_n = \emptyset, \forall n \geq N_0.$$

Moreover,

$$K \setminus V = M \subset W \Rightarrow K \subset V \cup W.$$

Thus $K \subset U := V \cup W$. By assumption,

$$\exists N_1 : A_n \subset U, \forall n \geq N_1 \Rightarrow A_n \cap B_n \subset A_n \subset U = V \cap W, \forall n \geq N_1.$$

However, $B_n \cap W = \emptyset, \forall n \geq N_0$. Thus, if we set $N := \max\{N_0, N_1\}$, then we obtain

$$A_n \cap B_n \subset V, \forall n \geq N.$$

□

Theorem 4.2.9. *(Thm. 5.2.4, p 221, Aubin[2], Thm. 1.1.4, p. 21, Aubin Frankowska[4],) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space X and $K \subset X$. If, for every neighborhood U of K*

$$\exists N \in \mathbb{N} : A_n \subset U, \forall n \geq N,$$

then

$$\limsup_{n\to\infty} A_n \subset cl(K).$$

Conversely, if X is a compact metric space, then, for every neighborhood U of $\limsup_{n\to\infty} A_n$,

$$\exists N \in \mathbb{N} : A_n \subset U, \forall n \geq N.$$

Proposition 4.2.10. *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be two sequences of subsets of a metric space X. Then the following hold true*

$$\begin{array}{ll}
(i) & \limsup_{n\to\infty}(A_n \cap B_n) \subset \limsup_{n\to\infty} A_n \cap \limsup_{n\to\infty} B_n \\
(ii) & \liminf_{n\to\infty}(A_n \cap B_n) \subset \liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n \\
(iii) & \limsup_{n\to\infty}(A_n \cup B_n) = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n \\
(iv) & \liminf_{n\to\infty}(A_n \cup B_n) \supset \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n \\
(v) & \limsup_{n\to\infty}(A_n \times B_n) \subset \limsup_{n\to\infty} A_n \times \limsup_{n\to\infty} B_n \\
(vi) & \liminf_{n\to\infty}(A_n \times B_n) = \liminf_{n\to\infty} A_n \times \liminf_{n\to\infty} B_n.
\end{array}$$

Proposition 4.2.11. *Let X and Y be metric spaces, $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are sequence in X and Y, respectively. If $f : X \to Y$ is a continuous function, then the following hold true*

$$(i) \qquad f\left(\limsup_{n\to\infty} A_n\right) \subset \limsup_{n\to\infty} f(A_n); \tag{4.8}$$

$$(ii) \qquad f\left(\liminf_{n\to\infty}\right) A_n \subset \liminf_{n\to\infty} f(A_n); \tag{4.9}$$

$$(iii) \qquad \limsup_{n\to\infty} f^{-1}(B_n) \subset f^{-1}\left(\limsup_{n\to\infty} B_n\right); \tag{4.10}$$

$$(iv) \qquad \liminf_{n\to\infty} f^{-1}(B_n) \subset f^{-1}\left(\liminf_{n\to\infty} B_n\right). \tag{4.11}$$

Proof. We prove (i) and the rest is transparent. We mainly use here Prop. 4.2.4. Let $y \in f(\limsup_{n\to\infty} A_n)$. Then $y = f(x)$ for some $x \in \limsup_{n\to\infty} A_n$. This implies,

$$\exists\{x_{n_k}\}, x_{n_k} \in A_{n_k} : x_{n_k} \to x.$$

Since, f is a continuous function, we have

$$f(x_{n_k}) \to f(x) = y.$$

Moreover, $f(x_{n_k}) \in f(A_{n_k})$. Hence,

$$y = f(x) \in \limsup_{n\to\infty} f(A_{n_k})$$

Hence,

$$f\left(\limsup_{n\to\infty} A_{n_k}\right) \subset \limsup_{n\to\infty} f(A_{n_k}).$$

□

Excercises 4.2.12. *Let $\{A_n\}$ be a sequence of sets in a metric space.*

1. *Show that the set $\liminf_{n\to\infty} A_n$ is a closed set.*

2. *Verify that the limit superior can be also written as*

$$\limsup_{n\to\infty} A_n = \bigcap_{N\geq 1} cl\left(\bigcup_{n\geq N} A_n\right).$$

3. *Verify the statements of Rem. 4.2.6.*

4. *Prove part (ii) of Prop. 4.2.4; i.e.* $\liminf_{n\to\infty} A_n = \{x \in X \mid x_n \in A_n : x_n \to x\}$.

5. *Prove part (ii) of Prop. 4.2.5; i.e.* $\liminf_{n\to\infty} A_n = \{x \in X \mid \forall\varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : \mathbf{B}_\varepsilon(x) \cap A_n \neq \emptyset, \forall n \geq N(\varepsilon)\}$.

6. *Prove statements (ii)-(iv) of Prop. 4.2.11.*

7. *Let X be a metric space and $\{A_n\} \subset \mathcal{P}(X)$ and $A \in \mathcal{P}(X)$. Then prove that*

$$\lim_{n\to\infty} A_n = A \quad \Leftrightarrow \quad \lim_{n\to\infty} dist(x, A_n) = 0, \forall x \in A \text{ and } \liminf_{n\to\infty} dist(x, A_n) > 0, \forall x \notin A.$$

4.2.2 Convergence w.r.t. the Hausdorff Metric

Definition 4.2.13 (Convergence in Hausdorff metric)**.** *Let $\{A_n\}$ be a sequence of closed subsets of X and $A \subset X$ be also closed. We say that A_n **converges to A w.r.t. the Hausdorff metric** iff*

$$\lim_{n\to\infty} h(A_n, A) = 0.$$

We denote this by

$$A_n \xrightarrow{h} A.$$

Remark 4.2.14. *It is easy to verify that: if* $\lim_{n\to\infty} h(A_n, A) = 0$*, then*

$$\forall\varepsilon > 0 : \exists N \in \mathbb{N} : A_n \subset \mathcal{U}_\varepsilon(A) = \{x \in X \mid dist(x, A) < \varepsilon\}, \forall n \geq N. \tag{4.12}$$

Note that, $A_n \subset \mathcal{U}_\varepsilon(A)$ implies that $dist(A_n, A) < \varepsilon$. This in turn implies that $A \subset \mathcal{U}_\varepsilon(A_n)$. We can now write (4.12) equivalently as

$$A \subset \bigcap_{\varepsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} \mathcal{U}_\varepsilon(A_m).$$

Theorem 4.2.15. *Let X be a metric space, $\{A_n\} \in \mathcal{P}_{cl}(X)$ and $A \in \mathcal{P}_{cl}(X)$. Then*

$$A_n \xrightarrow{h} A \quad \Rightarrow \quad A_n \to A$$

That is, Hausdorff-convergence implies Kuratowski-convergence.

Proof. We show that $A \subset \liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n \subset A$.

(i) By assumption

$$\lim_{n\to\infty} h(A_n, A) = 0 \Rightarrow \lim_{n\to\infty} h^*(A, A_n) = 0 \Rightarrow \lim_{n\to\infty} \sup_{x\in A} dist(x, A_n) = 0.$$

$\Rightarrow$

$$\forall x \in A : \lim_{n\to\infty} dist(x, A_n) = 0.$$

From this it follows that

$$A \subset \liminf_{n\to\infty} A_n.$$

(ii) Hence, according to Prop. 4.2.3, if we show that $\limsup_{n\to\infty} A_n \subset A$, then we are done. Assume that there is $\overline{x} \in \limsup_{n\to\infty} A_n$, but $\overline{x} \notin A$. Hence, $dist(\overline{x}, A) = \gamma > 0$, for some $\gamma \in \mathbb{R}$.

Now let

$$B := \{x \in X \mid dist(x, A) \leq \frac{\gamma}{2}\} = clU_{\frac{\gamma}{2}}(A)$$

Thus, $x \notin B$ and Rem. 4.2.14 implies that

$$\exists N \in \mathbb{N} : A_n \subset B, \forall n \geq N.$$

Consequently, we have

$$dist(x, A_n) \geq dist(x, B) \geq \frac{\gamma}{2} > 0, \forall n \geq N.$$

$\Rightarrow$

$$\liminf_{n\to\infty} dist(x, A_n) \geq \liminf_{n\to\infty} dist(x, B) \geq \frac{\gamma}{2} > 0$$

$\Rightarrow x \notin \limsup_{n\to\infty} A_n$, which is a contradiction.

Therefore, we have

$$\limsup_{n\to\infty} distA_n \subset A.$$

□

Remark 4.2.16. *However, the converse of Thm. 4.2.15 does not hold always true. To see this consider the sequence*

$$A_n := \begin{cases} \{0, \frac{1}{n}\}, & \textit{if } n \textit{ is even;} \\ \{0, n\}, & \textit{if } n \textit{ is odd;} \end{cases}$$

Note that:

- $x_n = \frac{1}{n} \in A_{2n}$ *and* $x_n \to 0 \Rightarrow 0 \in \limsup_{n\to\infty} A_n$. *Moreover, for* $x \in \mathbb{R}$ *arbitrary, we have*

$$dist(x, A_n) = \inf_{z\in A_n} \rho(x, z) = \inf_{z\in A_n} |x - z| = \begin{cases} \inf_{z\in\{0,\frac{1}{n}\}} |x - z|, & \textit{if } n \textit{ is even;} \\ \inf_{z\in\{0,n\}} |x - z|, & \textit{if } n \textit{ is odd;} \end{cases}$$

$\Rightarrow$

$$dist(x, A_n) = \begin{cases} \min\{|x|, |x - \frac{1}{n}|, & \textit{if } n \textit{ is even;} \\ \min\{|x|, |x - n|\}, & \textit{if } n \textit{ is odd;} \end{cases}$$

Hence,

$$\liminf_{n\to\infty} dist(x, A_n) = 0 \Leftrightarrow \left\{ \begin{array}{ll} \liminf_{n\to\infty} \min\{|x|, |x - \frac{1}{n}|, & \textit{if } n \textit{ is even;} \\ \liminf_{n\to\infty} \min\{|x|, |x - n|\}, & \textit{if } n \textit{ is odd;} \end{array} \right\} = 0.$$

$\Leftrightarrow$

$$\liminf_{n\to\infty} \min\{|x|, |x - \frac{1}{n}| = 0, \textit{ if } n \textit{ is even; and}$$
$$\liminf_{n\to\infty} \min\{|x|, |x - n|\} = 0, \textit{ if } n \textit{ is odd}$$

But these two limits are zero if and only if $x = 0$*; i.e.* $\limsup_{n\to\infty} A_n = \{0\}$. *Similarly,* $\liminf_{n\to\infty} A_n = \{0\}$. *Consequently,*

$$\lim_{n\to\infty} A_n = \{0\}.$$

• *Moreover,*

$$h(\{0\}, A_n) = \max\{h^*(\{0\}, A_n), h^*(A_n, \{0\})\}$$

Thus,

$$h^*(\{0\}, A_n) = \sup_{x\in\{0\}} dist(x, A_n) = dist(0, A_n) = 0.$$

But

$$h^*(A_n, \{0\}) = \sup_{x\in A_n} dist(x, \{0\}) = \sup_{x\in A_n} |x - 0| = \sup_{x\in A_n} |x| = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd;} \end{cases}$$

$\Rightarrow$

$$\lim_{n\to\infty} h^*(\{0\}, A_n) \neq 0.$$

Consequently,

$$\lim_{n\to\infty} h(A_n, \{0\}) \neq 0. \text{ That is, } A_n \nrightarrow \{0\}.$$

When do we have equality between Kuratowski and Hausdorff convergence? The answer is given in the following statement.

Proposition 4.2.17. *Let X be a metric space, $\{A_n\}$ be a sequence of compact subsets of X and $A \subset X$ compact. Then, if there is a compact subset K of X such that $A_n \subset K, \forall n$; and $\lim_{n\to\infty} A_n = A$, then $\lim_{n\to\infty} h(A_n, A) = 0$.*

Proof. By assumption and Prop. 4.2.3, we see that A is a compact set and, by Thm. 4.2.9, $A \subset K$. Assume that $h(A_n, A) \nrightarrow 0$; i.e. $A_n \overset{h}{\nrightarrow} A$. This implies

$$\lim_{n\to\infty} \max\{h^*(A_n, A), h^*(A, A_n)\} \neq 0 \Rightarrow \lim_{n\to\infty} \max\{h^*(A_n, A), h^*(A, A_n)\} > \gamma, \text{ for some } \gamma > 0.$$

$\Rightarrow$

$$\lim_{n\to\infty} h^*(A, A_n) \neq 0 \quad \text{or} \quad \lim_{n\to\infty} h^*(A_n, A) \neq 0.$$

(i) If $\lim_{n\to\infty} h^*(A, A_n) \neq 0$, then

$$\lim_{n\to\infty} \sup_{x\in A} dist(x, A_n) > 0$$

$\Rightarrow$ (by the compactness of A)

$$\forall n, \exists x_n \in A : dist(x_n, A_n) > 0.$$

But then $\{x_n\} \subset K$ and K is compact implying that there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \overline{x}$, where $\overline{x} \in A$. But

$$\forall n_k : \quad dist(x_{n_k}, A_{n_k}) > 0. \tag{4.13}$$

Moreover, since $A = \liminf_{n\to\infty} A_n$, by Prop. 4.2.4, there is a sequence $\{z_n\}$ with $z_n \in A_n$ such that $z_n \to \overline{x}$. Consequently,

$$dist(x_{n_k}, A_{n_k}) \leq \rho(x_{n_k}, z_{n_k}) \leq \rho(x_{n_k}, \overline{x}) + \rho(\overline{x}, z_{n_k}), \forall n \in \mathbb{N}.$$

However, from this follows that

$$\lim_{k\to\infty} dist(x_{n_k}, A_{n_k}) = 0$$

which is a contradiction to (4.13). Consequently, it must be that $\lim_{n\to\infty} h^*(A, A_n) = 0$.

(ii) If $\lim_{n\to\infty} h^*(A_n, A) \neq 0$, then there is a subsequence $\{A_l\}_{l\in L}$ of $\{A_n\}_{n\in\mathbb{N}}$, where $L \subset \mathbb{N}$ such that

$$\lim_{l\to\infty} h^*(A_l, A) = \lim_{l\to\infty} \sup_{y\in A_l} dist(y, A) > 0.$$

$\Rightarrow$

$$\forall l \in L, \exists x_l \in A_l : dist(x_l, A) > 0.$$

But, again $\{x_l\} \subset K$ and K is compact implies there is a convergent subsequence $\{x_{l_k}\}$ such that $x_{l_k} \to \overline{x}$, for some $\overline{x} \in X$. From this follows that $\overline{x} \in \limsup_{n\to\infty} A_n = A$ (see Prop. 4.2.4). However,

$$0 < dist(x_{l_k}, A) \leq \rho(x_{l_k}, \overline{x}), \forall l_k \Rightarrow 0 < dist(\overline{x}, A) = 0.$$

Which is a contradiction.

Therefore, from (i) and (ii), we conclude that $\lim_{n\to\infty} h(A_n, A) = 0$.

□

Proposition 4.2.18. *If $\{A_n\} \subset \mathcal{P}_{cl}(X)$, $A \in \mathcal{P}_{cl}(X)$ and $A_n \xrightarrow{h} A$, then*

$$A = \bigcup_{n\geq 1} cl\left(\bigcup_{m\geq n} A_m\right).$$

Proof. Since $A_n \xrightarrow{h} A$ implies $A_n \to A$, by Rem 4.2.6, we have that

$$\bigcap_{n\geq 1} cl\left(\bigcup_{m\geq n} A_m\right) \subset A.$$

Moreover, using Rem. 4.2.14, given any $n \geq 1$, for an arbitrary $\varepsilon > 0$,

$$\exists m \geq n : A \subset \mathcal{U}_\varepsilon(A_m) \Rightarrow A \subset \mathcal{U}_\varepsilon\left(\bigcup_{m\geq n} A_m\right).$$

Since, $\varepsilon > 0$ is arbitrary, we have

$$A \subset cl\left(\bigcup_{m\geq n} A_m\right).$$ This holds true for any $n \geq 1$. Consequently, $$A \subset \bigcap_{n\geq 1} cl\left(\bigcup_{m\geq n} A_m\right).$$

Therefore, the claim of the proposition follows.

□

A sequence of non-empty closed sets $\{A_n\}$ is said to be a **Cauchy sequence** in $< \mathcal{P}_{cl}(X), h >$ if

$$\forall \varepsilon > 0, \exists N : h(A_n, A_m) < \varepsilon, \forall n, m \geq N.$$

Theorem 4.2.19. *If $< X, \rho >$ is a complete metric space, then $< \mathcal{P}_{cl}, h >$ is also a compete metric space.*

Proof. Let $\{A_n\}$ be any Cauchy sequence of non-empty closed sets in $< \mathcal{P}_{cl}, h >$. Then we show that $\{A_n\}$ is convergent. According to Prop. 4.2.18, we need only to verify that $\{A_n\}$ converges to the set

$$A = \bigcap_{n\geq 1} cl\left(\bigcup_{m\geq n} A_m\right).$$

Thus, we have to show that: (i) A is closed. (Obvious!) (ii) A is non-empty. (iii) $A_n \xrightarrow{h} A$.

