On the Topology of Projective Shape Spaces

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Multiple view pinhole camera model

Figure: (Hartley, Zisserman: Multiple View Geometry)
Single view pinhole camera model

projective transformation

1
2
3
4

object hyperplane

film hyperplane
Projective shape space

Space of configurations

\((\mathbb{RP}^d)^k\).

Space of projective shapes is topological quotient

\((\mathbb{RP}^d)^k / \text{PGL}(d)\).

In homogeneous coordinates, this is the space of equivalence classes

\(X \sim DXB\)

with \(X\) a \(k \times (d + 1)\) configuration matrix with non-trivial rows, \(D\) non-singular \(k \times k\)-diagonal matrix, \(B \in \text{GL}(d + 1)\).
Invariants under $\text{PGL}(d)$

- linear dependencies of the rows of $X$, here called \textit{subspace constraints};

\[
\begin{align*}
\left[ x_A, x_i, x_j \right] \cdot \left[ x_A, x_k, x_l \right] \\
\left[ x_A, x_i, x_k \right] \cdot \left[ x_A, x_j, x_l \right]
\end{align*}
\]

- (generalized) cross-ratios:

\[
\frac{\left[ x_A, x_i, x_j \right] \cdot \left[ x_A, x_k, x_l \right]}{\left[ x_A, x_i, x_k \right] \cdot \left[ x_A, x_j, x_l \right]}
\]

with distinct $i, j, k, l \notin A \subset \{1, \ldots, k\}$, $|A| = d - 1$ and $[\cdot]$ denoting the determinant.
Motivation

We are interested in metric comparisons on space of projective shapes requires e.g. metric space, or even Riemannian manifold;

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**Problem:** \((\mathbb{RP}^d)^k / \text{PGL}(d)\) is not metrizable, not a manifold!

**Solution:** Use “good” subspace.
Desired subspace properties

1. metric space
   - (complete) Riemannian manifold
   - embedding into Euclidean space

2. respecting hierarchy of subspace constraints, containing all non-degenerate shapes

3. closed under reordering of points

4. maximal with this properties
Projective frames

Theorem (Mardia, Patrangenaru; 2005)

The subspace of shapes with first $d + 2$ points in general position is diffeomorphic to $(\mathbb{RP}^d)^{k - d - 2}$.

Idea: map a shape to its representant of type

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\cdots \\
x_{d+3} \\
x_k
\end{pmatrix}
\]

eliminating the $\text{GL}(d + 1)$-action.

This subspace is not closed under reordering for any $k \geq d + 3$!
Subspaces bounded by subspace numbers

We call a subspace $\mathcal{X} \subseteq (\mathbb{RP}^d)^k$ bounded by subspace numbers if there are numbers $n_i \in \mathbb{R}^+$, $i \in \{1, \ldots, d\}$, with $i \leq n_i \leq n_j$ for any $1 \leq i \leq j \leq d$ such that

$$\mathcal{X} = \{ p \in (\mathbb{RP}^d)^k : \text{any} (i-1)\text{-dimensional projective subspace contains less than } n_i \text{ points of } p \}.$$ 

$\mathcal{X} / \text{PGL}(d)$ fulfills properties (2) and (3)!
Subspaces bounded by subspace numbers

Theorem (FK, Kent & Hotz ; 2016)

Let \( \mathcal{X} \subseteq (\mathbb{RP}^d)^k \) be bounded by subspace numbers \((n_1, \ldots, n_d)\). Then, \( \mathcal{X} / \text{PGL}(d) \) is a differentiable Hausdorff manifold if and only if

\[
\lceil n_i - 1 \rceil + \lceil n_{d+1-i} - 1 \rceil < k \quad \text{for all} \quad 1 \leq i \leq d.
\]

Hausdorff: \( \mathcal{X} / \text{PGL}(d) \) does not contain shapes

\[
\begin{pmatrix}
X_1 & 0 \\
Y & X_2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
X_1 & Z \\
0 & X_2
\end{pmatrix}
\]

which are limit points of the sequence

\[
\begin{pmatrix}
X_1 & \frac{1}{n}Z \\
Y & X_2
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} X_1 & Z \\ \frac{1}{n}Y & X_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} X_1 & Z \\ \frac{1}{n}Y & X_2 \end{pmatrix}.
\]
(Manifold-valued) Charts are given by a generalization of the concept of frames:

$d = 2$: standardization to

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\vdots \\
x_{d+3} \\
x_k
\end{pmatrix}
\]

gives a diffeomorphism to \( (\mathbb{RP}^2)^{k-d-2} \)
(Manifold-valued) Charts are given by a generalization of the concept of frames:
\( d = 2 \): standardization to
\[
\begin{pmatrix}
1 \\
1 \\
1 \\
y \\
x_{d+4} \\
\vdots \\
x_k
\end{pmatrix}
\]
gives a diffeomorphism to \( \mathbb{R}^2 \times (\mathbb{RP}^2)^{k-d-3} \)
Space of Tyler regular shapes

Special case

\[ n_i = \frac{ik}{d + 1}. \]

Standardization (Mardia, Kent; 2006) to configuration \( X \) with

\[ X^t X = \frac{k}{d+1} I_{d+1} \quad \text{and} \quad \|x_i\| = 1 \text{ for all } 1 \leq i \leq k. \]

Ambiguity of discrete group from left and \( O(d + 1) \) from right remains. Consider \( XX^t \) to remove \( O(d + 1) \)-ambiguity.

\[ \rightarrow \text{embedding of covering of space of Tyler regular shapes into Euclidean space } (\text{Sym}(k), \| \cdot \|_F). \]

Remaining question: How to use this for (extrinsic) statistic?
Summary

- topology of projective shape space now understood
- subspace bounded by subspace numbers are differentiable Hausdorff manifolds under some conditions
- concept of frames generalized to obtain charts
- construction of metrics still open question
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Thank you!