(ii) Since $\{A_n\}$ a Cauchy sequence, given $\delta > 0$ (say $\delta = \frac{\varepsilon}{3}$, where $\varepsilon > 0$), for each $k \geq 0$, there is N_k such that

$$h(A_n, A_m) < \frac{\delta}{2^{k+1}}, \forall n, m \geq N_k.$$

- Now, for k=0,

$$\exists N_0 : h(A_n, A_m) < \frac{\delta}{2}, \forall n, m \geq N_0.$$

Then for any $n_0 \geq N_0$

$$h(A_n, A_{n_0}) < \frac{\delta}{2}, \forall n \geq N_0 \Rightarrow \sup_{x \in A_{n_0}} dist(x, A_n) < \frac{\delta}{2}, \forall n \geq N_0.$$

Thus, for any fixed $x_{n_0} \in A_{n_0}$

$$dist(x_{n_0}, A_n) < \frac{\delta}{2}, \forall n \geq N_0.$$

- For $k = 1$, there exists N_1 such that

$$h(A_n, A_m) < \frac{\delta}{2^2}, \forall n, m \geq N_1.$$

Then, for any $n_1 \geq \max\{N_0, N_1\}$ we have

$$dist(x, A_n) < \frac{\delta}{2^2}, \forall x \in A_{n_1}, \forall n \geq N_1.$$

Choose $x_{n_1} \in A_{n_1}$, then

$$\rho(x_{n_1}, x_{n_0}) \leq \rho(x_{n_1}, x) + \rho(x, x_{n_0}), \quad \text{for any } x \in A_n \text{ and } n \geq N_1.$$

$\Rightarrow$

$$\rho(x_{n_1}, x_{n_0}) < \frac{\delta}{2^2} + \frac{\delta}{2} = 3\frac{\delta}{2^2} = \frac{\varepsilon}{2^2}.$$

Proceeding in this way, for $n_k \geq \max\{N_0, N_1, \ldots, N_k\}$, we can choose $x_{n_{k+1}} \in A_{n_{k+1}}$ such that

$$\rho(x_{n_{k+1}}, x_{n_k}) < \frac{\varepsilon}{2^{k+1}}.$$

Thus the sequence $\{x_{n_k}\}$ is a Cauchy sequence. Since X is a complete metric space, there is $\overline{x} \in X$ such that $x_{n_k} \to \overline{x}$.

- Furthermore, for each $n \geq 1$, there is $n_{k_0} \geq n$ such that

$$x_{n_k} \in \bigcup_{m \geq n} A_m, \forall n_k \geq n_{k_0} \Rightarrow \overline{x} \in cl\left(\bigcup_{m \geq n} A_m\right).$$

This is true for all $n \geq 1$.

$$\overline{x} \in cl\left(\bigcup_{m \geq n} A_m\right), \forall n \geq 1 \Rightarrow \overline{x} \in \bigcap_{n \geq 1} cl\left(\bigcup_{m \geq n} A_m\right) = A.$$

Consequently, $A \neq \emptyset$.

(iii) Next, we show that $A_n \xrightarrow{h} A$.

- From (i), for each $n_0 \geq N_0$ and any $x_{n_0} \in A_{n_0}$ we obtain, by the continuity of $\rho(\cdot, x_0)$, that

$$\rho(\overline{x}, x_{n_0}) = \lim_{n_k \to \infty} \rho(x_{n_k}, x_{n_0}) \leq \lim_{n_k \to \infty} \sum_{i=1}^{n_k} \rho(x_{n_i}, x_{n_{i-1}}) < \lim_{n_k \to \infty} \sum_{i=1}^{n_k} \frac{\varepsilon}{2^i} = \varepsilon.$$

$\Rightarrow$

$$\forall n_0 \geq N_0, \forall x_{n_0} \in A_{n_0} : \rho(\overline{x}, x_{n_0}) < \varepsilon \Rightarrow A_{n_0} \subset B_\varepsilon(\overline{x}) \subset \mathcal{U}_\varepsilon(A), \forall n_0 \neq N_0.$$

$\Rightarrow$

$$\exists N_0 : h^*(A_n, A) < \varepsilon, \forall n \geq N_0. \tag{4.14}$$

- Conversely, let $x \in A$ be arbitrary, then

$$x \in cl\left(\bigcup_{m \geq N_0} A_m\right) \Rightarrow \exists m \geq N_0, \exists z \in A_m : \rho(x, z) < \frac{\varepsilon}{2}. \tag{4.15}$$

Moreover, the following holds true (see the proof of Lem. 4.1.7)

$$dist(x, A_n) \leq dist(x, A_m) + h(A_m, A_n).$$

Since $\{A_n\}$ is a Cauchy sequence, there is N_0' such that $h(A_n, A_m) < \frac{\varepsilon}{2}, \forall n \geq N_0'$. This, along with (4.15), yields that

$$dist(x, A_n) \leq dist(x, A_m) + h(A_n, A_m) \leq \rho(x, z) + h(A_n, A_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \forall n, m \geq max\{N_0, N_0'\}.$$

$\Rightarrow$

$$dist(x, A_n) < \varepsilon, \forall n \geq N_1' := max\{N_0, N_0'\} \Rightarrow x \in \mathcal{U}_\varepsilon(A_n), \forall n \geq N_1'.$$

Since, $x \in A$ is arbitrary, it follows that

$$A \subset \mathcal{U}_\varepsilon(A_n), \forall n \geq N_1' \Rightarrow h^*(A, A_n) < \varepsilon, \forall n \geq N_1'. \tag{4.16}$$

Finally, from (4.14) and (4.16) we conclude that

$$h(A, A_n) < \varepsilon, \forall n \geq N_1'.$$

Therefore, $A_n \xrightarrow{h} A$.

□

Excercises 4.2.20. *Verify the following statements.*

1. *if* $\lim_{n \to \infty} A_n = A$, *then*
$$\lim_{n \to \infty} h^*(A, A_n) = 0.$$

2. *Let* $\{A_n\}$ *be a sequence of closed sets and* $A \in \mathcal{P}_{cl}(X)$. *Then*

(i) if $\lim_{n\to} h^*(A_n, A) = 0$, *then* $\limsup_n A_n \subset A$*;*

(ii) if $\lim_{n\to} h^*(A, A_n) = 0$*, then* $\liminf_n A_n \supset A$*.*

3. *Let* X *be a metric space,* $\{A_n\} \subset \mathcal{P}_{cl}(X)$ *and* $A \in \mathcal{P}_{cl}(X)$ *such that*

$$\lim_{n\to\infty} h(A_n, A) = 0.$$

If $x_n \in A_n$ *and* $x_n \to \overline{x}$*, then* $\overline{x} \in A$*.*

4. *Let* X *be a metric space,* $\{A_n\} \subset \mathcal{P}_{cl}(X)$*,* $A \in \mathcal{P}_{cl}(X)$*. If* $A_n \xrightarrow{h} A$*, then* $\exists\{\varepsilon_k\}, \varepsilon_k \searrow 0$*, such that, for each* $k \in \mathbb{N} : dist(y, A) < \varepsilon_k, \forall y \in A_k$*.*

5. *If* $A_n \xrightarrow{h} A$*, then*

$$A = \bigcup_{n\geq 1} cl\left(\bigcup_{m\geq n} A_m\right) = \bigcap_{\varepsilon>0}\bigcup_{n\geq 1}\bigcap_{m\geq n} \mathcal{U}_\varepsilon(A_m).$$

6. *(see pp. 108-109 Aliprantis & Border [1]) Define* $\mathcal{U}_\varepsilon(F) = \{A \in \mathcal{P}_{cl}(X) \mid h(A, F) < \varepsilon\}$*.Then*

   (a) the collection $\{\mathcal{U}_\varepsilon(F) \mid F \in \mathcal{P}_{cl}(X), \varepsilon \in (0, \infty)\ \}$ *forms a base for a topology* τ_h *on* $\mathcal{P}_{cl}(X)$ *and this topology is first countable;*

   (b) if $< X, \rho >$ *is a compact metric space, then* $< \mathcal{P}_{cl}(X), \tau_h >$ *is a compact topological space;*

   (c) if $< X, \rho >$ *is a separable metric space, then* $< \mathcal{P}_{cl}(X), \tau_h >$ *is also a separable topological space.*

5 Set-Valued Maps

5.1 Introduction

In this chapter we, generally, assume X and Y to be at least Hausdorff topological spaces. But practically set-valued maps reveal interesting properties when X and Y are taken to be normed linear spaces.

Definition 5.1.1 (set-valued map)**.** *Let X and Y be topological spaces. If for each $x \in X$ there is a corresponding set $F(x) \subset Y$, then $F(\cdot)$ is called a **set-valued map** from X to Y. We denote this by*

$$F : X \rightrightarrows Y.$$

A function $f : X \to Y$ can be treated as a special set-valued map if we define $F(x) := \{f(x)\}$. For the sake of brevity we write 'SV-map' for 'set-valued map'*.

In some books a set-valued map from X to Y is denoted by $F : X \to 2^Y$, or $F : X \rightsquigarrow Y$, etc. But we exclusively use here the notation $F : X \rightrightarrows Y$. Furthermore, the terms 'set-valued map' (Aubin & Frankowska [4]),'point-to-set map' (Hogan [13]),'correspondences' (Aliprantis & Boder [1]), 'multivalued maps' (Robinson [19, 20]) and (Hu & Papagorgious [14]), 'multifunctions' (Castaing & Valadier [8]), are usually used interchangeably; while the first being frequently used in current literature.

In many cases we would like to know how a slight change in a parameter(s) of a given mathematical problem could affect the solution or solution set (or even the structure) of the problem. Currently, such a study is, in fact, very important as many useful mathematical problems are usually *approximately* solved on the computer. Thus sensitivity analysis could guarantee the acceptability of the obtained approximate solution(s), based on certain allowed error on the parameters of the problem. For instance, the characterization of the variation (due to, say, data perturbation) of solution sets of optimization problems, partial differential equations, etc., is done through set-valued maps. Consequently, Set-valued maps are indispensable tools in stability and sensitivity analysis of mathematical problems. Beside these, there are several other applications for set-valued maps.

5.1.1 Some Examples of Set-Valued Maps

Set-valued maps arise under several instances.

1. The inverse image of a non-bijective function

   Let the function $f : \mathbb{R} \to \mathbb{R}_+$ be given by $f(x) = x^2$. For $y \in \mathbb{R}_+$, the inverse image of y under f is either $x = -\sqrt{y}$ or $x = +\sqrt{y}$. That is,

   $$f^{-1}(y) = \{-\sqrt{y}, \sqrt{y}\}.$$

   Hence, for each $y \in \mathbb{R}$, $f^{-1}(y)$ represents more than one value. That is, f^{-1} is not single valued instead it is a *multivalued*.

*Actually, the short form 'SVM' would have been quite practical, but it has been widely used for 'Support Vector Machines' .

2.The *Subdifferential* map of a convex Function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, for $x \in \mathbb{R}^n$, the map

$$\partial f(x) := \{s \in \mathbb{R}^n \mid s^\top(y - x) \le f(y) - f(x), \forall y \in dom(f)\}$$

is called the **subdifferential** map of f at x.

2. Solution Sets of Metric Projections

Let $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define the map

$$\Pi_S(x) = \{y \in S \mid \|x - y\| = dist(x, S)\}$$

known us the **metric projection** of X onto S. Note that, if S is a closed convex set, then $\Pi_S(x)$ contains only a single element; otherwise $\Pi_S(x)$ a multivalued map.

3. The normal and tangent cone maps of a convex set Let $C \subset \mathbb{R}^n$ be a convex set. Then for $x \in C$

- the **normal cone** map: $N(x) = \{y \in \mathbb{R}^n \mid y^\top(z - x) \le 0, \forall x \in C\}$;
- the **tangent cone** map: $T_C(x) = cl\{y \in \mathbb{R}^n \mid y = \alpha(z-x), x \in C, \alpha \ge 0\} = cl\left(\bigcup_{\gamma>0} \frac{1}{\gamma}(C - x)\right)$.

4. The feasible and solution sets of a Parametric Optimization Problems

Let $f, h_i, g_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be functions. Then given the parametric optimization problem

$$\begin{array}{ll}(P(t)) & f(x,t) \to \inf \\ & h_i(x,t) = 0, i = 1, \ldots, m; \\ & g_j(x,t) \le 0, j = 1, \ldots, p.\end{array}$$

we have

- the **feasible set map** of (P(t)): $M(t) := \{x \in \mathbb{R}^n \mid h_i(x,t) = 0, i = 1, \ldots, m; g_j(x,t) \le 0, j = 1, \ldots, p\}$;
- the **optimal solution map** of (P(t)): $S(t) := \{x \in \mathbb{R}^n \mid w(t) = f(x,t)\}$;

where

$$w(t) := \inf_{x \in M(t)} f(x,t)$$

is the the **(marginal) value function** of (P(t)). Here, both $M(\cdot)$ and $S(\cdot)$ are set valued maps from T to X.

5.2 Basic Definitions

Definition 5.2.1 (Domain, Range and Graph). *Let X and Y be metric spaces and $F : X \rightrightarrows Y$. The* domain *of $F(\cdot)$, denoted by $Dom(F)$, is defined as:*

$$Dom(F) := \{x \in X \mid F(x) \ne \emptyset\};$$

the range *of $F(\cdot)$ is defined as*

$$Rang(F) := \bigcup_{x \in Dom(F)} F(x);$$

and the graph *of $F(\cdot)$, denoted by $Graph(F)$, is defined as*

$$Graph(F) := \{(x,t) \mid t \in F(x), x \in Dom(F)\}.$$

Definition 5.2.2. *Let X and Y be topological spaces.*

(i) A set valued map $F : X \rightrightarrows Y$ is said to be closed valued, open valued or compact valued if, for each $x \in X$, $F(x)$ is a closed, open or compact set, respectively, in Y. Furthermore, if Y is a topological linear space and $F(x)$ is a convex set in Y for each $x \in X$, then $F(\cdot)$ is called convex valued.

(ii) A set valued map $F : X \rightrightarrows Y$ is said to be a closed, open or compact set-valued map $Graph(F)$ is a closed, open or compact set w.r.t. the product topology of $X \times Y$. Furthermore, if X and Y are topological linear spaces; then $F(\cdot)$ called a convex set-valued map if $Graph(F)$ is a convex set in w.r.t. $X \times Y$.

Remark 5.2.3. *In Def. 5.2.2 care must be taken not to confuse closed valued maps and closed maps. The former refers to values of the map; while the latter refers to the graph of the map.*

5.2.1 Elementary Mathematical Operations with Set-Valued Maps

Definition 5.2.4 (closure, interior, convex-hull sv-maps)**.** *Let X and Y be topological spaces and $F : X \rightrightarrows Y$. Then*

- *the* **closure sv-map** *associated with F is the map*

$$cl(F):\ X \rightrightarrows Y, \text{ where } cl(F)(x) = cl(F(x)), \text{ for each } x \in X.$$

- *the* **interior sv-map** *associated with F is the map*

$$int(F):\ X \rightrightarrows Y, \text{ where } int(F)(x) = int(F(x)), \text{ for each } x \in X.$$

Moreover, if Y is a topological linear space, then

- *the* **convex-hull sv-map** *associated with F is the map*

$$conv(F):\ X \rightrightarrows Y, \text{ where } conv(F)(x) = conv(F(x)), \text{ for each } x \in X.$$

Example 5.2.5. *For instance, let $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = x^2$ and $F(x) = f^{-1}(x)$. Then $F:\ \mathbb{R}_+ \rightrightarrows \mathbb{R}$ and $F(x) = \{-\sqrt{x}, \sqrt{x}\}$. It follows that $conv(F(x)) = [-\sqrt{x}, \sqrt{x}]$, for each $x \in \mathbb{R}_+$.*

If $F:\ X \rightrightarrows Y$ and $A \subset X$, then the ***image of the set*** A under F is given by

$$F(A) = \bigcup_{x \in A} F(x).$$

Proposition 5.2.6 (Aubin & Frankowska [4])**.** *Let X and Y be topological spaces and $F:\ X \rightrightarrows Y$, $A, B \subset X$. Then*

(i) $F(A \cup B) = F(A) \cup F(B)$;

(ii) $F(A \cap B) \subset F(A) \cap F(B)$;

(iii) $F(X \setminus A) \supset Range(F) \setminus F(A)$;

(iv) $F(X \setminus A) \supset Range(F) \setminus F(A)$;

(v) if $A \subset B$, then $F(A) \subset F(B)$.

Proof. Trivial! ☐

Definition 5.2.7 (combination and composition of sv-maps). *Let X, Y and Z be topological spaces, F_1 and F_2 are two sv-maps from X to Y. Then*

- *the* **union map** *of F_1 and F_2 is the map $F_1 \cup F_2 : X \rightrightarrows Y$ given by $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$, for each $x \in X$;*
- *the* **intersection map** *of F_1 and F_2 is the map $F_1 \cap F_2 : X \rightrightarrows Y$ is given by $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$, for each $x \in X$.*
- *If $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$, then the production map of F_1 and F_2 the map $F_1 \times F_2 : X \rightrightarrows Y \times Z$ is given by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$, for each $x \in X$.*
- *If Y is a linear space, the* ***sum and difference*** *can be defined likewise. Thus*

$$(F_1 \pm F_2)(x) = F_1(x) \pm F_2(x).$$

Furthermore, if $F_1 : X \rightrightarrows Y$ and $F_2 : Y \rightrightarrows Z$, then **composition map** *of F_1 and F_2 is the map*

$$F_2 \circ F_1 : X \rightrightarrows Z$$

such that

$$(F_2 \circ F_1)(x) = \bigcup_{y \in F_1(x)} F_2(y).$$

Definition 5.2.8 (lower inverse of a SV-map). *Let $F : X \rightrightarrows Y$. For any $V \subset Y$ the (lower) inverse image of V under $F(\cdot)$ is denoted by $F^-(V)$ and is defined as:*

$$F^-(V) := \{x \in X \mid F(x) \cap V \neq \emptyset\} = \bigcup_{y \in V} F^-(y).$$

Definition 5.2.9 (upper inverse of a SV-map). † *Let $F : X \rightrightarrows Y$. Then for any $V \subset Y$, the upper inverse of V under $F(\cdot)$, denoted by $F^{+1}(V)$, is defined as:*

$$F^+(V) := \{x \in X \mid F(x) \subset V\}.$$

In Defs. 5.2.8 and 5.2.9, if $V = \emptyset$, then we have

$$F^+(V) = F^-(V) = \emptyset.$$

Thus, in general, we have

Proposition 5.2.10. *If $F : X \rightrightarrows Y$ and $V \subset Y$ any, then*

$$F^+(V) \subset F^-(V).$$

The above two definitions of inverses of a SV-map lead into two types of continuities - upper and lower semi-continuity.

Remark 5.2.11. *For a function $f : X \to Y$ and $V \subset Y$, we have*

$$f^+(V) = f^-(V) = f^{-1}(V).$$

†The terminologies *lower*- and *upper*-inverse are from Berge [6]; while in the book of Aubin & Frankowska [4] the former is simply termed *inverse*, and $F^{+1}(V)$ is termed the *core* of the set V under $F(\cdot)$. However, the naming 'weak inverse image' and 'strong inverse image', from Hu & Papageorgiou [14], could be more appropriate instead of 'upper-' and 'lower inverse', respectively.

Excercises 5.2.12. *Let $F: X \rightrightarrows Y$ and $V, W \subset Y$. Then verify the validity of the following statements.*

1. *(i) $F^{-}(V \cup W) = F^{-}(V) \cup F^{-}(W)$;*
   (ii) $F^{-}(V \cap W) \supset F^{-}(V) \cap F^{-}(W)$;
   (iii) $F^{+}(V \cup W) \subset F^{+}(V) \cup F^{+}(W)$;
   (iv) $F^{+}(V \cap W) = F^{+}(V) \cap F^{+}(W)$.
2. *Let $x^0 \in X$ and $S \subset Y$. Then*

   (i) iff $x^1 \in F^{+}(F(x^0))$, then $F(x^1) \subset F(x^0)$;
   (ii) $F(F^{+}(S)) \subset S$;
3. *Let $F_1, F_2: X \rightrightarrows Y$ and $S \subset Y$. Then*

   (i) $(F_1 \cup F_2)^{-}(S) = F_1^{-}(S) \cup F_2^{-1}(S)$;
   (ii) $(F_1 \cap F_2)^{-}(S) \subset F_1^{-}(S) \cap F_2^{-}(S)$;
   (iii) $(F_1 \cup F_2)^{+}(S) \subset F_1^{+}(S) \cup F_2^{+}(S)$;
   (iv) $(F_1 \cap F_2)^{+}(S) \supset F_1^{+}(S) \cap F_2^{+}(S)$;

5.3 Semi-Continuity of Set-Valued Maps

Assume that X and Y are Hausdorff topological spaces.

Definition 5.3.1 (upper semi-continuous SV-map)**.** *Let $F: X \rightrightarrows Y$ and $Dom(F) \neq \emptyset$. Then $F(\cdot)$ is said to be upper semi-continuous (u.s.c) at $x^0 \in X$ iff for any open set $V \subset Y$, where $F(x^0) \subset V$, there exists a neighborhood $U \subset X$ of x^0 such that*

$$\forall x \in U: \; F(x) \subset V, \; i.e. \, U \subset F^{+}(V).$$

The map $F(\cdot)$ is said to be u.s.c. on X if it is u.s.c. at every $x \in X$.

Definition 5.3.2 (lower semi-continuous SV-map)**.** *Let $F: X \rightrightarrows Y$. Then $F(\cdot)$ is said to be lower semi-continuous (l.s.c.) at $x^0 \in X$ iff for any open set $V \subset Y$ such that $F(x^0) \cap V \neq \emptyset$, there exists a neighborhood $U \subset X$ of x^0 such that*

$$\forall x \in U: \; F(x) \cap V \neq \emptyset; \; i.e. \; U \subset F^{-}(V).$$

The map $F(\cdot)$ is said to be l.s.c. on X if $F(\cdot)$ is l.s.c. at every $x \in X$.

A set-valued map which is both lower and upper semi-continuous is called **continuous**.

Example 5.3.3. *An upper semi-continuous map need not be lower semi-continuous and vice versa.*

1. *The set valued map* $F : \mathbb{R} \rightrightarrows \mathbb{R}$ *given by*

$$F(x) = \begin{cases} [1,4], & \text{if } x = 0 \\ [2,3], & \text{if } x \neq 0. \end{cases}$$

   is upper semi-continuous at $x = 0$*, but not lower semi-continuous* $x = 0$*. To see the upper semi-continuity at* $x = 0$*, let* V *be such that*

$$F(0) \subset V \Rightarrow [1,4] \subset V.$$

   Take any neighborhood U *of* $x = 0$*, then we have either* $F(x) = [1,4]$ *or* $F(x) = [2,3]$ *for* $x \in U$*. This implies*

$$\forall x \in U : F(x) \subset [1,4] \subset U.$$

   Hence, $F(\cdot)$ *is u.s.c. at* $x = 0$*. However, if* $r \in F(0) = [1,4]$ *such that* $3 < r < 4$ *and* $V = (r-\varepsilon, r+\varepsilon) \subset (3,4)$*, for a sufficiently small* $\varepsilon > 0$*, then*

$$F(0) \cap V = \emptyset.$$

   That implies that $F(\cdot)$ *is not lower semi-continuous at* $x = 0$*.*

2. *The set-valued map*

$$F(x) = \begin{cases} [1,4], & \text{if } x < 0 \\ [2,3], & \text{if } x \geq 0. \end{cases}$$

   is lower semi-continuous but not upper semi-continuous. (Exercise!)

5.3.1 Properties of Semi-Continuous Set-Valued Maps

We will make use of the following lemma repeatedly.

Lemma 5.3.4. *Let* $F : X \rightrightarrows Y$ *and* $Dom(F) = X$*. For any subset* $W \subset Y$ *we have*

$$(i) \quad X \setminus F^{-}(W) = F^{+}(Y \setminus W).$$

and

$$(ii) \quad X \setminus F^{+}(W) = F^{-}(Y \setminus W).$$

A. Properties of Upper Semi-Continuous Set-Valued Maps

Proposition 5.3.5. *Let* $F : X \rightrightarrows Y$ *and* $Dom(F) = X$*. Then the following statements are equivalent*

(i) $F(\cdot)$ *is u.s.c.;*

(ii) for each open set $V \subset Y$*,* $F^{+}(V)$ *is an open set in* X*;*

(iii) for each closed set $W \subset Y$*,* $F^{-}(W)$ *is a closed set in* X*.*

Proof. From Def. 5.3.1, $(i) \Rightarrow (ii)$.

(ii) $\Rightarrow$ (iii): For a closed subset W of Y we have $Y \setminus W$ is open in Y. Thus by (ii), $F^{+}(Y \setminus W)$ is open in X. Thus, by Lem. 5.3.4, we have $X \setminus F^{-}(W)$ is an open set in X. Hence, $F^{-}(W)$ is a closed set in X.

(iii) $\Rightarrow$ (i): Let $x^0 \in X$ be such that $F(x^0) \subset V$ for some open set $V \subset Y$. Hence, $Y \setminus V$ is closed in Y and $F(x^0) \cap (Y \setminus V) = \emptyset$. Using (iii) and Lem. 5.3.4, we have $X \setminus F^+(V)$ is closed in X and $x^0 \notin (X \setminus F^+(V))$. Hence, $x^0 \in F^+(V)$ and $F^+(V)$ is an open set implies

$$\exists U(x^0) : U(x^0) \subset F^+(V); \text{i.e., } F(\cdot) \text{ is u.s.c. at } x^0.$$

□

Proposition 5.3.6. *Let X and Y be metric spaces and $F : X \rightrightarrows Y$ and $x_0 \in X$. Then the following are equivalent*

(i) $F(\cdot)$ is u.s.c. at x_0;

(i) if $x_0 \in X$ and $\{x_n\}$ is any sequence such that $x_n \to x_0$ and $V \subset Y$ an open subset such that $F(x_0) \subset V$, then

$$\exists N \geq 1 : F(x_n) \subset V, \forall n \geq N.$$

Proof. (i) $\Rightarrow$ (ii): Follows by definition of u.s.c. and properties of convergence of sequences.

(ii) $\Rightarrow$ (i): Assume that $F(\cdot)$ is not u.s.c. at x^0 and arrive at a contradiction.

□

Corollary 5.3.7. *Let X and Y be metric spaces and $F : X \rightrightarrows Y$ and $x_0 \in X$. If $F(\cdot)$ is u.s.c. and $\{x_n\}$ is a sequence such that $x_n \to x_0$, then*

$$\limsup_{n\to\infty} F(x_n) \subset clF(x_0).$$

Proof. Follows from Prop. 5.3.6 and Thm. 4.2.9. □

Next we find some basic results on upper semi-continuity properties combination and composition of sv-maps.

Proposition 5.3.8. *Let $F : X \rightrightarrows Y$. If $F(\cdot)$ is u.s.c. and Y is a normal topological space, then $clF(\cdot)$ is u.s.c.*

Proposition 5.3.9. *Let X, Y, Z be topological spaces, $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$. If F and G are upper semi-continuous, then $G \circ F$ is u.s.c.*

Proposition 5.3.10. *Let $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Y$. Then*

(i) if $F_1(\cdot)$ and $F_2(\cdot)$ are u.s.c., then $F_1 \cup F_2$ is u.s.c.;

(i) Y is a normal topological space and if $F_1(\cdot)$ and $F_2(\cdot)$ are u.s.c. such that

$$F_1(x) \cap F_2(x) \neq \emptyset, \forall x \in X,$$

then $F_1 \cap F_2$ is u.s.c.

Proposition 5.3.11. *Let $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ be compact valued and Y be a topological linear space. If F_1 and F_2 are u.s.c., then $F_1 + F_2$ is also u.s.c.*

Proposition 5.3.12. *Let $F : X \rightrightarrows Y$ and Y be a complete normed linear space. If $F(\cdot)$ is u.s.c. and compact valued, then $cl(coF(\cdot))$ is u.s.c.*

B. Properties of Lower Semi-Continuous of Set-Valued Maps

Proposition 5.3.13. *Let $F : X \rightrightarrows Y$. If $F(\cdot)$ is an open map, then $F(\cdot)$ is l.s.c.*

Proof. Exercise! □

Proposition 5.3.14. *Let $F : X \rightrightarrows Y$ and $Dom(F) = X$. Then the following statements are equivalent*

(i) $F(\cdot)$ is l.s.c.;

(ii) for each open set $V \subset Y$, $F^{-}(V)$ is an open set in X;

(iii) for each closed set $W \subset Y$, $F^{+}(W)$ is a closed set in X.

Proof. (i) ⇒ (ii): Let V be an open set in Y and $\overline{x} \in F^{-}(V) \Rightarrow \overline{x} \notin X \setminus F^{-}(V) = F^{+}(Y \setminus V)$ (Lem. 5.3.4(i)). Hence,

$$\overline{x} \notin F^{+}(Y \setminus V) \Rightarrow F(\overline{x}) \nsubseteq Y \setminus V \Rightarrow F(\overline{x}) \cap V \neq \emptyset.$$

Since, by (i), $F(\cdot)$ is l.s.c. at x^0,

$$\exists U(x^0) : F(x) \cap V \neq \emptyset, \forall x \in U(x^0) \Rightarrow U(x^0) \subset F^{-}(V).$$

Consequently, $F^{-}(V)$ is an open set.

(ii) ⇒ (iii): Follows from Lem. 5.3.4(ii).

(iii) ⇒ (i): Let $\overline{x} \in X$ with $F(\overline{x}) \cap V \neq \emptyset$ and V is an open set in Y.
$\Rightarrow Y \setminus V$ is closed in $Y \Rightarrow F^{+}(Y \setminus V)$ is closed in X and $F(\overline{x}) \cap V \neq \emptyset$.
$\Rightarrow \overline{x} \notin F^{+}(Y \setminus V) = X \setminus F^{-}(V)$ and $X \setminus F^{-}(V)$ is a closed set (Lem. 5.3.4(i)).
$\Rightarrow \overline{x} \in F^{-}(V)$ and $F^{-}(V)$ is an open set.
⇒

$$\exists U(\overline{x}) : U(\overline{x}) \subset F^{-}(V) \Rightarrow \forall x \in U(\overline{x}) : F(x) \cap V \neq \emptyset.$$

Consequently, $F(\cdot)$ is l.s.c at x^0.

□

Proposition 5.3.15. *Let X and Y be metric spaces and $F : X \rightrightarrows Y$ and $x_0 \in X$. Then the following are equivalent*

(i) if $\{x_n\}$ is any sequence such that $x_n \to x_0$ and $V \subset Y$ an open subset such that $F(x_0) \cap V \neq \emptyset$, then

$$\exists N \geq 1 : F(x_n) \cap V \neq \emptyset, \forall n \geq N;$$

(ii) if $\{x_n\}$ is a sequence such that $x_n \to x_0$ and $y_0 \in F(x_0)$ arbitrary, then there is a sequence $\{y_n\}$ with $y_n \in F(x_n)$ such that $y_n \to y_0$;

(ii) $F(\cdot)$ is l.s.c. at x_0.

Proof.

(i) ⇒ (ii): Let $x_n \to x_0$ and $y_0 \in F(x_0)$. Hence, given $\varepsilon > 0$, then $\mathbf{B}_{\varepsilon}(y_0) \cap F(x_0) \neq \emptyset$. Hence, by (i)

$$\exists N : F(x_n) \cap \mathbf{B}_{\varepsilon}(y_0) \neq \emptyset, \forall n \geq N \Rightarrow \forall n \geq N, \exists y_n \in F(x_n) \cap \mathbf{B}_{\varepsilon}(y_0).$$

Consequently, for $n \in \{1, \dots, N-1\}$, choosing $y_n \in F(x_n)$, we will have a sequence $\{y_n\}$ such that $y_n \in F(x_n)$ and $y_n \to y_0$.

(ii) $\Rightarrow$ (iii): Assume that there is $x_0 \in X$ such that $F(\cdot)$ is not l.s.c. at x_0. This implies, by definition,

$$\exists V \subset Y \text{ open } : F(x_0) \cap V \neq \emptyset, \text{ but for any neighborhood } U(x_0) \text{ of } x_0, \exists x \in U(x_0) : F(x) \cap V = \emptyset.$$

In particular,

$$\forall n \in \mathbb{N}, \exists x_n \in \mathbf{B}_{\frac{1}{n}}(x_0) : F(x_n) \cap V = \emptyset.$$

This implies, if $y_0 \in F(x_0) \cap V$, there is not sequence no sequence $\{y_n\}$ such that $y_n \in F(x_n)$ and $y_n \to y_0$. But this contradicts (ii). Consequently, $F(\cdot)$ should be l.s.c. at x_0.

□

Sometimes the statement of Prop. 5.3.15(ii) is given as a definition for a lower semi-continuous set-valued map on metric spaces. As such the terminology '**open set valued map**' refers to lower semi-continuous maps. (see for instance Shimizu et. al. [24]).

Corollary 5.3.16. *Let X and Y be metric spaces and $F : X \rightrightarrows Y$ and $x_0 \in X$. If $F(\cdot)$ is l.s.c. at x_0, then for every sequence $x_n \to x_0$ we have*

$$F(x_0) \subset \liminf_{n\to\infty} F(x_n).$$

Proof. Follows from Prop. 5.3.15. □

Similarly, we have lower semi-continuity properties for combination and composition of sv-maps.

Proposition 5.3.17. *Let $F : X \rightrightarrows Y$. If $F(\cdot)$ is l.s.c. , then $clF(\cdot)$ is l.s.c.*

Proposition 5.3.18. *Let $F : X \rightrightarrows Y$ and Y be a topological linear space. If $F(\cdot)$ is l.s.c. and compact valued, then both $coF(\cdot)$ and $cl(coF(\cdot))$ are l.s.c.*

Proposition 5.3.19. *Let X, Y, Z be topological spaces, $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$. If F and G are lower semi-continuous, then $G \circ F$ is l.s.c.*

Proposition 5.3.20. *Let $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Y$. Then*

(i) if $F_1(\cdot)$ and $F_2(\cdot)$ are l.s.c. closed sv-maps, then $F_1 \cup F_2$ is l.s.c.;

(ii) if $F_1(\cdot)$ is l.s.c., $GraphF_2$ is open in $X \times Y$ and

$$(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x) \neq \emptyset, \forall x \in X,$$

then $F_1 \cap F_2$ is l.s.c.

(iii) if Y is linear topological space, $F_1(\cdot)$ is l.s.c., $F_2(\cdot)$ has open convex values and

$$(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x) \neq \emptyset, \forall x \in X,$$

then $F_1 \cap F_2$ is l.s.c.

Proposition 5.3.21. *Let $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ and Y be a topological linear space. If F_1 and F_2 are l.s.c., then $F_1 + F_2$ is also l.s.c.*

C. Outer , Inner, Upper and Lower Semi-Continuity

The terms "inner" and "outer" semi-continuous have been recently introduced by Rockafellar & Wets[22], but they also known as Kuratowski upper and lower semi-continuity, respectively (see Berger[7]).

Definition 5.3.22 (outer, inner semi-continuity, Rockafellar & Wets). *Let X and Y be metric spaces, $F: X \rightrightarrows Y$, $x^0 \in X$ and define*

$$\limsup_{x \to x^0} F(x) = \{y \mid \exists x_n \to x^0, \exists y^n \to y, y^n \in F(x^n)\} \tag{5.1}$$

$$\liminf_{x \to x^0} F(x) = \{y \mid \forall x_n \to x^0, \exists y^n \to y, y^n \in F(x^n)\}. \tag{5.2}$$

Then , $F: X \rightrightarrows Y$ is said to be

$$\text{(i)} \quad \textbf{outer semi-continuous at } x^0 \text{ if} \quad \limsup_{x \to x^0} F(x) \subset F(x^0); \tag{5.3}$$

$$\text{(ii)} \quad \textbf{inner semi-continuous at } x^0 \text{ if} \quad \liminf_{x \to x^0} F(x) \supset F(x^0). \tag{5.4}$$

Next, we would like to find out relations between inner and lower semi-continuity; as well as between outer and upper semi-continuity.

Proposition 5.3.23. *Let X and Y be metric spaces and $F: X \rightrightarrows Y$. Then $F(\cdot)$ is inner semi-continuous if and only if $F(\cdot)$ is lower semi-continuous.*

Proof. Follows from Prop. 5.3.15 and Def. 5.3.22. □

Proposition 5.3.24. *Let X and Y be metric spaces, $F: X \rightrightarrows Y$. Then $F(\cdot)$ is outer semi-continuous if and only if $F(\cdot)$ is a closed map; i.e. $Graph(F)$ is a closed set in $X \times Y$.*

Proof. "⇒": Suppose $F(\cdot)$ is outer semi-continuous and $(x^n, y^n) \in Graph(F)$ such that $(x^n, y^n) \to (x^0, y^0)$. This implies, $x^n \to x^0$, $y^n \to y^0$ and $y^n \in F(x^n)$. But, since $F(\cdot)$ is outer semi-continuous at x^0, we have $y^0 \in F(x^0)$. Consequently, $(x^0, y^0) \in Graph(F)$.

"⇐": Suppose $Graph(F)$ is a closed set in $X \times Y$. This implies, if there is a sequence $(x^n, y^n) \in Graph(F)$ such that $(x^n, y^n) \to (x^0, y^0)$, then $(x^0, y^0) \in Graph(F)$; i.e. $y^0 \in F(x^0)$.

□

Proposition 5.3.25. *Let X and Y be metric spaces and $F: X \rightrightarrows Y$ be closed valued. If $F(\cdot)$ is upper semi-continuous at x^0, then $F(\cdot)$ is outer semi-continuous at x^0.*

Proof. Assume that $F(\cdot)$ is not outer semi-continuous at x^0. This implies

$$\forall x^n \to x^0, \forall y^n \to y^0, y^n \in F(x^n), \text{ but } y^0 \notin F(x^0).$$

According to Cor. 5.3.7, for any $x^n \to x^0$ and $F(\cdot)$ is u.s.c, we have

$$\limsup_n F(x^n) \subset clF(x^0) = F(x^0).$$

And, by Prop. 4.2.4, we have

$$y^0 \in \limsup_n F(x^n) \subset F(x^0).$$

Thus, $y^0 \in F(x^0)$. But this is a contradiction. Therefore, $F(\cdot)$ is outer semi-continuous. □

Corollary 5.3.26. *(Closed Map, Aubin[2]) Let X and Y be metric spaces. If $F: X \rightrightarrows Y$ is closed valued and u.s.c., then $F(\cdot)$ is a closed map.*

Proof. Prop. 5.3.25 implies that $F(\cdot)$ is outer semi-continuous and, Prop. 5.3.24, yields that $F(\cdot)$ is a closed map. □

Remark 5.3.27. *Cor. 5.3.26 implies that, an upper semi-continuous closed valued map has a closed graph. However, the converse is not always true; i.e. the closedness of $F(\cdot)$ may not imply its upper semi-continuity, even if $F(\cdot)$ is compact valued. In other words,* ***there is an outer semi-continuous set valued map which is not upper semi-continuous.***

The example below indicates that the converse of Prop. 5.3.25 also may not be true.

Example 5.3.28 (Rem. 2.1, Kisielewicz [15])**.** *Let $F : \mathbb{R}_+ \rightrightarrows \mathbb{R}$ be given by*

$$F(x) := \begin{cases} \{0, \frac{1}{x}\}, & \text{if } x > 0; \\ \{0\}, & \text{if } x = 0. \end{cases}$$

Observe that, $F(\cdot)$ is compact valued.

- *$F(\cdot)$ is outer semi-continuous. To see this, let*

$$x^n = \frac{1}{n} \text{ and } y^n = \frac{1}{n}, \text{ then } y^n \in F(x_n).$$

In addition

$$x^n \to 0, y^n \to 0 \text{ and } 0 \in F(0).$$

Hence, $F(\cdot)$ is outer semi-continuous at $x^0 = 0$.

- *$F(\cdot)$ is not upper semi-continuous. Let $\varepsilon > 0$, then $\mathcal{U}_\varepsilon(F(0)) = (-\varepsilon, \varepsilon)$. For any $\delta > 0$, let $[0, \delta)$ be a neighborhood of $x^0 = 0$ in $\mathbb{R}_+$. Now take $\overline{\delta} \in [0, \delta)$ with*

$$0 < \overline{\delta} < \min\left\{\frac{1}{\varepsilon}, \delta, 1\right\}.$$

It follows that

$$F(\overline{\delta}) = \left\{0, \frac{1}{\overline{\delta}}\right\} \not\subseteq (-\varepsilon, \varepsilon), \text{ since } \frac{1}{\overline{\delta}} > \varepsilon.$$

This implies, $F(\cdot)$ is not upper semi-continuous at $x^0 = 0$.

The following statement guarantees the equivalence of outer semi-continuity and upper semi-continuity.

Proposition 5.3.29. *Let X be a metric space, Y be a compact metric space and $F : X \rightrightarrows Y$. If $F(\cdot)$ is a closed valued outer semi-continuous map, then $F(\cdot)$ is u.s.c.*

Proof. Suppose $F(\cdot)$ a closed valued closed map. Assume there is $x^0 \in X$ such that $F(\cdot)$ is not u.s.c. x^0. This implies, there is an open set V such that $F(x^0) \subset V$ and

$$\forall n \in \mathbb{N} : \exists x^n \in \mathbf{B}_{\frac{1}{n}}(x^0), \exists y^n \in F(x^n) : y^n \notin V. \tag{5.5}$$

Hence, $x^n \to x^0$ and, by the compactness of Y, there is a subsequence y^{n_k} such that $y^{n_k} \to y^0 \in Y$. This implies,

$$(x^{n_k}, y^{n_k}) \to (x^0, y^0) \text{ and } y^{n_k} \in F(x^{n_k}).$$

Since, $F(\cdot)$ is a closed map, we have $y^0 \in F(x^0) \subset V$. Since, V is an open set and $y^{n_k} \to y^0$, there is $k_{n_0} > 0$ such that

$$y^{n_k} \in V, \forall n_k \geq n_{k_0}.$$

But this is a contradiction to (5.5). Hence, $F(\cdot)$ should be u.s.c.

□

Observe that,in Prop. 5.3.29, $F(\cdot)$ is implicitly assumed to be compact valued.

Proposition 5.3.30 (Kisielewicz[15]). *Let X and Y be metric spaces and $F: X \rightrightarrows Y$. If $F(\cdot)$ is u.s.c. and compact valued on X, then, for every compact set $K \subset X$, $F(K)$ is a compact set in Y.*

Proof. Let $\{V_\alpha \mid \alpha \in \Omega\}$ be an open covering of $F(K)$. Let $x \in K$ be any, then $F(x)$ is compact and $F(x) \subset \bigcup_{\alpha\in\Omega} V_\alpha$. Hence, there is sub-covering $\{V_{\alpha_k} \mid k = 1, \ldots, n(x)\}$ such that

$$F(x) \subset \bigcup_{k=1}^{n(x)} V_{\alpha_k} := V_x \text{ and } V_x \text{ is an open set.}$$

By Prop. 5.3.5, $F^+(V_x)$ is an open set. Since, $x \in K$ is arbitrary, it follows that

$$K \subset \bigcup_{x\in K} F^+(V_x) \text{ and } K \text{ is a compact set.}$$

This implies there are $x_1, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} F^+(V_{x_i}).$$

Hence, using Prop. 5.2.6 and Exer. 5.2.12(2(ii)), we obtain that

$$F(K) \subset F\left(\bigcup_{i=1}^{m} F^+(V_{x_i})\right) = \bigcup_{i=1}^{m} F(F^+(V_{x_i})) \subset \bigcup_{i=1}^{m} V_{x_i}$$

□

Since, each V_{x_i} is a finite union of elements of $\{V_\alpha \mid \alpha \in \Omega\}$, then $\{V_\alpha \mid \alpha \in \Omega\}$ has a finite sub-cover for $F(K)$. Consequently, $F(K)$ is compact.

Proposition 5.3.31 (Kisielewicz[15]). *Let X and Y be metric spaces and $F: X \rightrightarrows Y$ be compact valued. Then $F(\cdot)$ is u.s.c. at each $x \in X$ if and only if, for every sequence $\{x_n\}, x^n \to x$ and every sequence $\{y^n\}, y^n \in F(x^n)$, there is a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y \in F(x)$.*

Proof. $" \Rightarrow "$: Suppose $F(\cdot)$ is u.s.c. at $x \in X$, $x_n \to x$ and $y^n \in F(x^n)$.
Set

$$K := \{x, x_1, x_2, \ldots, x_n, \ldots\}.$$

Then K is a compact set in X[‡]. Thus, by Prop. 5.3.30, $F(K)$ is a compact set in Y and $\{y^n\} \subset F(K)$. Consequently, there is a subsequence $\{y^{n_k}\}$ of $\{y^n\}$ such that $y^{n_k} \to y$ for some $y \in Y$. This implies, by Prop. 4.2.4, we have

$$y \in \limsup_n F(x^n)$$

Moreover, by Cor. 5.3.7, we have

$$\limsup_n F(x_n) \subset clF(x) = F(x), \text{ by the compactness of } F(x).$$

Consequently, $y \in F(x)$.

[‡]Note that, if the limit point x of the sequence $\{x_n\}$ is not an element of K, the compactness of K may not be guaranteed (Why?).

" $\Leftarrow$ " : Assume that $F(\cdot)$ is not u.s.c. at x^0. This implies there is an open set V, $F(x^0) \subset V$ such that

$$\forall n \in \mathbb{N} : \exists x^n \in \mathbf{B}_{\frac{1}{n}}(x^0) : F(x^n) \nsubseteq V \Rightarrow \forall n \in \mathbb{N} : \exists x^n \in \mathbf{B}_{\frac{1}{n}}(x^0), \exists y^n \in F(x^n) \text{ such that } y^n \notin V.$$

Hence, $x^n \to x^0$ and $y^n \in F(x^n)$ and $y^n \in Y \setminus V$. Consequently, for any convergent subsequence y^{n_k} with limit y, $y \in Y \setminus V$; i.e. $y \notin F(x)$. But this contradicts the assumption. Hence, $F(\cdot)$ should be u.s.c. at x^0.

□

Once again, Prop. 5.3.31 indicates the equivalence of outer and upper semi-continuity for compact valued set-valued maps. But, in general, upper semi-continuity is stronger than outer semi-continuity.

5.3.2 Local Uniform Boundedness

Definition 5.3.32 (local uniform boundedness)**.** *Let X and Y be metric spaces and $F : X \rightrightarrows Y$. Then $F(\cdot)$ is called* ***locally uniformly bounded*** *at $x^0 \in X$ iff there is a neighborhood $U(x^0)$ of x^0 such that the set*

$$\bigcup_{x \in U(x^0)} F(x)$$

is a bounded set in Y. And $F(\cdot)$ is called locally uniformly bounded iff it is locally uniformly bounded at every $x \in X$.

If Y is a finite dimensional or compact metric space, then $F(\cdot)$ is locally uniformly bounded implies that, for each $x \in X$, there is a neighborhood $U(x)$ of x such that

$$cl\left(\bigcup_{z \in U(x)} F(z)\right)$$

is bounded - thus, compact in Y. Thus, in some literature we find such a term like **locally uniformly compactness** being considered, but the local boundedness is more general .

The following statement includes a weaker form of the one given in Prop. 5.3.29.

Proposition 5.3.33. *Let X and Y be metric spaces and $F : X \rightrightarrows Y$. If $F(\cdot)$ is u.s.c. and compact valued, the $F(\cdot)$ locally uniformly bounded.*

Proof. Let $x^0 \in X$. Since $F(x^0)$ is compact. There is a abounded open set V such that $F(x^0) \subset V$ and $F(\cdot)$ is u.s.c. at x^0 imply that

$$\exists U(x^0) : \bigcup_{x \in U(x^0)} F(x) \subset V.$$

Consequently, $F(\cdot)$ is locally bounded. □

Proposition 5.3.34 (see also Hogan [13])**.** *Let X be a metric and Y be a compact metric spaces and $F : X \rightrightarrows Y$ be closed valued. If $F(\cdot)$ is closed and locally uniformly bounded, then Then $F(\cdot)$ is u.s.c.*

Proof. Given $F(\cdot)$ is a closed and locally uniformly bounded map, assume there is $x^0 \in X$ such that $F(\cdot)$ is not u.s.c. x^0. This implies, there is an open set V such that $F(x^0) \subset V$ and

$$\forall n \in \mathbb{N} : \exists x^n \in \mathbf{B}_{\frac{1}{n}}(x^0), \exists y^n \in F(x^n) : y^n \notin V.$$

Hence, $x^n \to x^0$ and $y^n \in F(x^n)$. By the local uniform boundedness,

$$\exists U(x^0) : \bigcup_{x \in U(x^0)} F(x) \text{ is bounded,}$$

and there is $N \in \mathbb{N}$ such that

$$x_n \in U(x^0), \forall n \geq N \Rightarrow y^n \in F(x^n) \subset \bigcup_{x \in U(x^0)} F(x), \forall n \geq N.$$

Consequently, $\{y^n \mid n \geq N\}$ is bounded; hence, there is $\{y^{n_k}\}$ such that $y^{n_k} \to y^0 \in Y \ldots$ (The rest of the proof is as in Prop. 5.3.29).

□

Local uniform boundedness property are useful in characterizing upper semi-continuity of set-valued maps with given structure.

5.3.3 Hausdorff Continuity

In this section we assume that X and Y are normed spaces and $F : X \rightrightarrows Y$.

Definition 5.3.35 (ε-upper semi-continuity)**.** *The map $F(\cdot)$ is said to be u.s.c. at $x^0 \in X$ in the ε sense if, for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\forall x \in \mathbf{B}_\delta(x^0), F(x) \subset F(x^0) + \mathbf{B}_\varepsilon$$

Definition 5.3.36 (ε-lower semi-continuity)**.** *The map $F(\cdot)$ is said to be l.s.c. at $x^0 \in X$ in the ε sense if, for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\forall x \in \mathbf{B}_\delta(x^0), F(x^0) \subset F(x) + \mathbf{B}_\varepsilon$$

Remark 5.3.37. *Using the Hausdorff metric, in particular h^* (see Rem. 4.1.6 and Lem. 4.1.13),*

- *$F(\cdot)$ is l.s.c. at $x^0 \in X$ in the ε sense is equivalent to*

  $$\forall \varepsilon > 0, \exists \delta > 0 : h^*(F(x), F(x^0)) < \varepsilon, \forall x \in \mathbf{B}_\delta(x^0);$$

  *In this case, $F(\cdot)$ is said to be **Hausdorff upper semi-continuous(H-u.s.c.)** at x^0.*

- *$F(\cdot)$ is u.s.c. at $x^0 \in X$ in the ε sense is equivalent to*

  $$\forall \varepsilon > 0, \exists \delta > 0 : h^*(F(x^0), F(x)) < \varepsilon, \forall x \in \mathbf{B}_\delta(x^0).$$

  *Here, $F(\cdot)$ is said to be **Hausdorff lower semi-continuous(H-l.s.c.)** at x^0.*

Hence, in the following, instead to ε-u.s.c. and ε-l.s.c. we simply say H-u.s.c. and H-l.s.c., respectively.

Definition 5.3.38 (Hausdorff Continuity)**.** *Let $x^0 \in X$. Then $F(\cdot)$ is said to be Hausdorff continuous (H-continuous) at x^0 if $F(\cdot)$ is both Hausdorff upper and lower semi-continuous at x^0.*

Proposition 5.3.39. *If $F(\cdot)$ is upper semi-continuous at x^0, then $F(\cdot)$ is H-u.s.c. at x^0.*

Proof. Exercise! □

The converse of Prop. 5.3.39 is not always true.

Example 5.3.40 (see Aubin & Cellina[3])**.** *Let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ given by*

$$F(x) = \{(x, y) \mid y = x\}.$$

Then $F(\cdot)$ is H-u.s.c., but it is not u.s.c.
Let $x^0 \in \mathbb{R}$ and $\varepsilon > 0$ and

$$F(x) \subset F(x^0) + \mathbf{B}_\varepsilon \Rightarrow F(x) \subset \mathbf{B}_\varepsilon\left((x^0, x^0)\right) \Rightarrow (x, x) \in \mathbf{B}_\varepsilon\left((x^0, x^0)\right) \Rightarrow \sqrt{2(x - x^0)^2} < \varepsilon.$$

Thus, if we choose $\delta = \frac{\varepsilon}{2}$, then

$$\forall x : |x - x^0| < \delta, F(x) \subset F(x^0) + \mathbf{B}_\varepsilon.$$

That is, $F(\cdot)$ is H-u.s.c. However, if $V = \{(x, y) \mid |y| < \frac{1}{x}\}$, then an open set in $\mathbb{R}^2$, but $F^+(V) = \{0\}$ which is a closed set in $\mathbb{R}$. Hence, according to Prop. 5.3.5, $F(\cdot)$ is not u.s.c.

Proposition 5.3.41. *If $F(\cdot)$ is H-u.s.c. and closed valued, then $F(\cdot)$ is a closed map; hence, $F(\cdot)$ is outer semi-continuous.*

Proof. Let $(x^n, y^n) \in Graph(F)$ such that $(x^n, y^n) \to (x^0, y^0)$. We want to show that $y^0 \in F(x^0)$; i.e. $(x^0, y^0) \in Graph(F)$. Then we have $x^n \to x^0$ and $y^n \to y^0$. Since, $F(\cdot)$ s H-u.s.c. at x^0,

$$\forall \varepsilon > 0, \exists \delta > 0 : h^*(F(x), F(x^0)) < \varepsilon, \forall x \in \mathbf{B}_\delta(x^0).$$

Hence, there is N such that $x^n \in \mathbf{B}_\delta(x^0), \forall n \geq N$. It follows that

$$\forall n \geq N : h^*(F(x^n), F(x^0)) < \varepsilon \Rightarrow \lim_{n\to\infty} h^*(F(x^n), F(x^0)) = 0.$$

This implies

$$\lim_{n\to\infty} dist(y^n, F(x^0)) = 0.$$

Since $dist(\cdot, F(x^0))$ is a continuous function and $F(x^0)$ is a closed set, we obtain

$$0 = \lim_{n\to\infty} dist(y^n, F(x^0)) = dist(y^0, F(x^0)) \Rightarrow y^0 \in clF(x^0) = F(x^0).$$

Therefore, $F(\cdot)$ is a closed map. The fact that $F(\cdot)$ is outer semi-continuous follows from Prop. 5.3.24. □

Observe that, the closed valuedness of an H-u.s.c. map is not enough to guarantee that is u.s.c. (see example 5.3.28). Thus, we have

Proposition 5.3.42. *If $F(\cdot)$ is H-u.s.c. and compact valued, then $F(\cdot)$ is u.s.c.*

Proof. We can either use here Prop. 5.3.31 or Prop. 5.3.34 for the proof. We use the latter. Let $x^0 \in X$ be any. Since $F(x^0)$ is compact, there is $\varepsilon > 0$ such that $\mathcal{U}_\varepsilon(F(x^0))$ is bounded and $F(x^0) \subset \mathcal{U}_\varepsilon(F(x^0)) = F(x^0) + \mathbf{B}_\varepsilon$. By H-u.s.c., there is $\delta > 0$ such that

$$\bigcup_{x \in \mathbf{B}_\delta(x^0)} F(x) \subset F(x^0) + \mathbf{B}_\varepsilon.$$

Consequently, $F(\cdot)$ is locally uniformly bounded and $F(\cdot)$ is compact valued. Therefore, by Prop. 5.3.34, $F(\cdot)$ is u.s.c. □

In contrast to H-u.s.c. we have the following result for H-l.s.c.

Proposition 5.3.43. *If $F(\cdot)$ is H-l.s.c., then $F(\cdot)$ is l.s.c.*

Proof. Exercise! ☐

But the converse requires stronger assumptions.

Proposition 5.3.44. *If $F(\cdot)$ l.s.c. and compact valued, then $F(\cdot)$ is H-l.s.c.*

Proof. Let $\varepsilon > 0$ be given and $x^0 \in X$ be any. Since $F(x^0)$ is compact, we have

$$F(x^0) \subset \bigcup_{y \in F(x^0)} B_{\frac{\varepsilon}{2}}(y) \Rightarrow \exists y_1, \dots, y_m : F(x^0) \subset \bigcup_{k=1}^{m} B_{\frac{\varepsilon}{2}}(y_k).$$

Hence, $F(x^0) \cap B_{\frac{\varepsilon}{2}}(y_k) \neq \emptyset$. By the lower semi-continuity of $F(\cdot)$, for each $k \in \{1, \dots, m\}$

$$\exists U_k(x^0) : F(x) \cap B_{\frac{\varepsilon}{2}}(y_k) \neq \emptyset, \forall x \in U_k(x^0).$$

Then

$$\bigcap_{k=1}^{m} U_k(x^0) \text{ is an open set. Thus, } \exists \delta > 0 : \mathbf{B}_\delta(x^0) \subset \bigcap_{k=1}^{m} U_k(x^0).$$

$\Rightarrow$

$$\forall x \in \mathbf{B}_\delta(x^0) : F(x) \cap B_{\frac{\varepsilon}{2}}(y_k) \neq \emptyset, \forall k \in \{1, \dots, m\}. \tag{5.6}$$

But

$$F(x) \cap B_{\frac{\varepsilon}{2}}(y_k) \neq \emptyset \Rightarrow y_k \in F(x) + \mathbf{B}_{\frac{\varepsilon}{2}} \Rightarrow \mathbf{B}_{\frac{\varepsilon}{2}}(y_k) \subset F(x) + \mathbf{B}_\varepsilon \text{ (Verify!)}$$

Hence, from (5.6), we have

$$\forall x \in \mathbf{B}_\delta(x^0) : \mathbf{B}_{\frac{\varepsilon}{2}}(y_k) \subset F(x) + \mathbf{B}_\varepsilon, \forall k \in \{1, \dots, m\} \Rightarrow \forall x \in \mathbf{B}_\delta(x^0) : \bigcup_{k=1}^{m} \mathbf{B}_{\frac{\varepsilon}{2}}(y_k) \subset F(x) + \mathbf{B}_\varepsilon.$$

Consequently,

$$\forall x \in \mathbf{B}_\delta(x^0) : F(x^0) \subset F(x) + \mathbf{B}_\varepsilon.$$

Therefore, $F(\cdot)$ is H-l.s.c. ☐

A Hausdorff continuous set-valued map behaves like a continuous single valued map.

Proposition 5.3.45. *Let X and Y be normed spaces, $F : X \rightrightarrows Y$. If $F(\cdot)$ is H-continuous, then for every convergent sequence $\{x^n\}$, $x^n \to x^0$, we have $F(x^n) \xrightarrow{h} F(x^0)$.*

Proof. Exercise! ☐

Excercises 5.3.46. *Prove the following statements*

1. *If $F : X \rightrightarrows Y$ is u.s.c., then the set $\{x \in X \mid F(x) = \emptyset\}$ is open in X.*
2. *If $F : X \rightrightarrows Y$ is l.s.c., then the set $\{x \in X \mid F(x) = \emptyset\}$ is closed in X.*

3. *For* $F_k: X \rightrightarrows Y, k \in \mathbb{N}$. *If for each* $x \in X$

$$\bigcap_{k\in\mathbb{N}} F_k(x) \neq \emptyset,$$

then, for any set $A \subset Y$,

$$\left(\bigcap_{k\in\mathbb{N}} F_k\right)^{-}(A) = \bigcap_{k\in\mathbb{N}} F_k^{-}(A).$$

4. *Let* X *be a metric space and* $S \subset X, S \neq \emptyset$. *The metric projection on to* S *is* $P_S : X \rightrightarrows S$ *given by*

$$P_S(x) = \{y \in S \mid dist(x, S) = \rho(x, y)\}.$$

Then $P_S(\cdot)$ *is compact valued and u.s.c. If* S *is also convex, then* $P_S(\cdot)$ *is also convex.*

5. *Let* $F : X \rightrightarrows Y$, *where* $X = [0, 1]$ *and* $Y = \mathbb{R}$ *given by*

$$F(x) = \begin{cases} [0,1], & \textit{if } 0 \leq x \leq 1; \\ [0,1), & \textit{if } x = 1. \end{cases}$$

Then $F(\cdot)$ *is* H*-u.s.c., but not u.s.c.*

6. *Prove Prop. 5.3.42 using Prop. 5.3.31.*

7. *Let* X *and* Y *be normed spaces and* $F: X \rightrightarrows Y$.

   (i) If $F(\cdot)$ *is* H*-u.s.c., then* $clF(\cdot)$ *is* H*-u.s.c.*

   (ii) If $F(\cdot)$ *is* H*-l.s.c., then* $clF(\cdot)$ *is* H*-l.s.c.*

8. *Let* X *and* Y *be normed spaces,* $F: X \rightrightarrows Y$ *and* $F(\cdot)$ *is compact valued. Then* $F(\cdot)$ *is continuous if and only if* $F(\cdot)$ *is* H*-continuous.*

9. *Prove or disprove the converse of Prop. 5.3.45.*

10. *Let* X *and* Y *be normed spaces,* $F_1, F_2 : X \rightrightarrows Y$. *Then prove that*

    (i) if F_1, F_2 *are* H*-u.s.c., then* $F_1 \cup F_2$ *is* H*-u.s.c.;*

    (ii) if F_1, F_2 *are* H*-l.s.c., then* $F_1 \cup F_2$ *is* H*-l.s.c.*

11. *If* $F(\cdot)$ *is H-l.s.c., then* $clF(\cdot)$ *and* $convF(\cdot)$ *are also H-l.s.c.*

5.4 Set-Valued Maps with Given Structures

In this section we assume all space X, Y, T, etc., to be finite dimensional, like $\mathbb{R}^n, \mathbb{R}^m$, and so on.

Set-valued maps which are defined using a parametric family of functions play a vital role in parametric optimization, in sensitivity and perturbation analysis of optimization problems. The main issue, behind set-valued maps with such given structures, is to characterize them through the topological properties of their defining functions. As such, one obtains u.s.c property under weaker assumptions, while the l.s.c. requires stronger ones.

Of interest are set-valued maps $F: X \rightrightarrows T$ and $M: X \rightrightarrows Y$ with structures:

$$F(x) := \{t \in T \mid g_j(x,t) = 0, j \in J;\ h_i(x,t) \leq 0, i \in I\};$$

and

$$M(x) := \{y \in Y \mid f_k(x,y) = 0, k \in K; G(x,y,t) \leq 0, t \in F(x)\}, x \in X,$$

under the following general assumptions

- $I = \{1, \ldots, p\}, J := \{1, \ldots, q\}$ and $K := \{1, \ldots, r\}$ are finite index sets;
- $X \subset \mathbb{R}^n, T \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^l$;
- $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in I$; $f_k : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, k \in K$; and $G : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ are continuous functions.

Proposition 5.4.1 (Thm. 3.1.1 Bank *et al.* [5]). *Let*

$$F(x) = \{t \in T \mid g_j(x,t) = 0, j \in J; h_i(x,t) \leq 0, i \in I\}.$$

If the sets X and T are closed and the functions $g_j, j \in J; h_i, i \in I$ are continuous, then $F(\cdot)$ is a closed set valued map.

Proof. Note that for $(x^n, t^n) \in Graph(F)$, we have $t^n \in F(x^n)$. This in turn implies,

$$g_j(x^n, t^n) = 0, j \in J \text{ and } h_i(x^n, t^n) \leq 0, i \in I.$$

Hence, if $(x^n, t^n) \to (x^0, t^0)$, it follows by the continuity of the $g_j's$ and $h_i's$ that

$$g_j(x^0, t^0) = 0, j \in J \text{ and } h_i(x^0, t^0) \leq 0, i \in I.$$

That is $t^0 \in F(x^0)$. Therefore, $F(\cdot)$ is a closed map. □

In fact, in Prop. 5.4.1, the upper semi-continuity of the functions $g_j, j \in J$, could have been sufficient.

Corollary 5.4.2. *Let X and T be closed sets and $F(x) = \{t \in T \mid h_i(x,t) \leq 0, i \in I;\ g_j(x,t) = 0, j \in J\}$ and the functions $h_i, i \in I;\ g_j, j \in J$ are continuous. If $F(\cdot)$ is locally uniformly bounded, then $F(\cdot)$ is u.s.c. and compact valued.*

Proof. Follows from Prop. 5.4.1 and Prop. 5.3.34. □

Remark 5.4.3. *In Cor. 5.4.2 if T is assumed to be a compact set, then*

$$F(x) = \{t \in T \mid g_j(x,t) = 0, j \in J;\ h_i(x,t) \leq 0, i \in I\}$$

will be locally uniformly bounded.

Thus, the upper semi-continuity of a SV-map with a given structure could be seen to hold true under somehow weaker assumptions. However, to guarantee the lower semi-continuity we need regularity conditions, like *Metric regularity* and *constraint qualifications*, etc.

Definition 5.4.4 (Slater Constraint Qualification(SCQ)). *Let, for each $i \in I = \{1, \ldots, m\}$, the function $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be continuous, $x^0 \in \mathbb{R}^n$ and $h_i(x^0, \cdot)$ be convex w.r.t. $t \in \mathbb{R}^m$. Then the Slater Constraint Qualification is said to be satisfied at x^0 if there is $t^* \in \mathbb{R}^m$ such that*

$$h_i(x^0, t^*) < 0, \forall i \in I.$$

Proposition 5.4.5. *Let, for each $i \in I = \{1, \dots, m\}$, the function $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $h_i(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ be continuous, $x^0 \in \mathbb{R}^n$, $h_i(x^0, \cdot)$ be convex w.r.t. $t \in \mathbb{R}^m$, and*

$$F(x) = \{t \in \mathbb{R}^m \mid h_i(x, t) \leq 0, i \in I\}.$$

If the SCQ holds at x^0, then $F(\cdot)$ is l.s.c. at x^0.

Proof. First not that $F(\cdot)$ is a closed and convex valued map. Now, let $V \subset \mathbb{R}^m$ be an open set and $F(x^0) \cap V \neq \emptyset$; i.e. $\exists \bar{t} \in F(x^0) \cap V$. Hence, for some $r > 0$, $\bar{t} \in B_r(\bar{t}) \subset V$. By the SCQ, there is t^* such that

$$h_i(x^0, t^*) < 0, \forall i \in I.$$

Since a convex function on $\mathbb{R}^m$ is continuous, we can find an open neighborhood $\widetilde{V}(t^*)$ such that

$$h_i(x^0, t) < 0, \forall i \in I, \forall t \in \widetilde{V}(t^*).$$

$\Rightarrow \forall t \in \widetilde{V}(t^*) : t \in F(x^0)$; i.e. $t^* \in int\ F(x^0)$.
Furthermore, we can find $\lambda \in (0, 1)$, sufficiently small, so that $t_\lambda := \lambda t^* + (1 - \lambda)\bar{t} \in B_r(\bar{t}) \subset V$. But, since $F(x^0)$ is a closed convex set, we also have $t_\lambda \in F(x^0)$. Consequently, $t_\lambda \in F(x^0) \cap V$ and

$$\forall i \in I : h_i(x^0, t_\lambda) \leq \lambda h_i(x^0, t^*) + (1 - \lambda) h_i(x^0, \bar{t}) < \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0.$$

Next, by the continuity of the $h_i's$, there is $U(x^0)$ such that

$$h_i(x, t_\lambda) < 0, \forall x \in U(x^0) \Rightarrow t_\lambda \in F(x), \forall x \in U(x^0),$$

and $t_\lambda \in V$. Consequently,

$$\forall x \in U(x^0) : t_\lambda \in F(x) \cap V \neq \emptyset.$$

Therefore, $F(\cdot)$ is l.s.c. at x^0. □

If (SCQ) does not hold, then the map $F(\cdot)$ might not be l.s.c.
Next, we would like to characterize semi-continuity when convexity is not available.

Definition 5.4.6 (Mangasarian-Fromovitz Constraint Qualification (MFCQ))**.** *Let $F(x) = \{t \in \mathbb{R}^m \mid h_i(x, t) \leq 0, i \in I\}$; the functions $h_i, i \in I$ be continuous in $\mathbb{R}^n \times \mathbb{R}^m$; for each $x \in \mathbb{R}^n$, $h_i(x, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable; and for $x^0 \in \mathbb{R}^n$, let $t^0 \in B(x^0)$. The Mangasarian-Fromovitz constraint qualification (MFCQ) is said to hold at (x^0, t^0) iff there exists a vector $\xi \in \mathbb{R}^m$ such that*

$$\xi^\top \nabla_t h_i(x^0, t^0) < 0, \forall i \in I(x^0, t^0);$$

where $I(x^0, t^0) = \{i \in I \mid h_i(x^0, t^0) = 0\}$. The vector ξ with the above property is known as an (MFCQ) vector.

Proposition 5.4.7. *If $t^0 \in B(x^0)$ and (MFCQ) holds at (x^0, t^0), then the map $F(\cdot)$ is lower semi-continuous at x^0.*

Proof. By the satisfaction of (MFCQ), for each $i \in I(x^0, t^0)$, there is λ_0^i such that

$$h_i(x^0, t^0 + \lambda\xi) = h_i(x^0, t^0) + \lambda \xi^\top \nabla_t h_i(x^0, t^0) + o(\lambda), \forall \lambda \in (0, \lambda_0^i),$$

which is the first order Taylor expansion of $h_i(x^0, \cdot)$ at t^0. Thus, using $t^0 \in F(x^0)$ and (MFCQ), we have (w.l.o.g.) that

$$h_i(x^0, t^0 + \lambda\xi) \leq 0, \forall \lambda \in (0, \lambda_0^i),$$

for each $i \in I(x^0, t^0)$. Since, the $h_i^{\prime}s$ are continuous, there is $\widetilde{U}_i(x^0)$ such that

$$h_i(x, t^0 + \lambda\xi) \leq 0, \forall x \in \widetilde{U}_i(x^0), \forall \lambda \in (0, \lambda_0^i),$$

$i \in I(x^0, t^0)$. Moreover,

$$h_i(x^0, t^0) < 0, i \in I \setminus I(x^0, t^0).$$

Hence, for each $i \in I \setminus I(x^0, t^0)$,

$$\exists U_i(x^0), \exists V_i(t^0) : h_i(x, t) < 0, \forall x \in U_i(x^0), \forall t \in V_i(t^0)$$

Now, set

$$U(x^0) = \bigcap_{i \in I(x^0, t^0)} \widetilde{U}_i(x^0) \cap \bigcap_{i \in I \setminus I(x^0, t^0)} U_i(x^0) \quad \text{and} \quad V(t^0) = \bigcap_{i \in I \setminus I(x^0, t^0)} V_i(t^0).$$

Then, for a sufficiently small λ_0 (say $\lambda_0 \leq \min\{\lambda_0^i \mid i \in I(x^0, t^0)\}$), we obtain $t^0 + \lambda\xi \in V(t^0), \forall \lambda \in (0, \lambda_0)$. It follows that, for each $i \in I$

$$h_i(x, t^0 + \lambda\xi) \leq 0, \forall x \in U(x^0), \forall \lambda \in (0, \lambda_0).$$

Now if $x^n \to x^0$, then for $\lambda_n \in (0, \lambda_0)$ and $\lambda_n \to 0$, we have $t^n = t^0 + \lambda_n \xi \in F(x^n)$ and $t^n \to t^0$. Therefore, by Prop. 5.3.15, $F(\cdot)$ is l.s.c.

□

In Prop. 5.4.7, if (MFCQ) is not satisfied at (x^0, t^0), then $F(\cdot)$ may fail to be l.s.c. at x^0.
Let us next characterize the lower semi-continuity of the map

$$M(x) = \{y \in Y \mid G(x, y, t) \leq 0, t \in F(x)\}, x \in \mathbb{R}^n.$$

For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, we define the **set of active constraints** as

$$E(x, y) := \{t \in F(x) \mid G(x, y, t) = 0\}.$$

Definition 5.4.8 (Extended Mangasarian-Fromowitz Constraint Qualification). *Let $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^q$, the function $G : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}$ be continuous, and $G(x, \cdot, t)$ differentiable w.r.t. y and $\nabla_y G(\cdot, \cdot, t)$ is continuous for each $t \in F(x)$. Then the* extended Mangasarian-Fromowitz constraint qualification *(EMFCQ) is said to be satisfied at (x^0, y^0) if there exists a vector $\xi \in \mathbb{R}^q$ such that*

$$\xi^\top \nabla_y G(x^0, y^0, t) < 0, \forall t \in E(x^0, y^0).$$

Proposition 5.4.9. *Let $F(\cdot)$ be u.s.c. and compact valued and $y^0 \in M(x^0)$. Then if (EMFCQ) is satisfied at (x^0, y^0), then $M(\cdot)$ is l.s.c. at x^0.*

Proof. Let $y^0 \in M(x^0)$ be arbitrary, $V(y^0)$ be any neighborhood of y^0 and EMFCQ be satisfied w.r.t. y at (x^0, y^0). Then there exists $\xi \in \mathbb{R}^q$ such that

$$\nabla_y G(x^0, y^0, t)\xi < 0, \forall t \in E(x^0, y^0).$$

For a fixed $t^l \in E(x^0, y^0)$, using Taylor's theorem, there is $\lambda_0^l := \lambda_0(t^l)$:

$$G(x^0, y^0 + \lambda\xi, t^l) = G(x^0, y^0, t^l) + \lambda \nabla_y G(x^0, y^0, t^l) + o(\lambda), \forall \lambda \in (0, \lambda_0^l).$$

$\Rightarrow$ (w.o.l.g.)

$$G(x^0, y^0 + \lambda\xi, t^l) \leq 0, \forall \lambda \in (0, \lambda_0^l).$$

(i) By the continuity of $G(\cdot,\cdot,\cdot)$, there are neighborhoods $U_1^l(x^0)$ and $W_1(t^l)$ (with an appropriate λ_0^l)such that

$$G(x, y^0 + \lambda\xi, t) \leq 0, \forall\lambda \in (0, \lambda_0^l), \forall x \in U_1^l(x^0), \forall t \in W_1(t^l).$$

(ii) Moreover, for each $\bar{t} \in F(x^0) \setminus E(x^0, y^0)$

$$G(x^0, y^0, \bar{t}) < 0.$$

Thus, (as above) for each fixed $\bar{t} \in F(x^0) \setminus E(x^0, y^0)$, there are neighborhoods $U_2^{\bar{t}}(x^0), W_2(\bar{t})$ and $V^{\bar{t}}(y^0)$ such that

$$G(x, y, t) < 0, \forall x \in U_2^{\bar{t}}(x^0), \forall y \in V^{\bar{t}}(y^0), \forall t \in W_2(\bar{t}).$$

The family $\{W_1(t^l), W_2(\bar{t}) \mid t^l \in E(x^0, y^0), \bar{t} \in B(x^0) \setminus E(x^0, y^0)\}$ forms an open covering of $F(x^0)$. By assumption, $F(x^0)$ is a compact set. Hence, there is a finite sub-covering $\{W(\bar{t}^l) \mid l = 1, \ldots, p\}$ of $F(x^0)$. Moreover, corresponding to this finite sub-covering we can find

- neighborhoods $\{U^l(x^0) \mid l = 1, \ldots, p\}$ of x^0, so that $U(x^0) := \bigcap_{l=1}^p U^l(x^0)$ is a neighborhood of x^0.
- a sufficiently small λ_0 (say $\lambda_0 := \min\{\overline{\lambda_0}^l \mid l = 1, \ldots, p\}$) is such a way that $y^0 + \lambda\xi \in V(y^0), \forall\lambda \in (0, \lambda_0)$,

so that

$$G(x, y^0 + \lambda\xi, t) \leq 0, \forall\lambda \in (0, \lambda_0), \forall x \in U(x^0), \forall t \in B(x^0) \subset \bigcup_{l=1}^p W(\bar{t}^l). \tag{5.7}$$

The map $F(\cdot)$ is u.s.c. This implies, there is a neighborhood $\widetilde{U}(x^0)$ such that

$$F(x) \subset \bigcup_{l=1}^p W(\bar{t}^l), \forall x \in \widetilde{U}(x^0).$$

Using this with (5.7) we obtain

$$G(x, y^0 + \lambda\xi, t) \leq 0, \forall\lambda \in (0, \lambda_0), \forall x \in U(x^0) \cap \widetilde{U}(x^0), \forall t \in B(x).$$

$\Rightarrow$

$$\forall x \in U(x^0) \cap \widetilde{U}(x^0), \forall\lambda \in (0, \lambda_0) : y^0 + \lambda\xi \in M(x).$$

Hence,

$$\forall x \in U(x^0) \cap \widetilde{U}(x^0), M(x) \cap V(y^0) \neq \emptyset.$$

Therefore, $M(\cdot)$ is a lower semi-continuous map.

□

Excercises 5.4.10. *Prove the following statements.*

1. *Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(x) = [a(x), b(x)]$, where $a, b : \mathbb{R} \to \mathbb{R}$, $a(x) \leq b(x), \forall x \in \mathbb{R}$, $a(\cdot)$ an u.s.c. and $b(\cdot)$ a lower semi-continuous functions. Then $F(\cdot)$ is a lower semi-continuous sv-map.*

2. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $F : \mathbb{R} \rightrightarrows \mathbb{R}^n$ *be given by*

$$F(t) = \{x \in \mathbb{R}^n \mid f(x) \leq t\}.$$

If f *is a convex function, then* $F(\cdot)$ *is a convex set-valued map.*

3. *Let* $g_i : \mathbb{R} \times \mathbb{R}^2, i = 1, \ldots, 4$ *given by*

$$\begin{aligned} h_1(x,t) &= -t_1 - 1; \\ h_2(x,t) &= t_1 - 1; \\ h_3(x,t) &= -t_2 - 1; \\ h_4(x,t) &= t_2 + xt_1; \end{aligned}$$

and

$$F(x) = \{t \in \mathbb{R}^2 \mid h_i(x,t) \leq 0, i = 1,2,3,4\}.$$

Then

(i) show that $F(\cdot)$ *is compact valued, closed and locally uniformly bounded (Hence, by Prop.5.3.34* $F(\cdot)$ *is u.s.c.); but,*

(ii) show that $F(\cdot)$ *is not l.s.c. at* $x = 0$*. (Hint: argue graphically). That is, the (SCQ) fails to hold at* $x = 0$*.*

6 Measurability of Set-Valued Maps

Unless explicitly specified, we assume here the spaces X, Y and Z to be are metric spaces and Ω to be a subset of a metric space.

6.1 Definitions and Properties of Measurable Set-Valued Maps

First we begin by recalling the definition of a σ-algebra and a measurable space.

Definition 6.1.1 (σ-Algebra)**.** *Let Ω be a non-empty set. A collection $\mathcal{F}$ of subsets of Ω is said to be a σ-algebra if*

(i) $\emptyset, \Omega \in \mathcal{F}$;

(ii) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$;

(iii) If $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ is any countable collection, then

$$\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{F}.$$

Definition 6.1.2 (a measurable space)**.** *A **measurable space** $(\Omega, \mathcal{F})$ is a non-empty set Ω along with a $\sigma-$algebra $\mathcal{F}$ defined on Ω.*

Let X be a complete metric space. Then smallest σ-algebra containing all open sets in X is called the **Borel σ-algebra** on X denoted by $\mathcal{B}(X)$. The measurable space $(X, \mathcal{B}(X))$ is also called the **Borel measurable space** on X. Moreover, if $A \in \mathcal{B}(X)$, then A is called **Borel measurable** w.r.t. X.

Definition 6.1.3 (measurable set-valued map)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space. A set valued map $F : \Omega \rightrightarrows Y$ is said to be **measurable** (or $\mathcal{F}$-measurable) on X if, for every open set $O \subset Y$, $F^-(O)$ is measurable; i.e. $F^-(O) \in \mathcal{F}$.*

Hence, for $F : \Omega \rightrightarrows Y$,

- if $F(\cdot)$ is measurable, then the sets $F^-(\emptyset)$ and $Dom(F) = F^-(Y)$ are measurable.
- if $B \subset Y$ and $F(x) = B, \forall x \in X$ (i.e. $F(\cdot)$ is a constant valued map), then $F(\cdot)$ is measurable. This follows from the fact that, for any open set $O \subset Y$, we have

$$F^-(O) = \begin{cases} \Omega, & \text{if } O \cap B \neq \emptyset; \\ \emptyset, & \text{if } O \cap B = \emptyset. \end{cases}$$

In Def. 6.1.3 if X is a complete metric space, $F : X \rightrightarrows Y$ and $\mathcal{F} = \mathcal{B}(X)$, then $F(\cdot)$ is said to be Borel-measurable.

Proposition 6.1.4. *Let X be a complete metric space, $F : X \rightrightarrows Y$ and $Dom(F) = X$. If $F(\cdot)$ is l.s.c., then $F(\cdot)$ is Borel-measurable.*

Proof. For $O \subset Y$ is open, Prop. 5.3.14, implies that $F^{-}(O)$ is open. Hence, $F^{-}(O) \in \mathcal{B}(X)$. $\square$

Proposition 6.1.5. *A set valued map $F: X \rightrightarrows Y$ is measurable if and only of the distance function*

$$\varphi(x) := dist(y, F(x)),$$

$\varphi : X \to \mathbb{R}_+$ is measurable, for each fixed $y \in \mathbb{R}^m$.

Proposition 6.1.6. *Let $(\Omega, \mathcal{F})$ be a measurable space, $F: \Omega \rightrightarrows Y$ be a closed valued map and Y be a separable metric space. Then the following statements are equivalent:*

(i) $F^{-}(C)$ is measurable for all closed sets $C \subset Y$;

(ii) $F^{-}(O)$ is measurable for all open sets $O \subset Y$;

(iii) $F^{-}(K)$ is measurable for all compact sets $K \subset Y$;

Proof.
(i) $\Rightarrow$ (ii): Let $O \subset Y$ be open. Since Y is a metric space, then O is an F_σ set; i.e. there is a countable family of closed sets $\{C_n \mid n \in \mathbb{N}\}$ such that

$$O = \bigcup_{n \in \mathbb{N}} C_n$$

Thus, by a property of F^{-} we have that

$$F^{-}(O) = F^{-}\left(\bigcup_{n \in \mathbb{N}} C_n\right) = \bigcup_{n \in \mathbb{N}} F^{-}(C_n).$$

But, for each $n \in N, F^{-}(C_n)$ is measurable. Consequently, $F^{-}(O)$ is a countable union of measurable sets and $\mathcal{F}$ is a σ-algebra imply that $F^{-}(O)$ is measurable.
(ii) $\Rightarrow$ (iii): Let $K \subset Y$ be a compact set. For each $n \in \mathbb{N}$ define the set

$$O_n := \left\{ y \in Y \mid dist(y, K) < \frac{1}{n} \right\}.$$

Then, it is easy to verify that O_n is open in Y, clO_n is compact and $clO_{n+1} \subset O_n$.
<u>Claim:</u>

$$F^{-}(K) = \bigcap_{n \in \mathbb{N}} F^{-}(O_n).$$

Let $x \in F^{-}(K)$. This implies, $F(x) \cap K \neq \emptyset$; i.e. $\exists y \in F(x) \cap K$. Hence, $dist(y, K) = 0$ so that $y \in O_n, \forall n \in N$. From this follows that

$$y \in F(x) \cap O_n, \forall n \in \mathbb{N} \Rightarrow x \in F^{-}(O_n), \forall n \in \mathbb{N} \Rightarrow x \in \bigcap_{n \in \mathbb{N}} F^{-}(O_n).$$

Consequently, $F^{-}(K) \subset \bigcap_{n \in \mathbb{N}} F^{-}(O_n)$. Conversely, let $z \in \bigcap_{n \in \mathbb{N}} F^{-}(O_n)$. This implies

$$z \in F^{-}(O_n), \forall n \in \mathbb{N} \Rightarrow F(z) \cap O_n \neq \emptyset, \forall n \in \mathbb{N}.$$

Hence,

$$\forall n \in \mathbb{N}, \exists y_n \in F(z) \cap O_n \Rightarrow \lim_{n \to \infty} dist(y_n, K) = 0.$$

Since, $\{y_n\}_{n\in\mathbb{N}} \subset clO_1$ and clO_1 is compact, there is a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to \overline{y}$, for some $\overline{y} \in Y$. Then, using the continuity of the distance function, we find that

$$dist(\overline{y}, K) = \lim_{k\to\infty} = dist(y_{n_k}, K) = dist(y_n, K) = 0.$$

Which implies $\overline{y} \in clK = K$. Since, $F(z)$ is closed and $\{y_n\} \subset F(z)$, we also have $\overline{y} \in F(z)$. From this follows that

$$\overline{y} \in F(z) \cap K \Rightarrow z \in F^-(K).$$

Hence,

$$\bigcap_{n\in\mathbb{N}} F^-(O_n) \subset F^-(K).$$

Consequently,

$$F^-(K) = \bigcap_{n\in\mathbb{N}} F^-(O_n).$$

By assumption, for each $n \in \mathbb{N}$, $F^-(O_n)$ is measurable. Hence, $F^-(K)$ is measurable.

(iii) $\Rightarrow$ (i): Let C be a closed set in Y. Since, Y is separable, there is a countable dense set $D = \{y_1, y_2, \ldots\}$. Given $\varepsilon > 0$, define the sets

$$K_n := \{y \in Y \mid \rho(y_n, y) \leq \varepsilon\}.$$

Then, for each $n \in \mathbb{N}$, K_n is a compact set and by density of D, we have

$$C \subset \bigcup_{n\in\mathbb{N}} K_n \Rightarrow C = \bigcup_{n\in\mathbb{N}} (C \cap K_n).$$

But, for each $n \in \mathbb{N}, C \cap K_n$ is compact. Hence, by assumption $F^-(C_n \cap K_n)$ is measurable. Hence,

$$F^-(C) = \bigcup_{n\in\mathbb{N}} F^-(C \cap K_n)$$

is measurable.

□

Corollary 6.1.7. *Let $F: \Omega \rightrightarrows Y$ be a closed valued map and Y be a separable metric space. If $F(\cdot)$ is measurable, then, for any closed set $C \subset Y$, $F^+(Y)$ is measurable.*

Proof. The set C is closed, implies $Y \setminus C$ is open. Then, by Prop. 6.1.6, $F^-(Y \setminus C)$ is measurable. But, by Lem. 5.3.4, we have

$$F^-(Y \setminus C) = X \setminus F^+(C).$$

Hence, $X \setminus F^+(C)$ is measurable. Therefore, $F^+(C)$ measurable. □

Proposition 6.1.8. *Let X be a complete and Y a separable metric spaces, $F: X \rightrightarrows Y$ and $Dom(F) = X$. If $F(\cdot)$ is u.s.c. and closed valued, then $F(\cdot)$ is Borel-measurable.*

Proof. Follows from Prop. 5.3.5 and Prop. 6.1.6. □

6.1.1 Operations with Measurable Set-Valued Maps

Proposition 6.1.9 (measurability of closure)**.** *The map $F : \Omega \rightrightarrows Y$ is measurable if and only if $clF(\cdot)$ is measurable.*

Proof. Use the fact that for any set B and an open set O we have

$$B \cap O \neq \emptyset \Leftrightarrow clB \cap O \neq \emptyset.$$

□

Corollary 6.1.10. *Let $F : \Omega \rightrightarrows Y$ and $G : \Omega \rightrightarrows Y$. If F is measurable and*

$$F(\omega) \subset G(\omega) \subset clF(\omega), \text{ and for each } \omega \in \Omega,$$

then $G(\cdot)$ is measurable.

Proof. Use the same idea as in Prop. 6.1.9. □

Proposition 6.1.11. *If $F_k : \Omega \rightrightarrows Y, k \in \mathbb{N}$, are measurable maps, then the union map*

$$F(\omega) := \bigcup_{k \in \mathbb{N}} F_k(\omega), \text{ for } \omega \in \Omega$$

is measurable.

Proof. Follows from the fact that

$$\left(\bigcup_{k \in \mathbb{N}} F_k \right)^{-} (O) = \bigcup_{k \in \mathbb{N}} F_k^{-}(O)$$

for an open set $O \subset Y$.

□

Proposition 6.1.12 (measurability of the intersection)**.** *If, for each $\alpha \in \Theta, F_\alpha : \Omega \rightrightarrows Y$ is measurable maps, then the intersection map*

$$F(\omega) := \bigcap_{\alpha \in \Theta} F_\alpha(\omega), \text{ for } \omega \in \Omega$$

is measurable.

Proposition 6.1.13. *Let Y be a normed topological space.*

(i) If $F_1 : \Omega \rightrightarrows Y$ and $F_2 : \Omega \rightrightarrows Y$ are measurable, then $F_1 + F_2$ is measurable.

(ii) If $F_1 : \Omega \rightrightarrows Y$ is measurable and $\gamma \in \mathbb{R}$, then $(\gamma F)(\cdot)$ is measurable; where

$$(\gamma F)(x) = \gamma F(x), \text{ for } x \in \Omega.$$

Proposition 6.1.14. *Let $G : \Omega \rightrightarrows \mathbb{R}^n$. If $G(\cdot)$ is a measurable, then the convex hull map $co\ G(\cdot)$ is measurable.*

Proof. Define the set

$$\Lambda = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \in \mathbb{Q}_+^{n+1} \;\middle|\; \sum_{k=1}^{n+1} \lambda_k = 1 \right\},$$

where $\mathbb{Q}_+$ represents the set of non-negative rational numbers. For $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \in \Lambda$ let

$$F_\lambda(\omega) := \sum_{k=1}^{n+1} \lambda_k F(\omega).$$

Then, by Prop. 6.1.13, for each $\lambda \in \Lambda$, $F_\lambda(\cdot)$ is measurable. Furthermore, the countability of Λ and Lem. 6.1.10 imply that

$$F(\omega) = \bigcup_{\lambda \in \Lambda} F_\lambda(\omega)$$

is measurable. Since

$$F(\omega) \subset co\ G(\omega) \subset cl\ F(\omega)$$

for each $\omega \in \Omega$, we conclude by Prop. 6.1.11 that $co\ G(\cdot)$ is measurable.

□

Proposition 6.1.15 (measurability of a product). *If $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$ are measurable set-valued maps, then $(F_1 \times F_2)(\cdot)$ is also a measurable set-valued map.*

Proof. The proof follows if we observe for an open set $O \subset Y \times Z$ that

$$(F_1 \times F_2)^-(O) = F_1^-(\pi_Y(O)) \cap F_2^-(\pi_Z(O));$$

where $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$ are projections maps onto Y and Z, respectively.

□

Proposition 6.1.16 (measurability of composition). *Let X, Y and Z be metric spaces. If $F : X \rightrightarrows Y$ is closed valued and measurable $G : Y \rightrightarrows Z$ is u.s.c., then $G \circ F$ is also measurable.*

Remark 6.1.17. *In Prop. 6.1.16, if the map $F_2(\cdot)$ fails to be u.s.c., then measurability of the composition is not guaranteed.*

6.2 Measurable Selections

Definition 6.2.1 (measurable selection). *Let $(\Omega, \mathcal{F})$ be a measurable space, X a metric space and $F : \Omega \rightrightarrows X$. A function $f : \Omega \to X$ is a selector of $F(\cdot)$ if*

$$f(\omega) \in F(\omega), \text{ for each } \omega \in \Omega.$$

A selector f is said to be a measurable selector if $f(\cdot)$ is measurable.

Lemma 6.2.2. *Let X be a separable metric space with $D = \{x_1, x_2, \ldots\}$ being a countable dense subset of X. If $A \subset X$, $A \neq \emptyset$, then for each fixed $n \in \mathbb{N}$*

$$A \cap \bigcup_{k \in \mathbb{N}} \mathbf{B}_{\frac{1}{n}}(x_k) \neq \emptyset.$$

Proposition 6.2.3 (Kuratowski-Ryll-Nardzewski selection Theorem, see Kisielewicz). *If $(\Omega, \mathcal{F})$ be a measurable space, X be a complete separable metric space, $F : \Omega \rightrightarrows X$ and $Dom(F) = X$ is a closed set-valued map, then $F(\cdot)$ has a measurable selector.*

Proof. Let $D = \{x_1, x_2, \ldots\}$ be a countable dense subset of X. By Lem.6.2.2, we have

$$F(\omega) \cap \bigcup_{k\in\mathbb{N}} cl\mathbf{B}_{\frac{1}{n+1}}(x_k) \neq \emptyset$$

for each fixed $n \in \mathbb{N}$. This implies,

$$\exists k \in \mathbb{N} : F(\omega) \cap \mathbf{B}_{\frac{1}{n+1}}(x_k) \neq \emptyset.$$

Let

$$k_n(\omega) := \min\{k \in \mathbb{N} \mid F(\omega) \cap clB_{\frac{1}{n+1}}(x_k) \neq \emptyset\}.$$

Define now, inductively, the set-valued maps

$$F_0(\cdot) = F(\cdot) \text{ and } F_{n+1}(\omega) = F_n(\omega) \cap clB_{n+1}(x_{k_n(\omega)}).$$

Now we have for each $\omega \in \Omega$,

$$F_n(\omega) \supset F_{n+1}(\omega), diam(F_n(\omega)) \leq \frac{1}{n} \to 0$$

and $F_n(\omega)$ is a closed set. Consequently, using Prop. 2.4.10, we have

$$\bigcap_{n\in\mathbb{N}} F_n(\omega) \text{ contains a single element; say } \{f(\omega)\} = \bigcap_{n\in\mathbb{N}} F_n(\omega) \subset F(\omega).$$

Hence, $f(\cdot)$ is a selector for $F(\cdot)$.
Claim: $f(\cdot)$ is measurable.

(i) Given $F_0 = F$ is measurable. By induction, assume that $F_n(\cdot)$ be measurable and $C \subset X$ is a closed set. Then

$$\begin{aligned}\{\omega \in \Omega \mid F_{n+1}(\omega) \cap C \neq \emptyset\} &= \{\omega \in \Omega \mid F_n(\omega) \cap clB_{n+1}(x_{k_n(\omega)}) \cap C \neq \emptyset\} \\ &= \bigcup_{k\in\mathbb{N}} \left(\{\omega \mid F_n(\omega) \cap clB_{n+1}(x_k) \cap C \neq \emptyset\} \cap \{\omega \in \Omega \mid k_n(\omega) = k\}\right).\end{aligned}$$

By induction assumption $\{\omega \mid F_n(\omega) \cap clB_{n+1}(x_k) \cap C \neq \emptyset\} \in \mathcal{F}$. Moreover

$$\{\omega \in \Omega \mid k_n(\omega) = k\} = \bigcap_{i=1}^{k-1} \left(\{\omega \in \Omega \mid F_n(\omega) \cap clB_{n+1}(x_i) = \emptyset\} \cap \{\omega \in \Omega \mid F_n(\omega) \cap clB_{n+1}(x_k) \neq \emptyset\}\right).$$

$\Rightarrow \{\omega \in \Omega \mid k_n(\omega) = k\} \in \mathcal{F}$. Consequently,

$$\{\omega \in \Omega \mid F_{n+1}(\omega) \cap C \neq \emptyset\} \in \mathcal{F}.$$

Hence, for each $n \in \mathbb{N}$, $F_n(\cdot)$ is measurable.

(ii) Let $G(\omega) = \{f(\omega)\}$. By Prop. 6.1.12, the measurability of $F_n(\cdot), n \in \mathbb{N}$, and

$$G(\omega) = \bigcap_{n\in\mathbb{N}} F_n(\omega)$$

we conclude that $G(\cdot)$ is measurable. Now, for a closed set $C \subset X$

$$G^{-}(C) = \{\omega \in \Omega \mid G(\omega) \cap C \neq \emptyset\} = \{\omega \in \Omega \mid \{f(\omega)\} \cap C \neq \emptyset\} = \{\omega \in \Omega \mid f(\omega) \in C\} = f^{-1}(C).$$

This implies, $f^{-1}(C) \in \mathcal{F}$. Therefore, $f(\cdot)$ is measurable.

□

Lemma 6.2.4. *Let X and Y be topological spaces and $F : X \rightrightarrows Y$. If $U \subset Y$ and $C \subset Y$ are any two subsets, then*

$$F^{-}(C) = F^{-}(C \cap U) \cup \left[F^{-}(C) \setminus F^{-}(U)\right].$$

Proof. Trivial!

□

Corollary 6.2.5 (Castaing representation, Aliprantis Border, Rockafellar/Wets)**.** *If $(\Omega, \mathcal{F})$ be a measurable space, X is a separable complete metric space, $F : \Omega \rightrightarrows X$, $Dom(F) = X$ and $F(\cdot)$ is closed valued, then there is a sequence of measurable selectors $\{f_n(\cdot)\}$ such that*

$$F(\omega) = cl\{f_n(\omega) \mid n \in \mathbb{N}\},$$

for each $\omega \in \Omega$.

Proof. Since X is a separable metric space, there is a countable dense subset D of X such that $D = \{x_1, x_2, \ldots, x_n, \ldots\}$. Thus, for each $n, k \in \mathbb{N}$ and $\omega \in \Omega$, define

$$F_{n,k}(\omega) = \begin{cases} F(\omega) \cap \mathbf{B}_{\frac{1}{2^k}}(x_n), & \text{if } F(\omega) \cap \mathbf{B}_{\frac{1}{2^k}}(x_n) \neq \emptyset, \\ F(\omega), & \text{otherwise.} \end{cases}$$

Then, by Lem 6.2.4, for any open subset $O \subset X$, we have that

$$F^{-}(O) = F^{-}\left(O \cap \mathbf{B}_{\frac{1}{2^k}}(x_n)\right) \cup \left[F^{-}(O) \setminus F^{-}\left(\mathbf{B}_{\frac{1}{2^k}}(x_n)\right)\right].$$

By the definition of the $F_n's$ we observe that

$$F_{n,k}^{-}(O) = F^{-}\left(O \cap \mathbf{B}_{\frac{1}{2^k}}(x_n)\right) \cup \left[F^{-}(O) \setminus F^{-}\left(\mathbf{B}_{\frac{1}{2^k}}(x_n)\right)\right].$$

But, the measurability of $F(\cdot)$ implies that $F^{-}(O \cap U_n) \cup [F^{-}(O) \setminus F^{-}(U_n)]$ is measurable. Consequently, for each $n, k \in \mathbb{N}$, $F_{n,k}(\cdot)$ is measurable and $Dom(F_{n,k}) = \Omega$. Using Prop. 6.1.9, we find that $clF_{n,k}(\cdot)$ is measurable and the assumptions of Prop. 6.2.3 are also satisfied. Hence, for each $n, k \in \mathbb{N}$, there is a measurable selector $f_{n,k} : \Omega \to X$ such that

$$f_{n,k}(\omega) \in clF_{n,k}(\omega) \text{ and } clF_n(\omega) \subset F(\omega), \omega \in \Omega. \tag{6.1}$$

for each fixed $\omega \in \Omega$.
Claim: For each $\omega \in \Omega, F(\omega) = cl\{f_{n,k}(\omega) \mid n, k \in \mathbb{N}\}$. From (6.1), the following inclusion is obvious

$$cl\{f_{n,k}(\omega) \mid n, k \in \mathbb{N}\} \subset F(\omega)$$

Now, let $x \in F(\omega)$, $\varepsilon > 0$ and $\mathbf{B}_\varepsilon(x)$ be a neighborhood of x. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{2^{k-1}} < \varepsilon$, and by the density of D, there is $n \in \mathbb{N}$ such that

$$x \in \mathbf{B}_{\frac{1}{2^k}}(x_n) \Rightarrow x \in F(\omega) \cap \mathbf{B}_{\frac{1}{2^k}}(x_n) \Rightarrow \omega \in F_{n,k}^{-}\left(\mathbf{B}_{\frac{1}{2^k}}(x_n)\right)$$

Thus,

$$f_{n,k}(\omega) \in clF_{n,k}(\omega) = clF(\omega) \cap cl\mathbf{B}_{\frac{1}{2^k}}(x_n) \Rightarrow f_{n,k}(\omega) \in cl\mathbf{B}_{\frac{1}{2^k}}(x_n).$$

Consequently,

$$\rho(f_{n,k}(\omega), x) \leq \rho(f_{n,k}(\omega), x_n) + \rho(x_n, x) \leq \frac{1}{2^k} + \frac{1}{2^k} \leq \frac{1}{2^{k-1}} < \varepsilon.$$

$\Rightarrow f_{n,k}(\omega) \in \mathbf{B}_\varepsilon(x)$. Since $\varepsilon > 0$ is arbitrary, we see that

$$x \in cl\{f_{n,k}(\omega) \mid n, k \in \mathbb{N}\}.$$

Therefore,

$$F(\omega) \subset cl\{f_{n,k}(\omega) \mid n, k \in \mathbb{N}\}.$$

Then the claim of follows by re-indexing the countable set $\{f_{n,k}(\omega) \mid n, k \in \mathbb{N}\}$. □

6.3 Measurability of Set-Valued Maps with given Structure

Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two measurable spaces. We say that a function $f : X \to Y$ is measurable (i.e. $(\mathcal{F}, \mathcal{G})$ measurable) if

$$\forall A \in \mathcal{G} : f^{-1}(A) \in \mathcal{F}.$$

In particular, a real valued function $f : X \to \mathbb{R}$ is measurable if

$$f^{-1}(A) \in \mathcal{F}$$

whenever A is a Borel set in $\mathbb{R}$.

Definition 6.3.1 (Caratheodory function). *Let $f : \Omega \times X \to \mathbb{R}$. Then f is said to be a* ***Caratheodory function*** *if*

(i) for each fixed $x \in X$, $f(\cdot, x) : \Omega \to \mathbb{R}$ is measurable; and

(ii) for each fixed $\omega \in \Omega$, $f(\omega, \cdot) : Y \to \mathbb{R}$ is continuous.

Remark 6.3.2. *Note that if $f : \Omega \times X \to \mathbb{R}$ is a Caratheodory function, then $-f$ is also a Caratheodory function.*

Definition 6.3.3 (Epigraphical and Domain maps). *Let $f : \Omega \times X \to \overline{\mathbb{R}}$. Then*

- *the* ***epigraphical map*** *associated with $f(\cdot, \cdot)$ is the set-valued map $E_f : \Omega \rightrightarrows X \times \mathbb{R}$ given by*

$$E_f(\omega) = \{(x, \lambda) \in X \times \mathbb{R} \mid f(\omega, x) \leq \lambda\}$$

- *the* ***domain map*** *associated with $f(\cdot, \cdot)$ is the set-valued map $D_f : \Omega \rightrightarrows X$ given by*

$$D_f(\omega) = \{x \in X \mid f(\omega, x) < \infty\}$$

Proposition 6.3.4. *Let $(\Omega, \mathcal{F})$ be a measurable space and X be a separable metric space. If $f : \Omega \times X \to \mathbb{R}$ is a Caratheodory function, then the set-valued map $F : \Omega \rightrightarrows X$ given by*

$$F(\omega) = \{x \in X \mid f(\omega, x) \leq 0\}, \omega \in \Omega$$

is measurable.

Proof. Since $F(\cdot)$ is closed valued and X is separable, we use Prop. 6.1.6. Thus, let D a countable dense subset of X and $C \subset X$ be a closed set. Then $C \cap D$ is a countable dense subset of C, say $D \cap C = \{x_1, x_2, \ldots\}$ and

$$\begin{aligned} F^-(C) &= \{\omega \in O \mid F(\omega) \cap C \neq \emptyset\} \\ &= \{\omega \in \Omega \mid f(\omega, x) \leq 0, \text{for some } x \in C\}. \end{aligned}$$

Since $f(\omega, \cdot)$ is continuous, for $\overline{x} \in C$, $f(\omega, \overline{x}) \leq 0$ implies there exists a neighborhood $U(\overline{x})$ such that $f(\omega, x) \leq 0, \forall x \in U(\overline{x})$. Hence, by the density of $D \cap C$ in C, there is $x_n \in U(\overline{x}) \cap (D \cap C)$ such that $f(\omega, x_n) \leq 0$. Consequently, we can write

$$F^-(C) = \{\omega \in \Omega \mid f(\omega, x_n) \leq 0, \text{for some } n \in \mathbb{N}\} \tag{6.2}$$

$$= \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid f(\omega, x_n) \leq 0\} \tag{6.3}$$

But, for each $n \in \mathbb{N}$, $\{\omega \in \Omega \mid f(\omega, x_n) \leq 0\} = f^{-1}(\cdot, x_n)[(-\infty, 0]$ is measurable, since $f(\cdot, x_n)$ is measurable. Consequently, $F^-(C) \in \mathcal{F}$. □

Proposition 6.3.5 (measurability of the epigraphical map). *Let* $(\Omega, \mathcal{F})$ *is a measurable space,* X *is a separable metric space. If* $f : \Omega \times X \to \mathbb{R}$ *is a Caratheodory function, then the epigraphical map* $E_f(\cdot) : \Omega \rightrightarrows X \times \mathbb{R}$ *of* f *is closed-valued and measurable.*

Proof. Since

$$E_f(\omega) = \{(x, \lambda) \in X \times \mathbb{R} \mid f(\omega, x) \leq \lambda\} = \{(x, \lambda) \in X \times \mathbb{R} \mid f(\omega, x) - \lambda \leq 0\},$$

defining $g(\omega, (x, \lambda)) := f(\omega, x) - \lambda$, we have that $g : \Omega \times X \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function on $\Omega \times (X \times \mathbb{R})$; i.e. $g(\cdot, (x, \lambda))$ is measurable for each fixed $(x, \lambda) \in X \times \mathbb{R}$ and $g(\omega, \cdot)$ is continuous for each fixed $\omega \in \Omega$. Therefore,

$$E_f(\omega) = \{(x, \lambda) \in X \times \mathbb{R} \mid g(\omega, (x, \lambda)) \leq 0\}$$

is measurable, according to Prop. 6.3.4.

□

Corollary 6.3.6 (measurability of the domain map). *Let* $(\Omega, \mathcal{F})$ *is a measurable space,* X *is a separable metric space. If* $f : \Omega \times X \to \mathbb{R}$ *is a Caratheodory function, then the domain map* $D_f(\cdot) : \Omega \rightrightarrows X$ *is measurable.*

Proof. Let $O \subset X$ be an open set. Then

$$\omega \in D_f^-(O) \Rightarrow D_f(\omega) \cap O \neq \emptyset.$$

$\Rightarrow \exists x \in D_f(\omega) \cap O, \exists \lambda \in \mathbb{R}$ such that

$$f(\omega, x) < \lambda < \infty \Rightarrow \omega \in E_f^-(O \times (-\infty, \lambda)).$$

Conversely, if $\omega \in E_f^-(O \times (-\infty, \lambda))$, then for some $x \in O$ we have

$$f(\omega, x) < \lambda < \infty \Rightarrow x \in D_f(\omega) \Rightarrow D_f(\omega) \cap O \neq \emptyset \Rightarrow \omega \in D_f^-(O).$$

Hence,

$$D_f^-(O) = E_f^-(O \times (-\infty, \lambda)).$$

Since $E_f^-(\cdot)$ is measurable, we conclude that $D_f^-(\cdot)$ is also measurable.

□

Proposition 6.3.7. *If* $f : \Omega \times X \to \mathbb{R}$ *is a Caratheodory function and* $x : \Omega \to X$ *is a measurable function, then the function* $g : \Omega \to \mathbb{R}$ *given by*

$$g(\omega) = f(\omega, x(\omega)), \omega \in \Omega,$$

is measurable.

Proof. Let $\alpha \in \mathbb{R}$ be any. We consider the set

$$\{\omega \in \Omega \mid g(\omega) < \alpha\} = \{\omega \in \Omega \mid f(\omega, x(\omega)) < \alpha\}.$$

Which we can also write as(see also Rockfellar/Wets)

$$\begin{aligned}
\{\omega \in \Omega \mid f(\omega, x(\omega)) < \alpha\} &= \{\omega \in \Omega \mid \exists \lambda \in \mathbb{R} : f(\omega, x) \leq \lambda, x(\omega) = x, \lambda < \alpha\} && (6.4)\\
&= \{\omega \in \Omega \mid \exists (x, \lambda) \in E_f(\omega), (x, \lambda) \in \{x(\omega)\} \times (-\infty, \alpha)\} && (6.5)\\
&= \{\omega \in \Omega \mid E_f(\omega) \cap R(\omega) \neq \emptyset\}, && (6.6)
\end{aligned}$$

where $R(\omega) := \{x(\omega)\} \times (-\infty, \alpha)$. Consequently, we obtain that

$$\{\omega \in \Omega \mid f(\omega, x(\omega)) < \alpha\} = Dom(E_f \cap R).$$

Since, $x(\cdot) : \Omega \to X$ is measurable and $(-\infty, \alpha)$ is a constant, the set value map $G : \Omega \rightrightarrows X \times \mathbb{R}$ is measurable, by Prop.6.1.15 also $E_f(\cdot)$ is measurable (Prop. 6.3.5). Thus, using Prop. 6.1.12 we conclude that the intersection map $(E_f \cap R)(\cdot)$ is measurable; hence, $Dom(E_f \cap R)$ is a measurable set in Ω. Therefore, $g(\cdot)$ is a measurable function. □

Proposition 6.3.8. *Let Ω be a complete metric space with $\mathcal{F}$ being the Borel σ-algebra and X a separable metric space. If $f : \Omega \times X \to \mathbb{R}$ is a lower semi-continuous function with respect to both Ω and X, then the set-valued map $F :\ \Omega \rightrightarrows X$ given by*

$$F(\omega) = \{x \in X \mid f(\omega, x) \leq 0\}, \omega \in \Omega$$

is Borel-measurable.

Proof. Uses the same ideas as in Prop. 6.3.4.

□

Proposition 6.3.9. *Let $(\Omega, \mathcal{F})$ be a measurable space and X be a separable metric space. If $g : \Omega \times X \to \mathbb{R}$ is a Caratheodory function, then the set-valued map $F :\ \Omega \rightrightarrows X$ given by*

$$F(\omega) = \{x \in X \mid g(\omega, x) = 0\}, \omega \in \Omega$$

is measurable.

Proof. We can write

$$F(\omega) = \{x \in X \mid g(\omega, x) \leq 0\} \cap \{x \in X \mid -g(\omega, x) \leq 0\}, \omega \in \Omega$$

Set

$$F_1(\omega) = \{x \in X \mid g(\omega, x) \leq 0\} \text{ and } F_2 = \{x \in X \mid -g(\omega, x) \leq 0\}$$

Then, Prop. 6.3.4 yields that both $F_1(\cdot)$ and $F_2(\cdot)$ are measurable. Therefore, by Prop. **??**, $F(\cdot)$ is measurable.

□

Proposition 6.3.10 (measurability of feasible set maps, see also Rockfellar/Wets). *Let $(\Omega, \mathcal{F})$ be a measurable space and X be a separable metric space. If $g_i : \Omega \times X \to \mathbb{R}, i \in I$, and $h_j : \Omega \times X \to \mathbb{R}, j \in J$ are Caratheodory functions, then the set-valued map $F :\ \Omega \rightrightarrows X$ given by*

$$F(\omega) = \{x \in X \mid h_j(\omega, x) \leq 0, j \in J; g_i(\omega, x) = 0, i \in I\}, \omega \in \Omega$$

is measurable.

Proof. Define

$$\begin{aligned} F_1(\omega) &= \{x \in X \mid h_j(\omega, x) \leq 0, j \in J\} && (6.7) \\ F_2(\omega) &= \{x \in X \mid g_i(\omega, x) = 0, i \in I\}. && (6.8) \end{aligned}$$

and apply Props. 6.3.4, 6.3.9 and 6.1.12.

□

Proposition 6.3.11 (measurability of Marginal Functions). *Let $(\Omega, \mathcal{F})$ be a measurable space and X be a complete separable metric space. If $f : \Omega \times X \to \mathbb{R}$ a Caratheodory function and $F : \Omega \rightrightarrows X$ is closed valued and measurable, then the marginal value function*

$$\varphi(\omega) := \sup_{x \in F(\omega)} f(\omega, x)$$

is measurable as a function $\varphi : \Omega \to \mathbb{R}$.

Proof. Since $\varphi(\cdot)$ is a real valued function, for $\gamma \in \mathbb{R}$, we consider the set

$$\{\omega \in \Omega \mid \varphi(\omega) \leq \gamma\} = \{\omega \in \Omega \mid \sup_{x \in F(\omega)} f(\omega, x) \leq \gamma\}.$$

Using Cor. 6.2.5, there is a sequence of measurable functions $\{x_n(\cdot)\}, x_n : \Omega \to X$ such that $F(\omega) = cl\{x_n(\omega) \mid n \in \mathbb{N}\}, \omega \in \Omega$. Hence,

$$\{\omega \in \Omega \mid \varphi(\omega) \leq \gamma\} = \{\omega \in \Omega \mid \sup_{x \in cl\{x_n(\omega) \mid n \in \mathbb{N}\}} f(\omega, x) \leq \gamma\}$$

Since $f(\omega, \cdot)$ is continuous, the above can be written as (why?)

$$\{\omega \in \Omega \mid \varphi(\omega) \leq \gamma\} = \{\omega \in \Omega \mid f(\omega, x_n(\omega)) \leq \gamma, \forall n \in \mathbb{N}\} = \bigcap_{n \in \mathbb{N}} \{\omega \in \Omega \mid f(\omega, x_n(\omega)) \leq \gamma\}.$$

Thus, Prop. 6.3.7 implies that, for each $n \in \mathbb{N}$, then function $g_n : \Omega \to \mathbb{R}$ given by $g_n(\omega) = f(\omega, x_n(\omega))$ is measurable. Consequently, $\{\omega \in \Omega \mid \varphi(\omega) \leq \gamma\}$ is a countable intersection of measurable sets. Therefore, $\varphi(\cdot)$ is measurable. □

Excercises 6.3.12. *Suppose $(\Omega, \mathcal{F})$ be a measruable space and X a metric space. Prove that if, for each $k \in \mathbb{N}, F_k : \Omega \rightrightarrows X$ is closed valued and measurable and, for each $x \in X$, $\bigcap_{k \in \mathbb{N}} F_k(x) \neq \emptyset$, then the set valued map*

$$\left(\bigcap_{k \in \mathbb{N}} F_k\right) : \Omega \rightrightarrows X$$

is measurable. (From this, it follows that the feasible set-valued map is measurable.)

---

7 Comments on Literature

There are several books and literature dealing with set-valued maps and their applications. But any one who wants to know about set-valued maps can begin with the list given below. However, this list is by no means exhaustive and is biased by my personal preferences and repeated citations in the literature.

- General set valued maps: Aubin & Cellina [3], Aubin & Frakowsk[4], Berge[6], HU & Papageoriou[14], Kiesielewichz[15], Rockafellar & Wets[22], Aliprantis & Border[1], Göfert et al. [10] etc.
- Set valued maps defined using parametric systems of functions: Bank et al.[5], Shimizu et al.[24], etc.
- Measurable set-valued maps: Rockafellar & Wets[22], Castaing & Valadier[8], etc.
- Differentiability properties of set-valued maps: Aubin[2], Göpfert[10], etc.
- Fixed Point Properties: Granas & Dugundji[12], Goebe & Kirk[11], etc.

Some of the books cited above still discuss other issues related with set valued maps. For instance, differential inclusions are discussed by Aubin & Cellina[3], Demiling[9], etc. Set valued maps as applied to optimization problems Shimizu et al.[24], Berger[7], etc. In an case, it worth paying attention to the literature cited in each of the books suggested above.

Bibliography

[1] C. D. Aliprantis, K. C. Border, Infinite Dimensional Analysis - A Hitchhiker's Guide (2nd. ed), Springer-Verlag, 1999.

[2] J.-P. Aubin, Mutational and Morphological Analysis, Birkhäuser, 1999.

[3] J.-P. Aubin, A. Cellina, Differential inclusions, Springer Verlag, Berlin 1984.

[4] J.-P. Aubin, H. Frankowska, Set valued analysis, Birkhäuser, Basel, 1990.

[5] B. Bank, J. Guddat, D. Klatte, B. Kummer, K. Tammer: Non-linear parametric optimization, Akademie-Verlag, Berlin, 1982.

[6] C. Berge, Topological Spaces. Oliver & Boyd, Edinburg, London, 1963.

[7] A. Berger, Beiträge zur Realisierung von Lösungsverfahren der nichtlinearen Optimierung. Ph.D. Dissertation, Technical University of Ilemanu, Fakulty of Mathematics and Natural Scinces, January, 1978.

[8] C. Castaing, M. Valadier, Convex analysis and measurable multifunctions. Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, 1977.

[9] K. Deimling, Multivalued Differential Equations, Walter de Gruyter & Co., 1992.

[10] A. Göfert, H. Riahi, C. Tammer, C. Zălinescu, Variational methods in partially ordered spaces. Springer, 2003.

[11] K. Goebel , W. A. Kirk, Topics in metric fixed point theory. Cambridge studies in advanced mathematics, V. 28, Cambridge University Press, 1990.

[12] A. Granas, J. Dugundji, Fixed point theory. Springer Monongrphs in Mathematics, Springer-Verlag, 2003.

[13] W. W. Hogan, Point-to-set maps in mathematical programming. SIAM Review, Vol. 15, No.3, pp. 591 - 603, 1973.

[14] S. Hu, N. S. Papageorgiou, Handbook of multivalued analysis: Volume I. Kluwer Academic Publishers, 1997.

[15] M. Kisielewicz, Differential Inclusions and Optimal Control. Polish Scientific Publishers & Kluwer Academic Publishers, 1991.

[16] D. Klatte, R. Henrion, Regularity and stability in non-linear semi-infinite optimization. Semi-infinite Programming, R. Reemtsen and J.-J. Rückmann (eds.), pp. 69-102, Kluwer Academic Pres, 1998.

[17] P. Kosmol, Optimierung und Approximation. Walter de Guyter & Co., 1991.

[18] G. Meinardus, Approximation of Functions: Theory and Numerical Methods. Springer-Verlag, 1967.

[19] Robinson, S. M., Some continuity properties of polyhedral multifunctions. Math. Prog. Study, V. 14, pp. 206-214, 1991.

[20] Robinson, S. M., Regularity and stability for convex multivalued functions. Math. of OR, V. 1, pp. 130-143, 1976.

[21] H. Royden, Real Analysis (3rd. edition). Macmillan Publishing Company, 1988.

[22] R. T. Rockafellar, R. J.-B. Wets, Varaitional Analysis, Springer Verlag, 1998.

[23] S. Shirali, H. L. Vasudeva, Metric Spaces. Springer-Verlag, 2006.

[24] K. Shimizu, Y. Ishizuka and J. Bard, Nondifferentiable and Two-Level Mathematical Programming. Kluwer Academic Publishers, 1997.

[25] J.-B. H. Urruty, C. Lemaréchal, Convex analysis and minimization algorithms I. Springer Verlag,1993.